# 10. L-functions and Dirichlet's Theorem

Let $k \in \mathbb{N}$ with $k \geq 2$ and let $\chi$ be a character mod $k$. For $Re(s) > 1$, define

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Let $\chi_0$ be the principle character.

Theorem 58: (1) If $\chi \neq \chi_0$, then $L(s,\chi)$ has an analytic continuation to $Re(s) > 0$.
(2) If $\chi = \chi_0$, then $L(s, \chi_0)$ has an analytic continuation to $Re(s) > 0$ with $s \neq 1$. At $s=1$, $L(s,\chi_0)$ has a simple pole with residue $\frac{\varphi(k)}{k}$.

Proof: Let

$$A(x) = \sum_{n \leq x} \chi(n)$$

and

$$E(\chi) = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise.} \end{cases}$$

By theorem 57(4), we have

$$A(x) = \begin{cases} \left[\frac{x}{k}\right]\varphi(k) + T(x) & \text{if } \chi = \chi_0, \\ \left[\frac{x}{k}\right] 0 + T(x) & \text{if } \chi \neq \chi_0, \end{cases}$$

with $|T(x)| \leq \varphi(k)$. It follows that

$$A(x) = E(\chi) \frac{\varphi(k)}{k} x + R(x)$$

with $|R(x)| \leq 2\varphi(k)$. Let $f(n) = \frac{1}{n^s}$. By Abel's summation,

$$\sum_{n \leq x} \frac{\chi(n)}{n^s} = \frac{A(x)}{x^s} + s\int_1^x \frac{A(u)}{u^{s+1}}\,du$$

$$= E(\chi)\frac{\varphi(k)}{k}\frac{1}{x^{s-1}} + \frac{R(x)}{x^s} + sE(\chi)\frac{\varphi(k)}{k}\left(\frac{-u^{s+1}}{s-1}\Big|_1^x\right) + s\int_1^x \frac{R(u)}{u^{s+1}}\,du$$

$$= E(\chi)\frac{\varphi(k)}{k}\left(x^{1-s} + \frac{s}{1-s}(x^{1-s} - 1)\right) + \frac{R(x)}{x^s} + s\int_1^x \frac{R(u)}{u^{s+1}}\,du. \quad \text{--- } (*)$$

(1) If $\chi \neq \chi_0$, then $E(\chi) = 0$. We see from $(*)$ that

$$\sum_{n\le x}\frac{\chi(n)}{n^s}=\frac{R(x)}{x^s}+s\int_1^x\frac{R(u)}{u^{s+1}}\,du.$$

By letting $x\to\infty$, since $|R(x)|\le 2\varphi(k)$, for $Re(s)>0$,

$$L(s,\chi)=s\int_1^\infty\frac{R(u)}{u^{s+1}}\,du.$$

Since the integral converges for $Re(s)>0$, $L(s,\chi)$ has an analytic continuation to $Re(s)>0$.

(2) If $\chi=\chi_0$, then $E(\chi)=1$. By $(*)$,

$$\sum_{n\le x}\frac{\chi(n)}{n^s}=\frac{\varphi(k)}{k}\left(x^{1-s}+\frac{s}{1-s}(x^{1-s}-1)\right)+\frac{R(x)}{x^s}+s\int_1^x\frac{R(u)}{u^{s+1}}\,du.$$

Consider $Re(s)>1$. By letting $x\to\infty$, we have

$$L(s,\chi_0)=\frac{\varphi(k)}{k}\frac{s}{s-1}+s\int_1^\infty\frac{R(u)}{u^{s+1}}\,du.$$

Since the integral converges for $Re(s)>0$, $L(s,\chi_0)$ has an analytic continuation to $Re(s)>0$, except a simple pole at $s=1$ with residue $\frac{\varphi(k)}{k}$. ∎

<u>Def</u> Let $\{\lambda_n\}_{n=1}^\infty$ be a strictly increasing sequence of positive real numbers. For $z\in\mathbb{C}$, a <u>Dirichlet series</u> attached to $\{\lambda_n\}_{n=1}^\infty$ is a series of the form

$$\sum_{n=1}^\infty a_n e^{-\lambda_n z}$$

where $\{a_n\}_{n=1}^\infty$ is a sequence of complex numbers.

