4. Riemann's ζ-function

For $s \in \mathbb{C}$, consider the series
$$\sum_{n=1}^{\infty} \frac{1}{n^s},$$
which converges absolutely if $\mathrm{Re}(s) > 1$.

Def) For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the Riemann ζ-function is defined by
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Note that
$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right).$$
Since a typical term in the above product is of the form
$$\frac{1}{p_1^{a_1 s} \cdots p_k^{a_k s}} = \frac{1}{(p_1^{a_1} \cdots p_k^{a_k})^s} = \frac{1}{n^s}$$
when $n = p_1^{a_1} \cdots p_k^{a_k}$; by the fundamental theorem of arithmetic, for $\mathrm{Re}(s) > 1$, we have
$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s).$$
This is the Euler product representation for $\zeta(s)$.

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Theorem 18: (a) For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have
$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \frac{\{u\}}{u^{s+1}}\,du,$$
where $\{u\} = u - \lfloor u \rfloor$. Deduce that
$$\lim_{s \to 1^+} (s-1)\zeta(s) = 1.$$
(b) $\zeta(s)$ has an analytic continuation to $\mathrm{Re}(s) > 0$ with $s \neq 1$. It is analytic except a simple pole at $s = 1$.

Proof: (a) Apply Abel's summation with $a_n = 1$ and $f(x) = \frac{1}{x^s}$. Then we have

$$\sum_{n\le x}\frac{1}{n^s} = \frac{\lfloor x\rfloor}{x^s} + s\int_1^x \frac{\lfloor u\rfloor}{u^{s+1}}\,du.$$

By letting $x\to\infty$, we see that for $\mathrm{Re}(s)>1$,

$$\begin{aligned}
\zeta(s) &= 0 + s\int_1^\infty \frac{\lfloor u\rfloor}{u^{s+1}}\,du \\
&= s\int_1^\infty \frac{u-\{u\}}{u^{s+1}}\,du \\
&= s\int_1^\infty \frac{u}{u^{s+1}}\,du - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,du \\
&= s\left(\frac{u^{1-s}}{1-s}\Big|_1^\infty\right) - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,du \\
&= \frac{s}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,du.
\end{aligned}$$

Thus

$$(s-1)\zeta(s) = s - s(s-1)\int_1^\infty \frac{\{u\}}{u^{s+1}}\,du.$$

Since $\{u\} = O(1)$, the above integral converges for $\mathrm{Re}(s)>0$. It follows that

$$\lim_{s\to 1^+}(s-1)\zeta(s) = 1.$$

(b) From (a), by the identity theorem for analytic functions, we see that

$$\zeta(s) = \frac{s}{s-1} - s\int_0^\infty \frac{\{u\}}{u^{s+1}}\,du.$$

Since

$$\int_0^\infty \frac{\{u\}}{u^{s+1}}\,du$$

converges for $\mathrm{Re}(s)>0$, $\zeta(s)$ has an analytic continuation to $\mathrm{Re}(s)>0$ with $s\neq 1$.

Theorem 19: $\zeta(s)$ has no zero in the region $\mathrm{Re}(s)\ge 1$.

Proof: If $\mathrm{Re}(s)>1$, we'll show in assignment 2 that $\zeta(s)\neq 0$. We recall that for $|u|<1$,

$$-\log(1-u) = \sum_{n=1}^{\infty} \frac{u^n}{n}$$

Thus

$$\log \zeta(s) = \log\left(\prod_p \left(1-\frac{1}{p^s}\right)^{-1}\right) = \sum_p \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{p^{ns}}\right)$$

Write $s=\sigma+it$ with $\sigma, t \in \mathbb{R}$. We see that

$$p^{-int} = e^{-int(\log p)} = \cos(-nt\log p) + i\sin(-nt\log p)$$
$$= \cos(nt\log p) - i\sin(nt\log p).$$

Thus $\mathrm{Re}(p^{-int}) = \cos(nt\log p)$. It follows

$$\mathrm{Re} \log \zeta(\sigma+it) = \sum_p \sum_{n=1}^{\infty} \frac{p^{-\sigma n}\cos(nt\log p)}{n}.$$

Note that for $\theta \in \mathbb{R}$, we have

$$0 \le 2(1+\cos\theta)^2$$
$$= 2(1+2\cos\theta+\cos^2\theta)$$
$$= 2+4\cos\theta+2\cos^2\theta$$
$$= 3+4\cos\theta+(2\cos^2\theta-1)$$
$$= 3+4\cos\theta+\cos(2\theta).$$

Thus

$$\sum_p \sum_{n=1}^{\infty} \frac{p^{-\sigma n}}{n}\left(3+4\cos(nt\log p)+\cos(2nt\log p)\right) \ge 0.$$

This implies that

$$\mathrm{Re}\left(3\log\zeta(\sigma) + 4\log(\sigma+it) + \log(\sigma+i2t)\right) \ge 0.$$

In particular, for $\sigma > 1$ and $t \in \mathbb{R}$, we have

$$|\zeta(\sigma)|^3 |\zeta(\sigma+it)|^4 |\zeta(\sigma+i2t)| \ge 1. \qquad (\star)$$

We recall that

$$\lim_{s\to 1^+} (s-1)\zeta(s) = 1.$$

Thus

$$\lim_{\sigma\to 1^+} |\zeta(\sigma)| = \lim_{\sigma\to 1^+} |(\sigma-1)^{-1}|.$$

Suppose that $(1+it_0)$ is a zero of $\zeta(s)$ with order $m \ge 1$, i.e. when $(\sigma+it) \to (1+it_0)$, we have

$$\zeta(\sigma+it) = ((\sigma+it)-(1+it_0))^m g(\sigma+it)$$

for some function g with $g(1+it_0)\neq 0$. Since $\zeta(s)$ has a pole at $s=1$, $t_0\neq 0$. Also, by taking $t=t_0$, we have

$$\lim_{\sigma\to 1^+}\frac{\zeta(\sigma+it_0)}{(\sigma-1)^m}=c_1\neq 0,$$

for some constant c_1. Thus

$$\lim_{\sigma\to 1^+}|\zeta(\sigma+it_0)|=\lim_{\sigma\to 1^+}|c_1(\sigma-1)^m|.$$

Also, since $(1+i2t_0)$ is not a pole of $\zeta(s)$, there exists c_2 such that

$$\lim_{\sigma\to 1^+}|\zeta(\sigma+i2t_0)|=c_2$$

Since $m\geq 1$, we have

$$\lim_{\sigma\to 1^+}|\zeta(\sigma)|^3|\zeta(\sigma+it_0)|^4|\zeta(\sigma+i2t_0)|$$

$$=\lim_{\sigma\to 1^+}\left|(\sigma-1)^{-3}c_1^4(\sigma-1)^{4m}c_2\right|=0,$$

$\geq$?

which contradicts (☆). Thus $\zeta(s)$ has no zero on $Re(s)>1$. ∎