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Valuation Rings

Def) Let K be a field. A valuation v on K is a map
$$v : K \to \mathbb{Z} \cup \{\infty\}$$
such that:

1) $v(a) = \infty \iff a = 0$;
2) $v(ab) = v(a) + v(b)$;
3) $v(a+b) \geq \min\{v(a), v(b)\}$.

Ex 1 Let $K = \mathbb{Q}$, let p be a prime number. For $n \in \mathbb{Z}$, $n \neq 0$, we write $n = p^k n'$, $p \nmid n'$. Define $v(n) = k$. This is called the p-adic valuation. $v(0) = \infty$.
For $\frac{m}{n} \in \mathbb{Q}$, $m \in \mathbb{Z}$, $n \in \mathbb{Z} \setminus \{0\}$, we define $v(m/n) = v(m) - v(n)$.
Why is this well-defined? Note $v(n_1 n_2) = v(n_1) + v(n_2)$.
Let's show that this is a valuation. 1 is immediate. 2:
$$v\left(\frac{m_1}{n_1}\frac{m_2}{n_2}\right) = v(m_1 m_2) - v(n_1 n_2) = v(m_1) + v(m_2) - v(n_1) - v(n_2) = \checkmark$$

3: $v(a) = d \iff a = p^d \frac{u}{v}$, $p \nmid u$, $p \nmid v$.
$v(ap^{-d}) = 0$ so if we write $ap^{-d} \frac{u}{v}$, $\gcd(u,v) = 1$
then $v(u) = v(v)$ & $\gcd(u,v) = 1$ means they can't both be divisible by p, so $v(u) = v(v)$
If $v(a) = d$, $v(b) = e$ then $a = p^d \frac{u}{v}$, $b = p^e \frac{u'}{v'}$, $p \nmid u, u', v, v'$.
So $d = \min(d,e) \Rightarrow a+b = p^d\left(\frac{u}{v} + p^{e-d}\frac{u'}{v'}\right)$

Eg $p = 3$ $\quad 9 + 18 = 9 + (3^3 - 9)$

Ex: Let K be the field of rational functions on $\mathbb{C}$, $\mathbb{C}(x)$.
Given $f \in K \setminus \{0\}$, define
$$v(f(x)) = \begin{cases} n & \text{if } f(x) \text{ has a zero at } x=0 \text{ of order } n, \\ -n & \text{" pole " } n, \\ 0 & \text{if } f(x) \text{ is analytic at } 0 \text{ and } f(0) \neq 0. \end{cases}$$
This gives a valuation. (Check).

Comparison

	starting ring	valuation	associated ideal	
$\mathbb{C}(x)$	$\mathbb{C}[x]$	order of 0 at $x=0$.	(x)	$v(p(x))=k \iff p(x)\in(x)^k$, $p(x)\notin(x)^{k+1}$
$\mathbb{Q}$	$\mathbb{Z}$	biggest k st $p^k \mid \cdot$	$p\mathbb{Z}$	$v(n)=k \iff n\in(p\mathbb{Z})^k$, $n\notin(p\mathbb{Z})^{k+1}$

Let K be a field and let

$$v: K \to \mathbb{Z}\cup\{\infty\}$$

be a valuation. We define the <u>valuation ring of v</u>:

$$\mathcal{O}_v = \{a\in K \,;\, v(a)\geq 0\}$$

Why is $\mathcal{O}_v$ a ring? Do it! (easy)

Remark 1: $\mathcal{O}_v$ is a local ring with maximal ideal $\mathcal{M}_v = \{a\in\mathcal{O}_v \,;\, v(a)>0\}$.
$\mathcal{M}_v$ is an ideal
If $x\in\mathcal{O}_v\setminus\mathcal{M}_v \Rightarrow x$ is a unit.

$$v(x)=0 \Rightarrow v(\tfrac{1}{x}x)=v(1)=0 \Rightarrow v(\tfrac{1}{x})+v(x)=0 \Rightarrow \tfrac{1}{x}\in\mathcal{O}_v \Rightarrow x\in\mathcal{O}_v^*$$

So $\mathcal{M}_v$ is the unique maximal ideal.