Theorem 59: If the Dirichlet series

$$\sum_{n=1}^\infty a_n e^{-\lambda_n z}$$

converges for $z=z_0$, then it converges uniformly for $Re(z-z_0)>0$ and $|\arg(z-z_0)|<a$ with $a<\frac{\pi}{2}$.

Proof: Without loss of generality, we may assume $z_0=0$. Since

$$\sum_{n=1}^\infty a_n$$

converges, for any $\varepsilon>0$, there exists $N=N(\varepsilon)\in\mathbb{N}$ such that if $l,m>N$ then

$$\left|\sum_{n=\ell}^{m} a_n\right| < \varepsilon.$$

Let

$$A_{\ell,m} = \sum_{n=\ell}^{m} a_n.$$

By taking the convention that $A_{\ell,\ell-1}=0$, we have

$$\sum_{n=\ell}^{m} a_n e^{-\lambda_n z} = \sum_{n=\ell}^{m}\left(A_{\ell,n} - A_{\ell,n-1}\right)e^{-\lambda_n z}$$
$$= \sum_{n=\ell}^{m-1} A_{\ell,n}\left(e^{-\lambda_n z} - e^{-\lambda_{n+1} z}\right) + A_{\ell,m}e^{-\lambda_m z}.$$

Thus for $Re(z)\geq 0$,

$$\left|\sum_{n=\ell}^{m} a_n e^{-\lambda_n z}\right| \leq \varepsilon\left(\sum_{n=\ell}^{m-1}\left|e^{-\lambda_n z} - e^{-\lambda_{n+1} z}\right| + 1\right).$$

Note that

$$e^{-\lambda_n z} - e^{-\lambda_{n+1} z} = z\int_{\lambda_n}^{\lambda_{n+1}} e^{-tz}\,dt.$$

Also, for $z=x+iy$ with $x,y\in\mathbb{R}$, we have $|e^{-tz}| = e^{-tx}$. Thus we have

$$\left|e^{-\lambda_n z} - e^{-\lambda_{n+1} z}\right| \leq |z|\int_{\lambda_n}^{\lambda_{n+1}} e^{-tx}\,dt$$
$$\leq \frac{|z|}{x}\left(e^{-\lambda_n x} - e^{-\lambda_{n+1} x}\right).$$

Thus

$$\left|\sum_{n=\ell}^{m} a_n e^{-\lambda_n z}\right| \leq \varepsilon\left(\frac{|z|}{x}\left(e^{-\lambda_\ell x} - e^{-\lambda_m x}\right) + 1\right).$$

Note that for $|\arg(z)|<\alpha$, we have $\frac{|z|}{x}<c$ for some $c=c(\alpha)$. Also we have

$$\left|e^{-\lambda_\ell x} - e^{-\lambda_m x}\right| \leq 2.$$

It follows that

$$\left|\sum_{n=\ell}^{m} a_n e^{-\lambda_n z}\right| < (2c+1)\varepsilon.$$

Thus the Dirichlet series converges for $Re(z)\geq 0$ and $|\arg(z)|\leq\alpha$. ∎

Theorem 60: Let

$$f(z)=\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

be a Dirichlet series with $a_n \in \mathbb{R}$ and $a_n \geq 0$ for all $n \in \mathbb{N}$. Suppose that the series converges for $Re(s)>\sigma_0$ with $\sigma_0 \in \mathbb{R}$ and suppose that $f(z)$ can be analytically continued in a neighbourhood of $\sigma_0$. Then there exists a real number $\varepsilon>0$ such that

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

converges for $Re(z)>\sigma_0-\varepsilon$.

