6. The ω and Ω functions

Def. For $n \in \mathbb{N}$, let $\omega(n)$ denote the number of distinct prime factors of n, and let $\Omega(n)$ denote the number of prime factors of n counted with multiplicity

Ex Let $n = 2^3 3^5 5^2$. Then $\omega(n) = 3$ and $\Omega(n) = 10$.

For $k \in \mathbb{N}$ and $x \in \mathbb{R}$, define

$$\tau_k(x) = \#\{n \le x ; \Omega(n) = k\}$$

and

$$\pi_k(x) = \#\{n \le x ; \omega(n) = \Omega(n) = k\}.$$

The latter counts the number of $n \le x$ which are square-free and have k prime factors. Note that

$$\pi(x) = \pi_1(x) = \tau_1(x).$$

Theorem 26 (Landau, 1900): For $k \in \mathbb{N}$, we have

$$\pi_k(x) \sim \tau_k(x) \sim \frac{1}{(k-1)!} \frac{x}{\log x} (\log\log x)^{k-1}.$$

Proof: Define

$$L_k(x) = \sum_{p_1 \cdots p_k \le x}' \frac{1}{p_1 \cdots p_k},$$

$$\Pi_k(x) = \sum_{p_1 \cdots p_k \le x}' 1,$$

and

$$\Theta_k(x) = \sum_{p_1 \cdots p_k \le x}' \log(p_1 \cdots p_k),$$

where Σ' signifies that the sum is taken over all k-tuples of primes $(p_1, \dots, p_k)$ with $p_1 \cdots p_k \le x$. Note that different k-tuples may correspond to the same product $p_1 \cdots p_k$. For $n \in \mathbb{N}$, let

$$c_n = c_n(k) = \#\{k\text{-tuples } (p_1, \dots, p_k) : p_1 \cdots p_k = n\}.$$

Thus we have

$$\Pi_k(x) = \sum_{n \le x} c_n \quad \text{and} \quad \Theta_k(x) = \sum_{n \le x} c_n \log n.$$

Note that

$$c_n = \begin{cases} 0 & \text{if } n \text{ is not a product of } k \text{ primes,} \\ k! & \text{if } n \text{ is square-free and } \omega(n) = \Omega(n) = k. \end{cases}$$

Also, $0 < c_n < k!$ if $\Omega(n) = k$ and n is not square free. It follows that

$$k!\,\pi_k(x) \le \Pi_k(x) \le k!\,\tau_k(x). \qquad \text{——— ①}$$

For $k \ge 2$, note that

$$\tau_k(x) - \pi_k(x) = \#\{n \le x \,;\, \Omega(n) = k \text{ and } n \text{ is not square-free}\}.$$

Thus

$$\tau_k(x) - \pi_k(x) = \sum_{\substack{p_1 \cdots p_k \le x \\ p_i = p_j \text{ some } i \ne j}}' 1 \le \binom{k}{2} \sum_{p_1 \cdots p_k \le x}' 1 = \binom{k}{2} \Pi_{k-1}(x). \qquad \text{——— ②}$$

By ① and ②, to prove that

$$\pi_k(x) \sim \tau_k(x) \sim \frac{1}{(k-1)!} \frac{x}{\log x} (\log\log x)^{k-1},$$

it suffices to show that for all $k \in \mathbb{N}$,

$$\Pi_k(x) \sim k \frac{x(\log\log x)^{k-1}}{\log x} \qquad \text{——— }(*)$$

Let $a_n = c_n$ and $f(u) = \log u$. By Abel's summation 2015 10 16

$$\Theta_k(x) = \sum_{n \le x} c_n \log n = \Pi_k(x) \log x - \int_1^x \frac{\Pi_k(u)}{u}\, du.$$

Observe that

$$\Pi_k(x) \le k!\,\tau_k(x) \le k!\, x.$$

Thus $\Pi_k(u) = O(u)$. It follows that

$$\Theta_k(x) = \Pi_k(x) \log x + O(x).$$

Thus to prove $(*)$, it suffices to show that for all $k \in \mathbb{N}$,

$$\Theta_k(x) \sim kx(\log\log x)^{k-1}.$$

We'll prove this by induction on k. For $k=1$

$$\Theta_1(x) = \theta(x) \sim x$$

by the Prime Number Theorem. Assume now that $\Theta_k(x) \sim kx(\log\log x)^{k-1}$.