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<u>Discrete Valuation Rings (DVRs) (dvrs)</u>

A ring of the form $\mathcal{O}_v$ is called a <u>discrete valuation ring</u>
We say that $\mathcal{O}_v$ is a ring, $\mathcal{O}_v$ is local, $\mathfrak{M}_v=\{a\,;\,v(a)>0\}$ unique maximal ideal.

Proposition: Let R be a dvr. Then R is a PID.

Proof: Let $I\trianglelefteq R$. If $I=(0)$ or $I=R$ then I is principal. So WLOG $(0)\subsetneq I\subseteq\mathcal{M}_v$. Pick $a\in I$ with $v(a)$ minimal ($a\neq 0$ note). We claim that $I=(a)$. To see this, suppose that $y\in I\setminus\{0\}$. Then $v(y)\geq v(a)$. Let $K=\mathrm{Frac}(R)$. Then $ya^{-1}\in K$ & $v(ya^{-1})=v(y)-v(a)\geq 0$ so $ya^{-1}\in\mathcal{O}_v=R$. So

$$y=(ya^{-1})a\in Ra \Rightarrow I\subseteq Ra \Rightarrow I=Ra.$$ ∎

Question: What are the possible Krull dimensions of a PID?
Answer: 0 and 1.
Why? Suppose that R is a PID & we have a chain $(0) \subset P \subset Q$, primes. Since R is a PID, $P = (x)$, $Q = (y)$. So $\exists a \in R$ such that $x = ay \in (x) \Rightarrow a \in (x)$ or $y \in (x)$. Cannot have $y \in (x)$ since $(x) \subset (y)$. So $a \in (x)$. Thus $a = xb$. So
$$x = ay = xby \Rightarrow 1 = by \Rightarrow (y) = R \quad \text{※} \qquad \square$$

Corollary: If R is a dvr then R is a PID & so R is Noetherian & has Kdim 1 (if v is non-trivial).

Proof: We saw R is a PID $\Rightarrow$ Noetherian and $\operatorname{Kdim} R \leq 1$. Now we have $m_v \neq \{0\}$ and is a prime ideal so $\operatorname{Kdim} \geq 1$.

Theorem: Let R be a dvr. Then R is integrally closed. $\leftarrow$ is integrally closed over Frac(R)

Proof: Let $K = \operatorname{Frac}(R)$ & let $x \in K$ be integral over R. Then $\exists n \geq 1$, $r_{n-1}, \ldots, r_0 \in R$ such that $x^n + r_{n-1}x^{n-1} + \cdots + r_0 = 0$.
Remark: $v(a_1 + \cdots + a_n) \geq \min\{v(a_1), \ldots, v(a_n)\}$ (induction).
We must show $v(x) \geq 0$. Assume, ~~for a contradiction~~, that $v(x) = c < 0$. Then
$$v(x^n) = nc, \quad v(r_i x^i) = v(r_i) + ic \geq ic.$$
Now $x^n = -r_{n-1}x^{n-1} \cdots - r_0$, so taking v yields
$$\begin{aligned} nc &= v(-r_{n-1}x^{n-1} - \cdots - r_0) \\ &\geq \min\{v(r_{n-1}x^{n-1}), \ldots\} \\ &\geq \min\{(n-1)c, \ldots, c, 0\} = (n-1)c \end{aligned}$$
So $nc \geq (n-1)c \Rightarrow c \geq 0$.