Proof: Without loss of generality, we may assume that $\sigma_0=0$. Since $f(z)$ is analytic in a neighbourhood of 0, by theorem 59, it is analytic for $Re(z)>0$. Since $f(z)$ is analytic for $Re(z)>0$ and is also analytic in a neighbourhood of 0, there exists $\varepsilon>0$ such that $f$ is analytic in $|z-1|\leq 1+\varepsilon$. We now consider the Taylor series expansion of $f(z)$ around 1 in $|z-1|\leq 1+\varepsilon$. Note that for $Re(z)>0$,

$$f^{(m)}(z)=\sum_{n=1}^{\infty} a_n(-\lambda_n)^m e^{-\lambda_n z}$$

This implies that

$$f^{(m)}(1)=\sum_{n=1}^{\infty} a_n(-\lambda_n)^m e^{-\lambda_n}.$$

Thus the Taylor expansion of $f(z)$ around 1 in $|z-1|\leq 1+\varepsilon$ is of the form

$$\sum_{m=0}^{\infty} \frac{f^{(m)}(1)}{m!}(z-1)^m.$$

We now consider $f(z)$ at the point $z=-\varepsilon$. We have

$$f(-\varepsilon)=\sum_{m=0}^{\infty}\left(\sum_{n=1}^{\infty} a_n(-\lambda_n)^m e^{-\lambda_n}\right)\frac{(-1-\varepsilon)^m}{m!}$$
$$=\sum_{m=0}^{\infty}\left(\sum_{n=1}^{\infty} a_n\lambda_n^m e^{-\lambda_n}\right)\frac{(1+\varepsilon)^m}{m!}.$$

Since $a_n \geq 0$ and all other terms are positive, we can switch the order of summation and obtain

$$f(-\varepsilon) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n} \left( \sum_{m=0}^{\infty} \frac{(\lambda_n)^m (1+\varepsilon)^m}{m!} \right)$$
$$= \sum_{n=1}^{\infty} a_n e^{-\lambda_n} e^{\lambda_n(1+\varepsilon)}$$
$$= \sum_{n=1}^{\infty} a_n e^{\lambda_n \varepsilon}$$
$$= \sum_{n=1}^{\infty} a_n e^{(-\lambda_n)(-\varepsilon)}.$$

Thus the series
$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$
converges to $f(z)$ at $z=-\varepsilon$. By theorem 59, it converges to $f(z)$ for $Re(z) > -\varepsilon$. ∎

Theorem 61: For $k \in \mathbb{N}$ with $k \geq 2$, let $\chi$ be a character mod $k$.
(1) $L(s,\chi)$ is nonzero for $Re(s) > 1$.
(2) If $\chi \neq \chi_0$ then $L(1,\chi)$ is nonzero.

Proof: (1) Note that $L(s,\chi)$ converges absolutely for $Re(s) > 1$. Since $\chi$ is completely multiplicative, $L(s,\chi)$ has an Euler product representation for $Re(s)>1$, which is
$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$
Since
$$\sum_p \left| \frac{\chi(p)}{p^s} \right|$$
converges for $Re(s)>1$, $L(s,\chi)$ is nonzero for $Re(s)>1$.
(2) We recall that for $|u|<1$,
$$-\log(1-u) = \sum_{n=1}^{\infty} \frac{u^n}{n}.$$
Thus for $Re(s)>1$, we have
$$\log L(s,\chi) = \log\left(\prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}\right)$$

$$= \sum_p -\log\left(1 - \frac{\chi(p)}{p^s}\right)$$

$$= \sum_p \sum_{n=1}^{\infty} \frac{\chi(p^n)}{np^{ns}}.$$

Let $l \in \mathbb{Z}$ with $(l,k) = 1$. By summing over all characters mod $k$, we have

$$\sum_{\chi \text{ char. mod } k} \overline{\chi}(l) \log L(s,\chi) = \sum_p \sum_{n=1}^{\infty} \frac{1}{np^{ns}} \sum_{\chi \text{ char mod } k} \overline{\chi}(l)\chi(p^n)$$

$$= \varphi(k) \sum_{n=1}^{\infty} \sum_{p^n \equiv l \pmod k} \frac{1}{np^{ns}}$$

by theorem 57(5). By taking $l = 1$ and exponentiating both sides, we have

$$\prod_{\chi \text{ char mod } k} L(s,\chi) = \exp\left(\varphi(k) \sum_{n=1}^{\infty} \sum_{p^n \equiv l \pmod k} \frac{1}{np^{ns}}\right)$$

Thus if $s \in \mathbb{R}$ with $s > 1$, we have

$$\prod_{\chi \text{ char mod } k} L(s,\chi) \geq 1.$$

We now split into cases depending on if $\chi$ is a real character or not. 2015 11 20