Consider Θ_{k+1}. Note that for $k \geq 1$,

$$\left(\sum_{p \leq x^{1/k}} \frac{1}{p}\right)^k \leq L_k(x) \leq \left(\sum_{p \leq x} \frac{1}{p}\right)^k.$$

By theorem 17,

$$\left(\sum_{p \leq x} \frac{1}{p}\right)^k \sim (\log\log x)^k$$

and

$$\left(\sum_{p \leq x^{1/k}} \frac{1}{p}\right)^k \sim (\log\log x^{1/k})^k$$

$$= (\log\log x - \log k)^k$$

$$\sim (\log\log x)^k.$$

Thus

$$L_k(x) \sim (\log\log x)^k.$$

It follows that

$$\Theta_{k+1}(x) - (k+1)x(\log\log x)^k = \Theta_{k+1}(x) - (k+1)xL_k(x) + o(x(\log\log x)^k).$$

Note that

$$k\log(p_1 \cdots p_{k+1}) = \log(p_1^k \cdots p_{k+1}^k)$$

$$= \log(p_2 \cdots p_{k+1}) + \log(p_1 p_3 \cdots p_{k+1}) + \cdots + \log(p_1 \cdots p_k)$$

where the sum above is over all k-subsets of $\{1, 2, \ldots, k, k+1\}$. Thus we have

$$k\Theta_{k+1}(x) = \sum_{p_1 \cdots p_{k+1} \leq x}' k\log(p_1 \cdots p_{k+1})$$

$$= \sum_{p_1 \cdots p_{k+1} \leq x}' \left(\log(p_2 \cdots p_{k+1}) + \cdots + \log(p_1 \cdots p_k)\right)$$

$$= (k+1)\sum_{p_1 \leq x} \sum_{p_2 \cdots p_{k+1} \leq x/p_1}' \log(p_2 \cdots p_{k+1})$$

$$= (k+1)\sum_{p_1 \leq x} \Theta_k\left(\frac{x}{p_1}\right).$$

Next, we put $L_0(x) = 1$ and note

$$L_k(x) = \sum_{p_1 \cdots p_k \leq x}' \frac{1}{p_1 \cdots p_k} = \sum_{p \leq x} \frac{1}{p_1} L_{k-1}\left(\frac{x}{p_1}\right).$$

Thus by the above two estimates,

$$\Theta_{k+1}(x) - (k+1)xL_k(x) = (k+1)\sum_{p_1 \le x}\left(\frac{1}{k}\Theta_k\left(\frac{x}{p_1}\right) - \frac{x}{p_1}L_{k-1}\left(\frac{x}{p_1}\right)\right).$$

By induction hypothesis

$$\Theta_k(y) - kyL_{k-1}(y) = o\left(y(\log\log y)^{k-1}\right).$$

Thus given $\varepsilon > 0$, there exists $x_0 = x_0(\varepsilon, k)$ such that for $y > x_0$,

$$\left|\Theta_k(y) - kyL_{k-1}(y)\right| \le \varepsilon y(\log\log y)^{k-1}.$$

Further, there exists $c = c(\varepsilon, k) > 0$ such that for $y \le x_0$,

$$\left|\Theta_k(y) - kyL_{k-1}(y)\right| \le c.$$

Thus for x sufficiently large (note: $\frac{x}{p_1} > x_0 \iff p_1 < \frac{x}{x_0}$)

$$\begin{aligned}\left|\Theta_{k+1}(x) - (k+1)xL_k(x)\right| &\le \frac{k+1}{k}\left(\sum_{p_1 < \frac{x}{x_0}} \varepsilon\frac{x}{p_1}\left(\log\log\frac{x}{p_1}\right)^{k-1} + \sum_{\frac{x}{x_0} \le p_1 \le x} c\right)\\ &\le 2\varepsilon x(\log\log x)^{k-1}\sum_{p_1 < \frac{x}{x_0}}\frac{1}{p_1} + 2cx\\ &\le 4\varepsilon x(\log\log x)^k + 2cx\\ &\le 5\varepsilon x(\log\log x)^k.\end{aligned}$$