Theorem: Let A be a Noetherian local domain of Krull dimension one, and let $P \trianglelefteq A$ be its maximal ideal. Then the following are equivalent:
1) A is a dvr;
2) A is integrally closed;
3) P is principal;
4) $\dim_k P/P^2 = 1$ where $k = A/P$ = residue field;
5) every non-zero ideal of A is of the form P^m for some $m \geq 0$;
6) $\exists x \neq 0$ in A such that every non-zero ideal of A is of the form (x^m) for some $m \geq 0$.

Why are dvrs useful?

General strategy in commutative algebra:

Input: Some Noetherian ~~ring~~ integral domain R & some problem.

Step 1: Show that you can reduce to the case where R is integrally closed by considering its integral closure.

Step 2: For each prime P of height 1 in R, we have $R \hookrightarrow R_P$ (dvr, Noe. Kdim 1)

ex Let p be a prime. Then consider

$$X = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z} \times \cdots$$
$$\cup\!|$$
$$\mathbb{Z}_p = \{([a_1]_p, [a_2]_{p^2}, \ldots) \ ; \ a_{k+1} \equiv a_k \bmod p^k\}$$

p-adic integers.

Remark: $\mathbb{Z}/p\mathbb{Z}$ can be given the discrete topology & it is compact. So X is a compact topological space & one can show that $\mathbb{Z}_p \subseteq X$ is a closed subset of X. So $\mathbb{Z}_p$ is a compact topological space under the subspace topology. It is also a ring.

Also $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$, $n \overset{1-1}{\longmapsto} ([n]_p, [n]_{p^2}, \ldots)$.

Also the image is dense.

$\mathbb{Z}_p$ is a dvr with maximal ideal $p\mathbb{Z}_p$, $p \mapsto ([0]_p, [p]_{p^2}, [p]_{p^3}, \ldots)$.

Proof (of thm)

$1 \Rightarrow 2$: ✓

$2 \Rightarrow 3$: Pick $a \in P \setminus P^2$. Why can we do this? Nakayama's lemma: $P = P^2 \Rightarrow \overset{P}{\underset{\shortparallel}{P}} = J(A)P$
$\Rightarrow P = (0)$ as P is f.g.

this isn't 'WLOG', it's proof by cases. And actually, it's not really proof by cases, it's proof by contradiction

Now $\sqrt{(a)} = P \Rightarrow \exists m \geq 1$ st $P^m \subseteq (a)$. Pick smallest n.

If $n = 1$ then $P \subseteq (a) \subseteq P \Rightarrow P = (a)$ ✓.

So WLOG $n > 1$. So $P^{n-1} \not\subseteq (a)$. Pick $b \in P^{n-1} \setminus (a)$. Let $x = a/b$. Then $x^{-1} \notin A$ as if it were then $x^{-1} = b/a \Rightarrow b \in (a)$ if $b/a \in A$ ※

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?

$3 \Rightarrow 4$. Suppose that $P = (x)$, $x \neq 0$. $P = Ax$ so $P/P^2 = Ax/P^2$. Now if $a \in A$ then $ax + P^2 = \bar{a}(x + P^2)$ where $\bar{a} \in A/P$ (P/P^2 is an A/P-module)

So P/P^2 is $k(x + P^2)$, 1-dim.

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we actually showed 4⇔3 rule

$4 \Rightarrow 5$: Notice that if P/P^2 is 1-dim then $\exists x \in P \setminus P^2$ st $x + P^2$ is a basis for P/P^2 as an $A/P = k$-v.s. (Claim $P = (x)$. Trick: Let $M = P/(x)$. Then $PM = (P^2 + (x))/(x) = P/(x) = M$. So $M = (0)$ by Nakayamas lemma ($P = \mathcal{J}(A)$, M is fin. gen. $\because A$ is Noetherian).

