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3. Abel's Summation Formula

Proposition 13 (Abel's summation formula): Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Let f be a function from $\{x \in \mathbb{R} ; x \geq 1\}$ to $\mathbb{C}$. For $x \in \mathbb{R}$, we write

$$A(x) = \sum_{n \leq x} a_n .$$

If f has a continuous first derivative for $x \geq 1$, then we have

$$\sum_{n \leq x} a_n f(n) = A(x) f(x) - \int_1^x A(u) f'(u) du .$$

Proof: Let $N = \lfloor x \rfloor$. Then

$$\sum_{n \leq N} a_n f(n) = A(1) f(1) + (A(2) - A(1)) f(2) + \cdots + (A(N) + A(N-1)) f(N)$$

$$= A(1)(f(1) - f(2)) + A(2)(f(2) - f(3)) + \cdots + A(N-1)(f(N-1) - f(N)) + A(N) f(N).$$

Note that for $i \in \mathbb{N}$ and $u \in \mathbb{R}$ with $i \leq u < (i+1)$, we have $A(u) = A(i)$.
Thus

$$A(i)(f(i) - f(i+1)) = -\int_i^{i+1} A(u) f'(u) du .$$

It follows that

$$\sum_{n \leq N} a_n f(n) = -\int_1^N A(u) f'(u) du + A(N) f(N) \qquad \text{———} \quad (1)$$

Also, for $x \geq u \geq N$, we have $A(u) = A(x)$. Thus

$$\int_N^x A(u) f'(u) du = A(x)(f(x) - f(N)) = A(x) f(x) - A(N) f(N),$$

$$\text{i.e.} \quad 0 = A(x) f(x) - A(N) f(N) - \int_N^x A(u) f'(u) du . \qquad \text{———} \quad (2)$$

Combining (1) and (2), we get

$$\sum_{n \leq x} a_n f(n) = \sum_{n \leq N} a_n f(n) = A(x) f(x) - \int_1^x A(u) f'(u) du . \quad \blacksquare$$

Def] Given $x \in \mathbb{R}$, we denote by $\{x\}$ the fractional part of x,

$$\{x\} = x - \lfloor x \rfloor .$$

We define Euler's constant γ by

$$\gamma = 1 - \int_1^\infty \frac{\{u\}}{u^2}\,du \quad (= 0.57721\ldots).$$

Theorem 14: We have

$$\sum_{n \le x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right).$$

Proof: Take $a_n = 1$ and $f(u) = \frac{1}{u}$. Then

$$A(x) = \sum_{n \le x} 1 = \lfloor x \rfloor.$$

By Abel summation,

$$\begin{aligned}
\sum_{n \le x} \frac{1}{n} &= \frac{\lfloor x \rfloor}{x} + \int_1^x \frac{\lfloor u \rfloor}{u^2}\,du \\
&= \frac{x - \{x\}}{x} + \int_1^x \frac{u - \{u\}}{u^2}\,du \\
&= 1 + O\left(\frac{1}{x}\right) + \int_1^x \frac{1}{u}\,du - \int_1^x \frac{\{u\}}{u^2}\,du \\
&= 1 + O\left(\frac{1}{x}\right) + \log x - \left(\int_1^\infty \frac{\{u\}}{u^2}\,du - \int_x^\infty \frac{\{u\}}{u^2}\,du\right) \\
&= \log x + \gamma + O\left(\frac{1}{x}\right) + \int_x^\infty \frac{\{u\}}{u^2}\,du
\end{aligned}$$

Note that

$$\int_x^\infty \frac{\{u\}}{u^2}\,du \le \int_x^\infty \frac{1}{u^2}\,du \le \frac{1}{x}.$$

Thus

$$\sum_{n \le x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right) \qquad \blacksquare$$

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Theorem 15: We have

$$\sum_{n \le x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

Proof: Let $a_n = 1$ and $f(n) = \log n$. By Abel's summation, we have

$$\sum_{n\le x}\log n = \lfloor x\rfloor \log x - \int_1^x \frac{\lfloor u\rfloor}{u}\,du$$
$$= (x-\{x\})\log x - \int_1^x \frac{u-\{u\}}{u}\,du$$
$$= x\log x + O(\log x) - (x-1) + \int_1^x \frac{\{u\}}{u}\,du$$
$$= x\log x - x + O(\log x).$$