Thus

$$\Theta_{k+1}(x) - (k+1)xL_k(x) = o\left(x(\log\log x)^k\right),$$

from which we conclude that

$$\Theta_{k+1}(x) \sim (k+1)x(\log\log x)^k. \quad \blacksquare$$

Theorem 27: We have

$$\sum_{n \le x} \omega(n) = x\log\log x + \beta x + o(x)$$

and

$$\sum_{n \le x} \Omega(n) = x\log\log x + \tilde{\beta}x + o(x)$$

where β is Merten's constant and

$$\tilde{\beta} = \beta + \sum_p \frac{1}{p(p-1)}.$$

2015 10 19

Proof: Let

$$S(x)=\sum_{n\le x}\omega(n) \quad \text{and} \quad T(x)=\sum_{n\le x}\Omega(n).$$

By theorem 17, we have

$$S(x)=\sum_{n\le x}\sum_{p\mid n}1=\sum_{p\le x}\left\lfloor\frac{x}{p}\right\rfloor = x\sum_{p\le x}\frac{1}{p}+O(\pi(x))$$
$$=x(\log\log x+\beta+o(1))+O(\pi(x))$$
$$=x\log\log x+\beta x+o(x).$$

Note that

$$T(x)-S(x)=\sum_{\substack{p^m\le x\\ m\ge 2}}\left\lfloor\frac{x}{p^m}\right\rfloor=\sum_{\substack{p^m\le x\\ m\ge 2}}\frac{x}{p^m}+O\left(\sum_{\substack{p^m\le x\\ m\ge 2}}1\right).$$

Note that $2^m\le p^m\le x$ and thus $m\le \log x/\log 2$. Also, $p^2\le p^m\le x$ implies that $p\le x^{\frac{1}{2}}$. Thus

$$T(x)-S(x)=x\sum_{\substack{p^m\le x\\ m\ge 2}}\frac{1}{p^m}+O\left(x^{\frac{1}{2}}\log x\right)$$
$$=x\left(\sum_p\left(\frac{1}{p^2}+\frac{1}{p^3}+\cdots\right)-\sum_{\substack{p^m>x\\ m\ge 2}}\frac{1}{p^m}\right)+O\left(x^{\frac{1}{2}}\log x\right).$$

Note that

$$\sum_{\substack{p^m>x\\ m\ge 2}}\frac{1}{p^m}\le\sum_{n\ge x}\left(\frac{1}{n^2}+\frac{1}{n^3}+\cdots\right)=O\left(\frac{1}{x}\right)=O(1).$$

Thus

$$T(x)-S(x)=x\left(\sum_p\frac{1}{p(p-1)}+o(1)\right)+O\left(x^{\frac{1}{2}}\log x\right).$$

By our estimate of $S(x)$, we have

$$T(x)=x\log\log x+\underbrace{\left(\beta+\sum_p\frac{1}{p(p-1)}\right)}_{\tilde{\beta}}x+o(x).$$ ∎

Def] Let $A\subseteq\mathbb{N}$. For $n\in\mathbb{N}$, let

$$A(n)=\{1,2,\ldots,n\}\cap A.$$

The upper asymptotic density of A, denoted by $\bar{d}(A)$, is defined by

$$\bar{d}(A)=\limsup_{n\to\infty}\frac{|A(n)|}{n}.$$

Similarly, we define $\underline{d}(A)$ the <u>lower asymptotic density</u> of A by
$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A(n)|}{n}.$$
If $\bar{d}(A) = \underline{d}(A)$, we say A has an <u>asymptotic density</u>
$$d(A) = \bar{d}(A) = \underline{d}(A).$$