Now let $I \neq (0)$ be ~~an~~ a proper ideal of A. Again, $\sqrt{I} = P$. So $\exists n$ st $P^n \subseteq I$.
In particular $\exists$ largest nat. m st $I \subseteq P^m$ (Why? $I \subseteq P^m \ \forall m \Rightarrow I \subseteq P^n$ & $P^n \subseteq I \Rightarrow I = P^n$.)
Since $I \subsetneq P^{m+1} = (x^{m+1})$, $\exists y \in I$ st $y \notin (x^{m+1})$. But $y \in I \subseteq (x^m)$ so $y = ax^n$. Notice $a \notin (x) = P$ $\because y \notin (x^{m+1})$.
So $a \in A \setminus P \Rightarrow a$ is a unit, so $(y) = (x^m) \Rightarrow (x^m) = (y) \subseteq I \subseteq P^m = (x^m)$. ✓

$5 \Rightarrow 6$: Pick $x \in P \setminus P^2$ (as before by Nakayama's). Now $(x) = P^m$ for some $m \geq 0$.
Notice $m \geq 1$ $\because x \in P$ & $m < 2$ $\because x \notin P^2$. So $P = (x)$. So $P^m = (x^m)$.

$6 \Rightarrow 1$: We'll define a map $v : A \setminus \{0\} \to \mathbb{N}_{\geq 0}$ by $v(a) = m$ where $m \geq 0$ is the unique non-neg integer st $a \in P^m$ & $a \notin P^{m+1}$. Why can't a be in $(x^m)\ \forall m$? See A4. So v is well-defined. We extend

$$v : K \to \mathbb{Z} \cup \{\infty\}$$

by $v(\frac{a}{b}) = v(a) - v(b)$ for $a, b \in A$, $b \neq 0$, $v(0) = \infty$ (As in examples, this is well-defined.) Then we claim v is a valuation & A is the valuation ring of v.

(Remark: Since A is local and $P = (x)$, if $a \in A \setminus \{0\}$ then $(a) = (x^m)$ so $a = ux^m$, u a unit, $v(u) = 0$. If $a, b \in A \setminus \{0\}$, $a = ux^m$, $b = u'x^n$

$$\frac{a}{b} = u(u')^{-1}x^{m-n}, \quad v\left(\frac{a}{b}\right) = m - n$$

Notice $v(\frac{a}{b}) \geq 0 \Leftrightarrow m - n \geq 0 \Leftrightarrow \frac{a}{b} = u(u')^{-1}x^d$, $d \geq 0$ "$m-n$" $\Leftrightarrow \frac{a}{b} \in A$.)
So $A = \{\frac{a}{b} \in K^*;\ v(\frac{a}{b}) \geq 0\} \cup \{0\}$.

We should check that v is a valuation:

$\alpha, \beta \in K \setminus \{0\}$. $\alpha = u_1 x^{d_1}$, $\beta = u_2 x^{d_2}$; $\alpha\beta = u_1 u_2 x^{d_1 + d_2}$ $\quad d_i \in \mathbb{Z}$, $u_i \in A^*$

$v(\alpha\beta) = d_1 + d_2 = v(\alpha) + v(\beta)$

$\alpha + \beta = u_1 x^{d_1} + u_2 x^{d_2} = x^{d_1}(\underbrace{u_1 + u_2 x^{d_2 - d_1}}_{\in A})$ if $d_1 \leq d_2$

so $v(\alpha + \beta) = d_1 + v(u_1 + u_2 x^{d_2 - d_1}) \geq d_1$. 5

Dedekind Domains

A Dedekind domain is just an integral domain A with the following properties:
1) A is Noetherian;
2) A has Krull dimension 1;
3) A is integrally closed.

Remark: If P is a maximal ideal of $A \Rightarrow A_P$ is a dvr.

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eg $\mathbb{Z}$, $k[t]$ for k a field
More generally, if $K \overset{<\infty}{\supseteq} \mathbb{Q}$ finite extension of $\mathbb{Q}$
Then $\mathcal{O}_K :=$ integral closure of $\mathbb{Z}$ in K is a Dedekind domain.

