12. The Circle Method

We recall that the number $g(k)$ is determined by "small" numbers of special form. Thus a more interesting question is to estimate $G(k)$, defined to be the least integer $s=s(k)$ such that every sufficiently large integer is the sum of at most s k-powers of natural numbers. Note $G(k) \le g(k)$.

Conjecture: $G(k) = \max\{k+1, \Gamma_0(k)\}$, where $\Gamma_0(k)$ is the least integer s such that for every prime p and $m \in \mathbb{N}$, $n = x_1^k + \cdots + x_s^k$ has a solution in mod p with $(x_1, p) = 1$.

For large k, Wooley (1992) showed

$$G(k) \le k\log k + O(k\log\log k).$$

One can consider a more refined question: for fixed $k \in \mathbb{N}$ with $k \ge 2$, let

$$R_s(n) = R_{s,k}(n) = \#\{n = x_1^k + \cdots + x_s^k,\ x_i \in \mathbb{N}\ (1 \le i \le s)\}.$$

Note that if the above equality holds, then $x_i \le n^{1/k}$. Also, the sum $x_1^k + \cdots + x_s^k$ ranges from s to sn. Thus we expect that $R_s(n)$ is of size

$$\underbrace{(n^{1/k})^s}_{\text{the choices for } x_1,\dots,x_s} \cdot \underbrace{(sn-s)^{-1}}_{\text{the "probability" that their sum is } n} \asymp n^{s/k-1}$$

That is, we expect

$$R_s(n) \sim C(s,k;n)\, n^{s/k-1}$$

for some $C(s,k;n) > 0$. Let $\tilde{G}(k)$ be the least integer $s=s(k)$ such that the above asymptotic formula holds for every sufficiently large integer n. Note that for $G(k)$, we only need $R_s(n) > 0$. Thus we have

$$G(k) \le \tilde{G}(k).$$

To estimate $R_s(n)$, we apply the exponential function. For $\alpha \in \mathbb{R}$ let

$$e(\alpha) = e^{2\pi i \alpha}.$$

We have $e(\alpha)e(\beta) = e(\alpha+\beta)$. Moreover, for $h \in \mathbb{Z}$, we have the following orthogonal relation:

$$\int_0^1 e(\alpha h)\, d\alpha = \begin{cases} 1 & \text{if } h = 0, \\ 0 & \text{if } h \in \mathbb{Z}\setminus\{0\}. \end{cases}$$

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Define

$$p = n^{1/\alpha} \quad \text{and} \quad f(\alpha) = \sum_{1 \le x \le p} e(\alpha x^k).$$

It follows that

$$\int_0^1 f(\alpha)^s e(-n\alpha)\, d\alpha = \int_0^1 \Big(\sum_{x \le p} e(\alpha x^k)\Big)^s e(-n\alpha)\, d\alpha$$

$$= \sum_{x_1 \le p} \cdots \sum_{x_s \le p} \int_0^1 e(\alpha x_1^k) \cdots e(\alpha x_s^k) e(-n\alpha)\, d\alpha$$

$$= \sum_{x_1 \le p} \cdots \sum_{x_s \le p} \int_0^1 \underbrace{e\big(\alpha(x_1^k + \cdots + x_s^k - n)\big)\, d\alpha}_{= \begin{cases} 1 & \text{if} \\ 0 & \text{otherwise} \end{cases}}$$

Note that as α runs between 0 and 1, $e(\alpha)$ runs through the unit circle. This is why we call this approach the "circle method".

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Define $\tilde{G}(k)$ to be the least integer $s = s(k)$ such that the expected asymptotic formula holds for every sufficiently large n.

Conjecture: $\tilde{G}(k) = \max\{k+1, \Gamma_0(n)\}$ where $\Gamma_0(k)$ is the least integer s such that for every prime p and $m \in \mathbb{N}$, $n = x_1^k + \cdots + x_s^k$ has a solution in mod p^m with $(x, p) = 1$.