<u>Ex</u> (1) If A is the set of primes, then $\underline{d}(A) = \bar{d}(A) = 0$.
(2) If $A = \{n\in\mathbb{N};\ n\equiv 0 \pmod 5\}$, then $\underline{d}(A) = \bar{d}(A) = \frac{1}{5}$.
(3) If $A = \{n\in\mathbb{N};\ n$ is not of the form k^2+1 for some $k\in\mathbb{N}\}$, then $\underline{d}(A) = \bar{d}(A) = 1$.
(4) If $A = \{a\in\mathbb{N};\ (2k)! < a < (2k+1)!$ for some $k\in\mathbb{N}\}$, then for $n=(2k+1)!$, any a satisfying $(2k)! < a < (2k+1)!$ will be counted. Thus
$$1 \geq \frac{|A((2k+1)!)|}{(2k+1)!} \geq \frac{(2k+1)! - (2k)!}{(2k+1)!} = \frac{2k}{2k+1}.$$
As $k\to\infty$, we have
$$\frac{|A((2k+1)!)|}{(2k+1)!} \longrightarrow 1$$
and thus $\bar{d}(A) = 1$. On the other hand, if $n=(2k)!$, then we only count a up to $(2k-1)!$. Thus
$$0 \leq \frac{|A((2k)!)|}{(2k)!} \leq \frac{(2k-1)!}{(2k)!} = \frac{1}{2k}.$$
As $k\to\infty$, we have
$$\frac{|A((2k)!)|}{(2k)!} \longrightarrow 0$$
and thus $\underline{d}(A) = 0$.

<u>Def</u> Let $f(n)$ and $F(n)$ be functions from $\mathbb{N}\to\mathbb{R}^+$. We say that $f(n)$ has <u>normal order</u> $F(n)$ if for any $\varepsilon>0$, the set
$$A(\varepsilon) = \{n\in\mathbb{N};\ (1-\varepsilon)F(n) < f(n) < (1+\varepsilon)F(n)\}$$
has the property that $\underline{d}(A(\varepsilon)) = 1$.

<u>Def</u> Let $f(n)$ and $F(n)$ be functions from $\mathbb{N}\to\mathbb{R}^+$. We say that $f(n)$ has <u>average order</u> $F(n)$ if
$$\sum_{i=1}^{n} f(i) \sim \sum_{i=1}^{n} F(i).$$

Ex (1) Let
$$f(n)=\begin{cases} 1 & \text{if } n\neq k! \text{ for all } k\in\mathbb{N},\\ n & \text{if } n=k!.\end{cases}$$
Then f has normal order 1, but not average order 1.

(2) Let
$$f(n)=\begin{cases} 2 & \text{if } n\equiv 1 \pmod 2,\\ 0 & \text{if } n\equiv 0 \pmod 2.\end{cases}$$
Then f has average order 1 but does not have normal order 1.

(3) Let
$$f(n)=\begin{cases} \log n+(\log n)^{\frac{1}{2}} & \text{if } n\equiv 1 \pmod 2,\\ \log n-(\log n)^{\frac{1}{2}} & \text{if } n\equiv 0 \pmod 2.\end{cases}$$
Then f has both normal and average order n.

From Theorem 27, we see that $\omega(n)$ and $\Omega(n)$ have average order $\log\log n$. (Note:
$$\sum_{n\le x}\log\log n\sim x\log\log x.)$$
We now prove that they have normal order $\log\log n$.

