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2. Möbius Function and von Mangoldt Function

Notation: Let f and g be functions from $\mathbb{N}$ or $\mathbb{R}^+$ to $\mathbb{R}$, and suppose that g maps to $\mathbb{R}^+$.

(1) $f = O(g)$ means that there exist $c, C \in \mathbb{R}^+$ such that for $x > c$ we have $|f(x)| \le C g(x)$.

(2) $f = o(g)$ means that
$$\lim_{x\to\infty} \frac{|f(x)|}{g(x)} = 0.$$

(3) $f(x) \sim g(x)$ means that
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$
(read: f is asymptotic to g)

Ex: $20x = O(x)$, $\sin(x) = O(1)$,
$$\int_x^\infty \frac{1}{u^2}\,du = O\left(\frac{1}{x}\right),$$
$x = o(x^2)$, $\sin x = o(\log x)$,
$$\frac{x}{(\log x)^2} = o\left(\frac{x}{\log x}\right),$$
$x+1 \sim x$, $x + \sqrt{x} \sim x$

By the prime number theorem,
$$\pi(x) \sim \frac{x}{\log x}.$$

Def] The Möbius function μ is defined by
$$\mu(n) = \begin{cases} (-1)^r & \text{if } n \text{ is a product of } r \text{ distinct primes,} \\ 0 & \text{otherwise} \end{cases}$$

Ex. $\mu(12) = \mu(2^2 3) = 0$, $\mu(15) = \mu(3\,5) = (-1)^2 = 1$,
$\mu(30) = \mu(2\cdot 3\cdot 5) = (-1)^3 = -1$

Proposition 10: We have
$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise} \end{cases}$$

Proof: If $n=1$ then the proposition is true. If $n>1$, let

$$n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$$

be the unique factorization of n into distinct prime powers. Set

$$N = p_1 \cdots p_r .$$

(N is called the radical of n.) Since $\mu(d)=0$ unless d is square-free, we have

$$\sum_{d|n} \mu(d) = \sum_{d|N} \mu(d).$$

Note that the divisors of N are in one-to-one correspondence with subsets of $\{p_1, \ldots, p_r\}$. Thus the latter sum contains 2^r summands. The number of k element subsets is $\binom{r}{k}$ and the corresponding divisor d of such a set satisfies $\mu(d)=(-1)^k$. Thus

$$\sum_{d|N} \mu(d) = \sum_{k=0}^{r} \binom{r}{k}(-1)^k = (1-1)^r = 0.$$

Thus

$$\sum_{d|n} \mu(d) = 0$$

if $n>1$. ☒

Proposition 11 (Möbius Inversion Formula): We have

$$f(n) = \sum_{d|n} g(d)$$

if and only if

$$g(n) = \sum_{d|n} \mu(d) f\left(\tfrac{n}{d}\right).$$

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Notes on AI QS:

$\mathbb{F}_q[t]$, $|f| = q^{\deg(f)}$,

$$\sum_{f \in \mathbb{F}_q[t] \text{ monic}} T^{\deg f} = \prod_{\substack{v \in \mathbb{F}_q[t] \\ \text{monic, irred}}} \left(1 + T^{\deg v} + T^{2\deg v} + \cdots\right)$$

$$= \prod_{\substack{v \in \mathbb{F}_q[t] \\ \text{monic, irred}}} \left(1 - T^{\deg v}\right)^{-1}$$

$$= \prod_{d=1}^{\infty} \left(1 - T^{d}\right)^{-N_d}$$

Proof: ($\Leftarrow$): Suppose

$$g(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right).$$

Then we have

$$\begin{aligned}\sum_{d|n} g(d) &= \sum_{d|n} \sum_{e|d} \mu(e) f\left(\frac{d}{e}\right) \\ &= \sum_{est=n} \mu(e) f(s) \qquad s = \tfrac{d}{e} \\ &= \sum_{s|n} f(s) \underbrace{\sum_{e|\frac{n}{s}} \mu(e)}_{= \begin{cases} 1 & \text{if } n=s \\ 0 & \text{otherwise} \end{cases}} \\ &= f(n).\end{aligned}$$

($\Rightarrow$): Suppose

$$f(n) = \sum_{g|n} g(d).$$

Then we have

$$\begin{aligned}\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) &= \sum_{d|n} \mu(d) \sum_{e|\frac{n}{d}} g(e) \\ &= \sum_{des=n} \mu(d) g(e) \\ &= \sum_{e|n} g(e) \sum_{d|\frac{n}{e}} \mu(d) = g(n).\end{aligned}$$

∎

Def) For $n \in \mathbb{N}$, the von Mangoldt function, denoted by $\Lambda(n)$, is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Also for $x \in \mathbb{R}$, we define

$$\Theta(x) = \sum_{p \le x} \log p = \log\left(\prod_{p \le x} p\right).$$

and

$$\psi(x) = \sum_{p^k \le x} \log p = \sum_{n \le x} \Lambda(n).$$

Note that

$$\psi(x) = \sum_{p \le x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p.$$

Also, $p^2 \le x$ is equivalent to $p \le x^{\frac{1}{2}}$, $p^3 \le x$ is equivalent to $p \le x^{\frac{1}{3}}$, and etc. Thus we have

$$\psi(x) = \Theta(x) + \Theta(x^{\frac{1}{2}}) + \Theta(x^{\frac{1}{3}}) + \cdots$$