Idea: Divide $[0,1)$ into two parts: major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$ where $\mathfrak{M}$ contains $\alpha \in [0,1)$ that are "close" to a rational number of "small" denominators and $\mathfrak{m} = [0,1) \setminus \mathfrak{M}$.

<u>The major arcs</u>: Consider $\alpha = \frac{a}{q} \in \mathbb{Q}$ with $(a,q) = 1$. Write $x = yq + r$ with $1 \le r \le q$. We have

$$f\Big(\frac{a}{q}\Big) = \sum_{1 \le x \le p} e\Big(\frac{a}{q} x^k\Big)$$

$$= \sum_{\frac{1-r}{q} \le y \le \frac{p-r}{q}} e\Big(\frac{a}{q}(yq+r)^k\Big)$$

$$= \sum_{r=1}^{q} \sum_{\frac{1-r}{q} \le y \le \frac{p-r}{q}} e\Big(\frac{ar^k}{q}\Big) \sim \frac{p}{q} \sum_{r=1}^{q} e\Big(\frac{ar^k}{q}\Big).$$

(Need $q \le p$ for $\sim$ to be true.)

To extend the above estimate to $\alpha \in [0,1)$ that is "close" to $\frac{a}{q}$.

$$f\left(\frac{a}{q}+cp^{-k}\right) = \sum_{x \le p} e\left(\left(\frac{a}{q}+cp^{-k}\right)x^k\right) = \sum_{x \le p} e\left(\frac{a}{q}x^k\right) e\left(cp^{-k}x^k\right).$$

To approximate $f(\frac{a}{q}+cp^{-k})$ by $f(\frac{a}{q})$, we need $e(cp^{-k}x^k)$ to be "close" to 1. Since $p^{-k}x^k \le 1$, it suffices to choose c to be "small". This motivates the following definition of the major arcs.

Let $\delta \in \mathbb{R}$ with $0 < \delta < \frac{1}{5}$, and let $a, q \in \mathbb{N} \cup \{0\}$. We define the major arcs

$$\mathcal{M} = \bigcup_{\substack{0 \le a < q \le p^\delta \\ (a,q)=1}} \mathcal{M}(q,a)$$

where

$$\mathcal{M}(a,q) = \{\alpha \in [0,1) \, ; \, |\alpha - \tfrac{a}{q}| \le p^{\delta - k}\}.$$

The remaining set

$$\mathfrak{m} = [0,1) \setminus \mathcal{M}$$

is called the minor arcs. One can show

$$\int_{\mathcal{M}} f(\alpha)^s e(-n\alpha)\, d\alpha \sim C(s,k,n)\, n^{s/k - 1}.$$

The minor arcs: A trivial bound for $f(\alpha)$ is

$$|f(\alpha)| = \left|\sum_{x \le p} e(\alpha x^k)\right| \le p.$$

Suppose that we can show that $\sup_{\alpha \in \mathfrak{m}} |f(\alpha)| \ll p^{1-\upsilon}$ for some $\upsilon = \upsilon(k) > 0$. Then it follows that

$$\begin{aligned} \left|\int_{\mathfrak{m}} f(\alpha)^s e(-n\alpha)\, d\alpha\right| &\le \left(\sup_{\alpha \in \mathfrak{m}} |f(\alpha)|\right)^s \int_{\mathfrak{m}} 1 \, d\alpha \\ &\ll (p^{1-\upsilon})^s \\ &\ll p^{s-k-\upsilon} \\ &\ll n^{s/k - 1 - \upsilon/s}. \end{aligned}$$

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The analysis of the major arcs: Let $a, q \in \mathbb{N} \cup \{0\}$. Suppose that $0 < a < q$ and $(a,q) = 1$. For $\alpha \in \mathcal{M}(q,a) \subseteq \mathcal{M}$, we have $q \le p^\delta$ $(0 < \delta < 1/5)$ and $|\alpha - \frac{a}{q}| \le p^{\delta - k}$. Write $\alpha = \frac{a}{q} + \beta$, we have

$$f(\alpha) = \sum_{1\le x\le P} e(\alpha x^k)$$
$$= \sum_{r=1}^{q} \sum_{\frac{1-r}{q} \le y \le \frac{P-r}{q}} e\left(\left(\frac{a}{q}+\beta\right)(qy+r)^k\right)$$
$$= \sum_{r=1}^{q} e\left(\frac{ar^k}{q}\right) \sum_{\frac{1-r}{q} \le y \le \frac{P-r}{q}} e\left(\beta(qy+r)^k\right).$$