Theorem 28: We have
$$\sum_{n\le x}(\omega(n)-\log\log x)^2=O(\log\log x).$$

2015 10 21

Note that
$$\sum_{n\le x}\log\log n=\sum_{n\le x^{1/2}}\log\log n+\sum_{x^{1/2}<n\le x}\log\log n$$
$$=O\left(x^{\frac{1}{2}}\log\log x\right)+\sum_{x^{1/2}<n\le x}\log\log n.$$
Note that
$$\sum_{x^{1/2}<n\le x}\log\log n\ge(\log\log x-\log 2)\sum_{x^{1/2}<n\le x}1=x\log\log x+O\left(x^{\frac{1}{2}}\log\log x\right),$$
$$\sum_{x^{1/2}<n\le x}\log\log n\le\log\log x\sum_{x^{1/2}<n\le x}1=x\log\log x+O\left(x^{\frac{1}{2}}\log\log x\right).$$
Thus

$$\sum_{n\le x}\log\log n = x\log\log x + O(x^{\frac{1}{2}}\log\log x).$$

Thus the average order of $\omega(n)$ is $\log\log n$.

Proof (of Theorem 28): We have

$$\sum_{n\le x}(\omega(n)-\log\log x)^2 = \sum_{n\le x}\omega^2(n) - 2\log\log x\sum_{n\le x}\omega(n) + \underbrace{(\log\log x)^2\sum_{n\le x}1}_{x(\log\log x)^2+O((\log\log x)^2)}.$$

By theorem 27, we have

$$2\log\log x\sum_{n\le x}\omega(n) = 2x(\log\log x)^2 + O(x\log\log x).$$

We now consider the sum of $\omega^2(n)$. We have

$$\begin{aligned}
\sum_{n\le x}\omega^2(n) &= \sum_{n\le x}\left(\left(\sum_{p\mid n}1\right)\left(\sum_{q\mid n}1\right)\right)\\
&= \sum_{n\le x}\left(\sum_{\substack{pq\mid n\\ p\ne q}}1 + \sum_{p\mid n}1\right)\\
&= \sum_{\substack{pq\le x\\ p\ne q}}\sum_{\substack{n\le x\\ pq\mid n}}1 + \sum_{n\le x}\omega(n)\\
&= \sum_{\substack{pq\le x\\ p\ne q}}\left\lfloor\frac{x}{pq}\right\rfloor + O(x\log\log x)\\
&= x\sum_{\substack{pq\le x\\ p\ne q}}\frac{1}{pq} + O(x) + O(x\log\log x).
\end{aligned}$$

Also, we have

$$\sum_{\substack{pq\le x\\ p\ne q}}\frac{1}{pq} = \sum_{pq\le x}\frac{1}{pq} - \sum_{p^2\le x}\frac{1}{p^2} = \sum_{pq\le x}\frac{1}{pq} + O(1).$$

Observe that

$$\left(\sum_{p\le x^{1/2}}\frac{1}{p}\right)^2 \le \sum_{pq\le x}\frac{1}{pq} \le \left(\sum_{p\le x}\frac{1}{p}\right)^2.$$

By theorem 17, we have

$$\left(\sum_{p\le x}\frac{1}{p}\right)^2 = (\log\log x)^2 + O(\log\log x)$$

and

$$\left(\sum_{p\le x^{1/2}} \frac{1}{p}\right)^2 = \left(\log\log x^{\frac{1}{2}} + O(1)\right)^2$$
$$= \left(\log\log x - \log 2 + O(1)\right)^2$$
$$= (\log\log x)^2 + O(\log\log x).$$

Thus
$$\sum_{pq\le x} \frac{1}{pq} = (\log\log x)^2 + O(\log\log x).$$

Combining the above estimates, we have
$$\sum_{n\le x} \omega^2(n) = x(\log\log x)^2 + O(x\log\log x).$$

It follows that
$$\sum_{n\le x} (\omega(n)-\log\log x)^2 = \sum_{n\le x}\omega^2(n) - 2\log\log x \sum_{n\le x}\omega(n) + (\log\log x)^2 \sum_{n\le x} 1$$
$$= \left(x(\log\log x)^2 + O(x\log\log x)\right) - 2x(\log\log x)^2 + x(\log\log x)^2$$
$$= O(x\log\log x).$$ ∎

A3 ↳ be more careful everywhere

Corollary 29: Let $\delta > 0$. Then
$$\#\{n\le x \,;\, |\omega(n) - \log\log n| > (\log\log n)^{\frac{1}{2}+\delta}\}$$
is $o(x)$. Thus the normal order of $\omega(n)$ is $\log\log n$.