Theorem: In a Dedekind domain R every nonzero ideal I has a factorization into prime ideals
$$I = P_1^{m_1} \cdots P_r^{m_r}$$
Moreover, this factorization is unique up to permutation of factors.

Historically, Dedekind domains arise in number theory with FLT.

How will we prove this theorem?
Strategy:
1) Use primary decomposition: $I = Q_1 \cap \cdots \cap Q_s$, Q_i primary
2) Show that in a Dedekind domain, $(0) \neq Q$ primary $\Rightarrow Q = P^n$, P maximal
3) So 1 & 2 $\Rightarrow I = P_1^{m_1} \cap \cdots \cap P_s^{m_s}$, P_i distinct
4) Show $P_1^{m_1} \cap \cdots \cap P_s^{m_s} = P_1^{m_1} \cdots P_s^{m_s} = I$
5) Use local rings to prove uniqueness

Step 4 follows from a basic ring theoretic ~~fact~~ remark:

Remark: If $I_1, \ldots, I_s \trianglelefteq R$ are pairwise comaximal then
$$I_1 \cap \cdots \cap I_s = I_1 \cdots I_s.$$

Proof: We have $I_1 \cdots I_s \subseteq I_1 \cap \cdots \cap I_s$. For insight, let's look at $s=2$.
$I_1 + I_2 = R$, $a \in I_1$, $b \in I_2$, $a+b=1$. If $x \in I_1 \cap I_2$ then

$$x = x\,1 = xa + xb \in I_1 I_2 + I_1 I_2 = I_1 I_2$$

In general, I_j, $\bigcap_{j \neq i} I_j$ are comaximal. We know $\forall j\ \exists a_j \in I_i$ & $b_j \in I_j$ st $a_j + b_j = 1 \Rightarrow I_1$ & $I_2 \ldots I_s$ are comaximal ???
So

$$1 = (a \quad ? \qquad = x + b_2 \quad b_n \in I_1 \cap \cdots \cap I_n \quad ???$$

For each j $\exists c_j \in I_j$ & $d_j \in \prod_{k \neq j} I_k$ st $c_j + d_j = 1$
So if $x \in I_1 \cap \cdots \cap I_n \Rightarrow$

$$x = x \cdot 1 = x(c_1 + d_1) \cdots (c_n + d_n) \in I_1 \cdots I_n \Rightarrow I_1 \cap \cdots \cap I_n = I_1 \cdots I_n. \quad \square$$

Proposition: Let A be a Noetherian integral domain of Krull dimension 1. Then the following are equivalent:
1) A is integrally closed ($\Leftrightarrow A$ is a Dedekind domain);
2) Every primary ideal in A is of the form P^m, $m \geq 1$, P prime;
3) If $(0) \neq P \trianglelefteq A$ is a prime ideal then A_P is a dvr.

Proof: We first show $(1) \Leftrightarrow (3)$.
Assume 1. If $P \trianglelefteq A$ is a prime ideal then A_P is integrally closed (This is part of a more general fact: If R is integrally closed in $K = \mathrm{Frac}(R)$ & if $S \subseteq R$ is mult. closed then $S^{-1}R$ is integrally closed in K.)
To see this, suppose that $x \in K$ is integrally ~~closed~~ over $S^{-1}R$.

$$\Rightarrow x^n + (r_{n-1} s_{n-1}^{-1}) x^{n-1} + \cdots + r_0 s_0^{-1} = 0$$

let $s = s_1 \cdots s_{n-1}$

$$\Rightarrow x^n + (r_{n-1}' s^{-1}) x^{n-1} + \cdots + r_0' s^{-1} = 0$$

$$\Rightarrow s^n x^n + r_{n-1}' s^{n-1} x^{n-1} + \cdots + r_0' s_{n-1} = 0$$

$$\Rightarrow (sx)^n + r_{n-1}' (sx)^{n-1} + \cdots + (r_1' s^{n-2})(sx)^1 + r_0' s^{n-1} = 0$$