Since $2^m \le p^m \le x$, we see that $\Theta(x^{\frac{1}{m}}) = 0$ provided that

$$m > \frac{\log x}{\log 2}$$

Thus

$$\psi(x) = \sum_{k=1}^{\left\lfloor \frac{\log x}{\log 2} \right\rfloor} \Theta(x^{\frac{1}{k}}).$$

Since

$$\Theta(x) = \sum_{p \le x} \log p \le x \log x$$

we see that

$$\sum_{k=2}^{\left\lfloor \frac{\log x}{\log 2} \right\rfloor} \Theta(x^{\frac{1}{k}}) \le \sum_{k=2}^{\left\lfloor \frac{\log x}{\log 2} \right\rfloor} x^{\frac{1}{k}} \log(x^{\frac{1}{k}})$$

$$\le x^{\frac{1}{2}} \sum_{k=2}^{\left\lfloor \frac{\log x}{\log 2} \right\rfloor} \frac{1}{k} \log x$$

$$= O\left(x^{\frac{1}{2}} (\log x)^2\right)$$

Thus

$$\psi(x) = \Theta(x) + O\left(x^{\frac{1}{2}} (\log x)^2\right) \qquad (3)$$

We have seen in Theorem 6 that

$$\pi(x) \le c_1 \frac{x}{\log x}$$

for some $c_1 > 0$. Thus

$$\Theta(x) = \sum_{p \le x} \log p \le \pi(x) \log x \le c_1 x$$

Combining this with (3), we see that

$$\psi(x) \le c_2 x$$

for some $c_2>0$. Also, we have seen in the proof of Theorem 6 that

$$2^n \le \binom{2n}{n} \quad \text{and} \quad \binom{2n}{n} \Big| \prod_{p\le 2n} p^{r_p}$$

with $p^r \le 2n < p^{r_p-1}$. It follows that

$$n\log 2 = \log(2^n) \le \log\binom{2n}{n} \le \sum_{p\le n} \underbrace{\left\lfloor \frac{\log 2n}{\log p} \right\rfloor}^{r_p} \log p = \psi(2n).$$

For $x\ge 2$, let $n\in\mathbb{N}$ with $2n\le x<(2n+2)$. Then we have

$$\psi(x) \ge \psi(2n) \ge n\log 2 > \frac{x-2}{2}\log 2$$

Thus there exists $c_3>0$ such that $\psi(x)>c_3x$. Combining this with (5), we see that $\theta(x)>c_4x$ for some $c_4>0$.

Theorem 12: We have

$$\pi(x) \sim \frac{\theta(x)}{\log x} \sim \frac{\psi(x)}{\log x}.$$

Remark: By theorem 12, to prove

$$\pi(x) \sim \frac{x}{\log x},$$

it suffices to show that $\psi(x)\sim\theta(x)\sim x$.

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Proof: We have seen that $\psi(x)=\theta(x)+O(x^{\frac{1}{2}}(\log x)^2)$. Since $\theta(x)>c_4x$, it follows that

$$\frac{\theta(x)}{\log x} \sim \frac{\psi(x)}{\log x}.$$

Thus it suffices to show that

$$\pi(x) \sim \frac{\theta(x)}{\log x}.$$

Note that

$$\theta(x) = \sum_{p\le x} \log p \le \pi(x)\log x.$$

Thus

$$\pi(x) \ge \frac{\theta(x)}{\log x}, \quad \text{ie} \quad \frac{\pi(x)}{\frac{\theta(x)}{\log x}} \ge 1.$$

Note that for any $\delta > 0$, we have

$$\begin{aligned}\theta(x) &= \sum_{p \le x} \log p \\ &\ge \log(x^{1-\delta}) \sum_{x^{1-\delta} < p \le x} 1 \\ &\ge (1-\delta)\log x \left(\pi(x) - \pi(x^{1-\delta})\right).\end{aligned}$$

Thus

$$\theta(x) + (1-\delta)\log x\,(x^{1-\delta}) \ge (1-\delta)(\log x)\pi(x)$$

$$\text{i.e. } \frac{\theta(x)}{(1-\delta)\log x} + x^{1-\delta} \ge \pi(x).$$

$$\text{i.e. } \frac{1}{1-\delta} + \frac{x^{1-\delta}\log x}{\theta(x)} \ge \frac{\pi(x)\log x}{\theta(x)}.$$

Given any $\varepsilon > 0$, we can choose $\delta > 0$ so that

$$\frac{1}{1-\delta} < 1 + \frac{\varepsilon}{2}.$$

Since $\theta(x) > c_4 x$ for some $c_4 > 0$, there exists $x_0 \in \mathbb{R}$ such that for $x > x_0$,

$$\frac{x^{1-\delta}\log x}{\theta(x)} < \frac{\varepsilon}{2}$$

Thus for $x > x_0$,

$$\frac{\pi(x)\log x}{\theta(x)} < 1 + \varepsilon.$$

Since

$$1 \le \frac{\pi(x)\log x}{\theta(x)} < 1 + \varepsilon,$$

by choosing ε to be arbitrarily close to 0, the result follows $\blacksquare$