(2-1) Suppose that $L(1,\chi) = 0$ with $\chi$ a non-real character. Since $L(s,\overline{\chi}) = \overline{L(s,\chi)}$ for $s \in \mathbb{R}$ with $s > 1$, we have

$$L(1,\overline{\chi}) = \overline{L(1,\chi)} = 0.$$

We also recall that for $s \in \mathbb{R}$ with $s > 1$, we have

$$\prod_{\chi \text{ char mod } k} L(s,\chi) \geq 1. \qquad \text{———— } (*)$$

We have seen in theorem 58 that $L(s,\chi_0)$ has a simple pole at $s = 1$ and $L(s,\chi)$ does not have a pole at $s = 1$ for any $\chi \neq \chi_0$. Thus as $s \to 1^+$ on the real line, we have

$$\prod_{\chi \text{ char mod } k} L(s,\chi) = O\left((s-1)^{-1}(s-1)^2\right) = O(s-1),$$

which contradicts $(*)$. Thus $L(1,\chi) \neq 0$ for $\chi$ a non-real character.

(2-2) Suppose that $L(1,\chi) = 0$ with $\chi$ a real character. For $\operatorname{Re}(s) > 1$, define

$$g(s) = \frac{\zeta(s) L(1,\chi)}{\zeta(2s)}$$

Consider the Euler product representation of $g(s)$ for $Re(s)>1$. We have

$$g(s) = \prod_p \frac{(1-\frac{1}{p^{2s}})}{(1-\frac{1}{p^s})(1-\frac{\chi(p)}{p^s})}$$

$$= \prod_p \frac{1+\frac{1}{p^s}}{1-\frac{\chi(p)}{p^s}}$$

$$= \prod_p \left(1+\frac{1}{p^s}\right)\left(\sum_{\ell=0}^{\infty} \frac{\chi(p^\ell)}{p^{\ell s}}\right)$$

$$= \prod_p \left(1+\sum_{\ell=1}^{\infty} \frac{\chi(p^{\ell-1})+\chi(p^\ell)}{p^{\ell s}}\right)$$

$$= \prod_p \left(1+\sum_{\ell=1}^{\infty} \frac{b(p^\ell)}{p^{\ell s}}\right)$$

where

$$b(p^\ell) = \chi(p^{\ell-1}) + \chi(p^\ell). \qquad (\ell \geq 1)$$

Since $\chi$ is a real character, $\chi(p) \in \{0, \pm 1\}$. Since $\chi$ is multiplicative, we have

$$b(p^\ell) = \chi(p^{\ell-1}) + \chi(p^\ell) = \begin{cases} 0 & \text{if } \chi(p)=0, \\ 2 & \text{if } \chi(p)=1, \\ 0 & \text{if } \chi(p)=-1. \end{cases}$$

In all cases, we have $b(p^\ell) \geq 1$ for all $\ell \geq 1$. Thus

$$g(s) = 1+\sum_{n=2}^{\infty} \frac{a_n}{n^s} \qquad \text{——— } (**)$$

with $a_n \in \mathbb{R}$ and $a_n \geq 0$ $(n \geq 2)$. Since the zero of $L(1,\chi)$ eliminates the pole of $\zeta(s)$ at $s=1$, and since $\zeta(2s)$ is nonzero and analytic for $Re(s) > \frac{1}{2}$, then $g(s)$ has an analytic continuation to $Re(s) > \frac{1}{2}$. By theorem 60, we conclude that the series defining $g$ converges to $g$ for $Re(s) > \frac{1}{2}$. As $s \to \frac{1}{2}^+$ on the real axis, since $\zeta(2s)$ has a pole at $s = \frac{1}{2}$, we see that

$$g(s) = O(s-\tfrac{1}{2}),$$

which contradicts $(**)$ as $g(s) \geq 1$ for $Re(s) > \frac{1}{2}$. Thus $L(1,\chi) \neq 0$ for $\chi$ a real character. ∎

Theorem 62: Let $\ell, k \in \mathbb{Z}$ with $k \geq 2$ and $(\ell, k) = 1$. Then the series

$$\sum_{p \equiv \ell \pmod{k}} \frac{1}{p}$$

diverges. This implies that there are infinitely many primes $p$ with $p \equiv \ell \pmod{k}$.