So sx is integral over $R \Rightarrow sx \in R \Rightarrow x = rs^{-1} \in S^{-1}R$
So A integrally closed $\Rightarrow A_P$ integrally closed $\Rightarrow A_P$ is a dvr.
Now let's do $3 \Rightarrow 1$:
Let $C \subseteq K = \mathrm{Frac}(A)$ be the integral closure of A.
So $A \subseteq C \subseteq K$. Want $A = C$.

a bunch of equivalences
which are the same

Let $f: A \to C$ be inclusion. We show f is onto.
So suppose $\exists c \in C \setminus A$. Let P be a maximal ideal of A.
Then $A_P \subseteq S^{-1}C$ where $S = A \setminus P \subseteq C$.
Now A_P is integrally closed & observe that $S^{-1}C$ is integral over A_P.
$S^{-1}c \in S^{-1}C$??? $c^n + a_{n-1}c^{n-1} + \cdots + a_0$
$\Rightarrow s^{-n}c^n + a_{n-1}s^{-n}c^{n-1} + \cdots + a_0 s^{-n}$
$= (s^{-1}c)^n + (a_{n-1}s^{-1})(s^{-1}c)^{n-1} + \cdots + a_0 s^{-n} = 0$
$\Rightarrow s^{-1}c$ is integral over A_P
Since A_P is integrally closed & $S^{-1}C$ is integral over A_P
$\Rightarrow A_P = S^{-1}C$ $\forall P$ Suppose $\exists c \in C \setminus A$
$\Rightarrow c \in S^{-1}C$ so $c = ax^{-1}$ where $x \in A \setminus P$
So for each maximal ideal P
$\exists a_P \in A$ and $x_P \in A \setminus P$ st $c = a_P x_P^{-1}$ ie $x_P c = a_P \in A$
Let $J = \{ r \in A ; rc \in A \} \trianglelefteq A$.
Notice if $J \neq A$ then $J \subseteq P$ for some maximal ideal P.
But $x_P \in J$ & $x_P \notin P \Rightarrow J = A \Rightarrow 1 \in J \Rightarrow c \in A$.

Next we show $2 \Leftrightarrow 3$. 2015 04 c

Suppose first that 2 holds. Let $(0) \neq P$ be a prime ideal. (So P is maximal) Consider the local ring A_P. Let J be a non-zero ideal of A_P.
$$J \trianglelefteq A_P \longleftrightarrow J = (J \cap A) A_P \ ; \ I = J \cap A$$
What is the radical of $J \trianglelefteq A_P$? So $\sqrt{J} = PA_P$.
Since A_P is Noetherian, $\exists n$ st $(PA_P)^n \subseteq J$. Then $I \supseteq P^n$
So what is $\sqrt{I}$? $\sqrt{I} = P$.
Recall that if $\sqrt{I}$ is maximal then I is primary. So (by (2))
$\exists m$ st $I = P^m$. But $J = IA_P = P^m A_P = (PA_P)^m \Rightarrow A_P$ is a dvr.

Finally we show $(1) \Rightarrow (2)$.

So suppose (1) holds & so A is a Dedekind domain. Let $(0) \neq I$ be a primary ideal. Let $P = \sqrt{I}$, maximal. Then A_P is a dvr (by (3)). Thus $IA_P = (PA_P)^m = P^m A_P$ some $m \geq 1$.
Goal: show $I = P^m$. Then we're done.
We know we have a bijection between proper ideals of A & S-saturated ideals of A contained in P, where $S = A \setminus P$.
$$J \trianglelefteq A_P \longmapsto J \cap A \trianglelefteq A$$

playing tennis really
messes me up. I shouldn't
do exercise

Since $IA_P = P^m A_P$
$\implies \underbrace{IA_P \cap A}_{=I} = \underbrace{P^m A_P \cap A}_{=P^m}$
Claim:
We'll show that I & P^m are S-saturated. Suppose that $xs \in I$.
I is primary, $\therefore$ either $x \in I$ or ~~$s^n \in I \subseteq P$~~ (as $s \notin P$, P prime)
So I is S-saturated. Similarly P^m is S-saturated
So $IA_P = P^m A_P \implies I = P^m$ by the bijective ■