Proof: The number of $n \le x^{\frac{1}{2}}$ is $o(x)$. Also, for $x^{\frac{1}{2}} < n \le x$, we have
$$\log\log x \ge \log\log n > \log\log x - \log 2.$$
Thus to prove the corollary, it suffices to show that
$$E(x) = \#\{n\le x;\, |\omega(n) - \log\log x| > (\log\log x)^{\frac{1}{2}+\delta}\}$$
is $o(x)$. By theorem 28, we have
$$E(x)\cdot(\log\log x)^{1+2\delta} \le \sum_{n\le x}(\omega(n) - \log\log x)^2 = O(x\log\log x).$$
It follows that
$$E(x) = O\left(\frac{x\log\log x}{(\log\log x)^{1+2\delta}}\right) = o(x).$$ ∎

Corollary 30: The normal order of $\Omega(n)$ is $\log\log n$.

Proof: By theorem 27, we have

$$\sum_{n\le x}\left(\Omega(n)-\omega(n)\right)=O(x).$$

Thus

$$\#\{n\le x;\ \Omega(n)-\omega(n)>(\log\log n)^{\frac{1}{2}+\delta}\}=o(x).$$

Then the result follows from corollary 29. ■

Remark: Since the average order of $\omega(n)$ is $\log\log n$, which is $\sim \log\log x$ for almost all n, we can view

$$\sum_{n\le x}(\omega(n)-\log\log x)^2$$

as the 'square of the standard deviation' of $\omega(n)$.
On assignment 3, we'll prove

$$\sum_{n\le x}(\omega(n)-\log\log x)^2\sim x\log\log x.$$

Thus the standard deviation of $\omega(n)$ is about $\sqrt{\log\log n}$.
Let

$$G(\gamma)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\gamma}e^{-\frac{t^2}{2}}\,dt,$$

the Gaussian normal distribution. In 1934, Erdös and Kac proved that

$$\lim_{x\to\infty}\frac{1}{x}\#\left\{n\le x;\ \frac{\omega(n)-\log\log n}{\sqrt{\log\log n}}\le\gamma\right\}=G(\gamma).$$

See probability number theory.

2015 10 23

Let $d(n)$ be the number of positive divisors of n. If

$$n=p_1^{a_1}\cdots p_r^{a_r}$$

with $a_1,\dots,a_r\in\mathbb{N}$ and $p_1,\dots,p_r$ are distinct primes, then

$$\omega(n)=r,\quad \Omega(n)=a_1+\cdots+a_r,\quad\text{and}\quad d(n)=(a_1+1)\cdots(a_r+1).$$

Theorem 31: For $\varepsilon>0$, define the set

$$S(\varepsilon)=\left\{n\in\mathbb{N};\ 2^{(1-\varepsilon)\log\log n}<d(n)<2^{(1+\varepsilon)\log\log n}\right\}.$$

Then $S(\varepsilon)$ has asymptotic density 1.

Proof: Note that for $a \in \mathbb{N}$,

$$2 \leq (1+a) \leq 2^a.$$

Thus we have

$$2^{\omega(n)} \leq d(n) \leq 2^{\Omega(n)}.$$

Then the result follows from Corollaries 29 and 30. ▪

Remark: Recall that

$$\sum_{n \leq x} d(n) \sim x \log x.$$

Thus the average order of $d(n)$ is $\log n$. However, by the above theorem, for almost all n, $d(n)$ satisfies

$$(\log n)^{\log 2 - \varepsilon} < d(n) < (\log n)^{\log 2 + \varepsilon}$$

for any $\varepsilon > 0$ (exercise).