Since $e(\cdot)$ is smooth, we have

$$\sum_{\frac{1-r}{q} \le y \le \frac{P-r}{q}} e\left(\beta(yq+r)^k\right) \sim \int_{\frac{1-r}{q}}^{\frac{P-r}{q}} e\left(\beta(zq+r)^k\right)dz$$
$$\sim \int_{-r/q}^{P-r/q} e\left(\beta(zq+r)^k\right)dz \sim \frac{1}{q}\int_0^P e(\beta\gamma^k)d\gamma.$$

Define

$$S(q,a) = \sum_{r=1}^{q} e\left(\frac{ar^k}{q}\right)$$

and

$$v(\beta) = \int_0^P e(\beta\gamma^k)d\gamma.$$

Then one can show that for $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$

$$f(\alpha) = \frac{1}{q} S(q,\beta) v\left(\alpha - \frac{a}{q}\right) + P^{2\delta}.$$

Since $f(\alpha) \sim \frac{1}{q} S(q,\beta)$, we have

$$\int_{\mathfrak{M}} f(\alpha)^s e(-n\alpha)d\alpha = \sum_{1\le q\le P^\delta} \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} \int_{|\alpha-\frac{a}{q}|=|\beta|\le P^{\delta-k}} f(\alpha)^s e\left(\frac{-na}{q}\right)e(-n\beta)d\beta$$
$$\sim \sum_{1\le q\le P^\delta} \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} \left(q^{-1}S(q,a)\right)^s e\left(\frac{-na}{q}\right) \int_{|\beta|\le P^{\delta-k}} v(\beta)^s e(-n\beta)d\beta.$$

For $Q>0$, define

$$G_s(n,Q) = \sum_{1\le q\le Q} \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} \left(q^{-1}S(q,a)\right)^s e\left(\frac{-na}{q}\right)$$

and

$$J_s(n,Q) = \int_{-Q}^{Q} v(\beta)^s e(-n\beta)d\beta.$$

Then one can show

$$\int_0^1 f(\alpha)^s e(-n\alpha)d\alpha \sim \int_{\mathfrak{M}} f(\alpha)^s e(-n\alpha)\,d\alpha = G_s(n,P^\delta) J_s(n,P^{\delta-k}) + O(P^{s-k-u}),$$

for some $u>0$. Define the singular series
$$\mathfrak{S}_s(n)=\sum_{q=1}^{\infty}\sum_{\substack{a=0\\(a,q)=1}}^{q-1}\left(q^{-1}S(q,a)\right)^s e\left(\frac{-na}{q}\right)$$
and singular integral
$$J_s(n)=\int_{-\infty}^{\infty}\nu(\beta)^s e(-n\beta)\,d\beta$$
One can show that for $s>2^k$ there exists $w>0$ such that
$$\int_{\mathfrak{M}} f(\alpha)^s e(-n\alpha)\,d\alpha=\mathfrak{S}_s(n)J_s(n)+O(P^{s-k-w}).$$
One can show that for $s\geq 2$,
$$J_s(n)=\frac{\Gamma\left(1+\frac{1}{k}\right)^s}{\Gamma(s/k)}n^{\frac{s}{k}-1}$$
where Γ is the Gamma function, defined by
$$\Gamma(x)=\int_0^{\infty}t^{x-1}e^{-t}\,dt.$$
It remains to show that $1\ll\mathfrak{S}_s(n)\ll 1$. One can show that
$$\mathfrak{S}_s(n)=\prod_p\sigma(p)$$
where
$$\sigma(p)=\sum_{n=0}^{\infty}A(p^n,n)$$
where
$$A(q,n)=\sum_{\substack{a=0\\(a,q)=1}}^{q-1}\left(q^{-1}S(q,a)\right)^s e\left(-\frac{na}{q}\right).$$
Indeed, $\sigma(p)$ corresponds to the "p-adic solutions" of $x_1^k+\dots+x_s^k=n$.
More precisely, let
$$M_n(q)=\#\{m_1^k+\dots+m_s^k\equiv n \pmod q,\ 1\leq m_i\leq q\}.$$
Then we have
$$\sigma(p)=\lim_{h\to\infty}p^{h(1-s)}M_n(p^h)>0.$$
It follows that
$$\int_{\mathfrak{M}} f(\alpha)^s e(-n\alpha)\,d\alpha=\mathfrak{S}_s(n)\frac{\Gamma\left(1+\frac{1}{k}\right)^s}{\Gamma(s/k)}n^{\frac{s}{k}-1}+O(n^{s-k-1-w})$$
for some $w>0$.