(2 pages back)

Proof (of theorem)
Let $(0) \neq I \subsetneq A$. Then I has a primary decomposition
$$I = Q \cap \cdots \cap Q_s.$$
By our last theorem, since A is a Dedekind domain, we have that every primary ideal is a prime power, so $\exists P_1, \_\_, P_s$ prime ideals (maximal) st $Q_i = P_i^{m_i}$. So
$$I = Q_1 \cap \cdots \cap Q_s = P_1^{m_1} \cap \cdots \cap P_s^{m_s}$$
WLOG the P_i are distinct, as $P^{m_i} \cap P^{m_j} = P^{\max(m_i, m_j)}$ so get smaller expression.
Now $P_1^{m_1}, \ldots, P_s^{m_s}$ are pairwise comaximal.
This is part of a more general remark:
If P & Q are comaximal then P^m & Q^m are comaximal
Why? P & Q are $\Rightarrow x + y = 1$, $x \in P$ $y \in Q$
Now $(x+y)^{m+n} = 1$

$$\underbrace{x^{m+n} + \binom{m+n}{1} x^{m+n-1} y + \cdots + \binom{m+n}{m} x^n y^m}_{\in P^n} + \underbrace{\cdots + y^{m+n}}_{\in Q^m}$$

Therefore $P_1^{m_1} \cap \cdots P_s^{m_s} = P_1^{m_1} \cdots P_s^{m_s}$.
So it remains to show uniqueness. To see this, suppose that
$I = P_1^{m_1} \cdots P_s^{m_s} = Q_1^{n_1} \cdots Q_t^{n_t}$, P_i distinct primes, Q_i "
Claim 1: $\{P_1, \_\_\} = \{Q_1, \_\_\}$
To see this, if $Q_j \notin \{P_1, \ldots, P_s\}$ then $P_1^{m_1} \cdots P_s^{m_s} = Q_1^{n_1} \cdots \subseteq Q_j$
So $\exists i$ st $P_i \subseteq Q_j$. But P_i max. so $P_i = Q_j$.
By symmetry, Claim 1 holds.

So now we can ~~assume~~ consider

$$I = P_1^{m_1} \cdots P_s^{m_s} = P_1^{n_1} \cdots P_s^{n_s}.$$

Now it suffices to show that $m_i = n_i$.

Let's look at ~~the~~ this in the local ring A_{P_i}

$$IA_{P_i} = (P_1^{m_1} \cdots P_s^{m_s})A_{P_i} = (P_1^{n_1} \cdots P_s^{n_s})A_{P_i}$$

$$\| \qquad\qquad \|$$

$$\underbrace{(P_1A_{P_i})}_{A_{P_i}}^{m_1} \cdots (P_iA_{P_i})^{m_i} \cdots \underbrace{(P_sA_{P_i})}_{A_{P_i}}^{m_s} \qquad \text{similar}$$

Hence $(P_iA_{P_i})^{m_i} = (P_iA_{P_i})^{n_i}$

Say $m_i < n_i$. Then

$$(P_iA_{P_i})^{m_i} = (\quad)^{m_i+1} = \cdots = (\quad)^{n_i}$$

$$= \underbrace{(\quad)}_{J}\underbrace{(\quad)}_{\text{fin gen..}}^{m_i}$$

by NAKAYAMA!!!!!!!

$$(P_iA_{P_i})^{m_i} = 0$$

$\#$ since $P_i \neq 0$ and A is int dom ∎