Remark: For $x \in \mathbb{R}$, let

$$\pi(x; k, \ell) = \#\{p \leq x;\ p \text{ is a prime and } p \equiv \ell \pmod{k}\}.$$

Then using similar method used by Newman for his proof of the prime number theorem, one can prove that

$$\pi(x; k, \ell) \sim \frac{1}{\varphi(k)} \frac{x}{\log x}$$

(proved by Vallée-Poussin). In the case when $k$ is "small", the Siegal-Walfisz theorem gives a refinement of the above result. More precisely, define

$$\psi(x; k, \ell) = \sum_{\substack{n \leq x \\ n \equiv \ell \pmod{k}}} \Lambda(n)$$

If $k \leq (\log x)^N$ for some $N \in \mathbb{N}$, then

$$\psi(x; k, \ell) = \frac{x}{\varphi(k)} + O(x \exp(-c_N (\log x)^{\frac{1}{2}})).$$

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Proof: We have seen in the proof of theorem 61 that

$$\frac{1}{\varphi(k)} \sum_{\chi \text{ char mod } k} \overline{\chi}(\ell) \log L(s, \chi) = \sum_{n=1}^{\infty} \sum_{p^n \equiv \ell \pmod{k}} \frac{1}{np^{ns}}. \quad \text{--- } (*)$$

We recall that

$$E(\chi) = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0. \end{cases}$$

As $s \to 1^+$ on the real axis by theorems 58 and 61, $(s-1)^{E(\chi)} L(s, \chi)$ tends to a finite nonzero limit. Thus $E(\chi) \log(s-1) + \log L(s, \chi)$ tends to a limit. It follows that as $s \to 1^+$ on the real axis, we have

$$\log L(s, \chi) = -E(\chi) \log(s-1) + O(1).$$

Thus we have

$$\frac{1}{\varphi(k)} \sum_{\chi \text{ char mod } k} \overline{\chi}(\ell) \log L(s, \chi) = \frac{1}{\varphi(k)} \log L(s, \chi_0) + \frac{1}{\varphi(k)} \sum_{\chi \neq \chi_0 \text{ char mod } k} \overline{\chi}(\ell) \log L(s, \chi)$$

$$= -\log(s-1) + O(1).$$

Combining this with (*), we have

$$\sum_{n=1}^{\infty} \sum_{p^n \equiv \ell \pmod{k}} \frac{1}{np^{ns}} = -\frac{1}{\varphi(k)} \log(s-1) + O(1).$$

Thus

$$\sum_{p \equiv \ell \pmod{k}} \frac{1}{p^s} + \sum_{n=2}^{\infty} \sum_{p^n \equiv \ell \pmod{k}} \frac{1}{np^{ns}} = -\frac{1}{\varphi(k)} \log(s-1) + O(1).$$

Note that for $\mathrm{Re}(s) \geq 1$ and $s \in \mathbb{R}$,

$$\begin{aligned}
\sum_{n=2}^{\infty} \sum_{p^n \equiv \ell \pmod{k}} \frac{1}{np^{ns}} &\leq \frac{1}{2} \sum_{n=2}^{\infty} \sum_{p^n \equiv \ell \pmod{k}} \frac{1}{p^{ns}} \\
&= \frac{1}{2} \sum_{m=2}^{\infty} \left( \frac{1}{m^{2s}} + \frac{1}{m^{3s}} + \cdots \right) \\
&\leq \frac{1}{2} \sum_{m=2}^{\infty} \frac{1}{m^{2s}} \left( \frac{1}{1 - \frac{1}{m^s}} \right) \\
&\leq \sum_{m=2}^{\infty} \frac{1}{m^{2s}} \\
&\leq \sum_{m=2}^{\infty} \frac{1}{m^2} \\
&\leq \frac{\pi^2}{6}.
\end{aligned}$$

Thus

$$\sum_{p \equiv \ell \pmod{k}} \frac{1}{p^s} = -\frac{1}{\varphi(k)} \log(s-1) + O(1).$$

As $s \to 1^+$ on the real axis, the quantity $-\frac{1}{\varphi(k)} \log(s-1) \to \infty$. It follows that

$$\sum_{p \equiv \ell \pmod{k}} \frac{1}{p}$$

diverges ∎