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We have "proved" that for $s \geq 2^k+1$ (indeed $s \geq 2k+1$ is enough) that

$$\int_{\mathfrak{M}} f(\alpha)^s e(-n\alpha)\,d\alpha \sim C(s;k;n)\,p^{s-k}$$

for some $C(s;k;n)>0$. It remains to show

$$\int_{\mathfrak{m}} f(\alpha)^s e(-n\alpha)\,d\alpha = o(p^{s-k}).$$

<u>The analysis of the minor arcs:</u> Our goal is to estimate

$$f(\alpha)=\sum_{x\leq p} e(\alpha x^k),$$

but it is difficult. However, if $k=1$, then it is a geometric series. We observe that

$$\begin{aligned}|f(\alpha)|^2 &= f(\alpha)f(-\alpha)\\ &= \sum_{x\leq p}\sum_{y\leq p} e(\alpha(y^k-x^k))\\ &= \sum_{x\leq p}\sum_{x+h\leq p} e(\alpha((x+h)^k-x^k)).\end{aligned}$$

Note that $(x+h)^k-x^k$ is a polynomial of degree $k-1$ in x. If we iterate the powers $k-1$ times, we get a geometric series (degree 1 in x).

Weyl's Inequality:

$$\sup_{\alpha\in\mathfrak{m}} |f(\alpha)| \ll p^{1-\delta 2^{-k}+\varepsilon}$$

For

$$f(\alpha)=\sum_{x\leq p} e(\alpha x^k),$$

consider

$$\begin{aligned}\int_0^1 |f(\alpha)|^{2t}\,d\alpha &= \sum_{x_1,y_1\leq p}\cdots\sum_{x_t,y_t\leq p}\int_0^1 e(\alpha((x_1^k-y_1^k)+\cdots+(x_t^k-y_t^k)))\,d\alpha\\ &= \#\{x_1^k+\cdots+x_t^k=y_1^k+\cdots+y_t^k,\ x_i,y_i\leq p\}\\ &\asymp p^{2t-k}.\ \text{(expected)}\end{aligned}$$

Hua's Lemma:

$$\int_0^1 |f(\alpha)|^{2^k} d\alpha \ll P^{2^k - k + \varepsilon}.$$

By combining Weyl's inequality with Hua's lemma, we see that for $s \geq 2^k + 1$, we have

$$\begin{aligned} \int_{\mathfrak{m}} f(\alpha)^s e(-n\alpha)\, d\alpha &\leq \left(\sup_{\alpha \in \mathfrak{m}} |f(\alpha)| \right)^{s-2^k} \int_0^1 |f(\alpha)|^{2^k} d\alpha \\ &\ll \left(P^{1-\delta 2^{1-k} + \varepsilon} \right)^{s-2^k} P^{2^k - k + \varepsilon} \\ &\ll P^{s-k-\delta 2^{1-k}(s-2^k)+\varepsilon} \\ &= o(P^{s-k}) \end{aligned}$$