Also, we have

$$\sum_{n\le x}\log n = \log(\lfloor x\rfloor !)$$
$$= \sum_{p\le x}\left(\sum_{k=1}^{\infty}\left\lfloor \frac{x}{p^k}\right\rfloor\right)\log p$$
$$= \sum_{p^k\le x}\left\lfloor \frac{x}{p^k}\right\rfloor \log p$$
$$= \sum_{n\le x}\left\lfloor \frac{x}{n}\right\rfloor \Lambda(n)$$
$$= \sum_{n\le x}\left(\frac{x}{n} - \left\{\frac{x}{n}\right\}\right)\Lambda(n)$$
$$= x\sum_{n\le x}\frac{\Lambda(n)}{n} + O\left(\sum_{n\le x}\Lambda(n)\right)$$

Since

$$\sum_{n\le x}\Lambda(n) = \psi(x) = O(x),$$

we have

$$\sum_{n\le x}\log n = x\sum_{n\le x}\frac{\Lambda(n)}{n} + O(x).$$

Thus

$$x\sum_{n\le x}\frac{\Lambda(n)}{n} = x\log x + O(x),$$

i.e. $$\sum_{n\le x}\frac{\Lambda(n)}{n} = \log x + O(1).$$

Theorem 16: We have

$$\sum_{p\le x}\frac{\log p}{p} = \log x + O(1).$$

Proof: By theorem 15, we have

$$\sum_{p\le x}\frac{\log p}{p}=\sum_{n\le x}\frac{\Lambda(n)}{n}-\sum_{m\ge 2}\sum_{p^m\le x}\frac{\log p}{p^m}$$
$$=\log x+O(1)-\sum_{m\ge 2}\sum_{p^m\le x}\frac{\log p}{p^m}.$$

Note that

$$\sum_{m\ge 2}\sum_{p^m\le x}\frac{\log p}{p^m}\le\sum_{p}\left(\frac{1}{p^2}+\frac{1}{p^3}+\cdots\right)\log p$$
$$\le\sum_{p}\frac{\log p}{p(p-1)}$$
$$\le\sum_{n=2}^{\infty}\frac{\log n}{n(n-1)}=O(1).$$

Combining the above two inequalities, the theorem follows. ∎

Theorem 17: There exists $\beta\in\mathbb{R}$ such that

$$\sum_{p\le x}\frac{1}{p}=\log\log x+\beta+O\left(\frac{1}{\log x}\right).$$

Proof: Let

$$a_n=\begin{cases}\frac{\log p}{p} & \text{if } n=p \text{ is a prime}\\ 0 & \text{otherwise}\end{cases}$$

and $f(n)=\frac{1}{\log n}$. Write

$$A(x)=\sum_{n\le x}a_n.$$

By Abel's summation,

$$\sum_{p\le x}\frac{1}{p}=\frac{A(x)}{\log x}+\int_1^x\frac{A(u)}{u(\log u)^2}\,du.$$

By theorem 16, we can write

$$A(x)=\sum_{p\le x}\frac{\log p}{p}=\log x+\gamma(x),$$

where

$$\gamma(x)=\sum_{p\le x}\frac{\log p}{p}-\log x=O(1).$$

Also, we note that $A(u)=0$ for $1\le u<2$. Thus

$$\sum_{p\le x}\frac{1}{p} = 1+O\left(\frac{1}{\log x}\right)+\int_2^x \frac{\log u+\gamma(u)}{u(\log u)^2}\,du$$
$$= 1+\int_2^x \frac{1}{u\log u}\,du+\int_2^x \frac{\gamma(u)}{u(\log u)^2}\,du+O\left(\frac{1}{\log x}\right).$$

It follows that

$$\sum_{p\le x}\frac{1}{p} = 1+(\log\log x-\log\log 2)+\int_2^x \frac{\gamma(u)}{u(\log u)^2}\,du+O\left(\frac{1}{\log x}\right)$$
$$= \log\log x+(1-\log\log 2)+\int_2^\infty \frac{\gamma(u)}{u(\log u)^2}\,du-\int_x^\infty \frac{\gamma(u)}{u(\log u)^2}\,du+O\left(\frac{1}{\log x}\right)$$
$$= \log\log x+\beta+O\left(\frac{1}{\log x}\right)$$

where

$$\beta = 1-\log\log 2+\int_2^\infty \frac{\gamma(u)}{u(\log u)^2}\,du$$

is a constant. ∎

Remark: The constant β is called Merten's constant. One can show that

$$\beta = \gamma+\left(\sum_p \log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right) = 0.261497\ldots,$$

where γ is Euler's constant.