2015 03 06 (

So $A[\frac{1}{b}] \leftarrow$ GK dim $\leq d$

$\quad\ \ \cup\!|$

$k[c_1, \ldots, c_{d+1}] \cong k[x_1, \ldots, x_{d+1}] \leftarrow$ GKdim $= d+1$

But the following fact gives us ~~that~~ our contradiction.

Fact: If $R \subseteq S$, R, S f.g. Then GKdim $R \leq$ GKdim S. } Why? Pick V gen space for R. add a set of gens for S to obtain a gen space $W \supseteq V$ for S. Then $\dim V^n \leq \dim W^n \ \forall n$.

2015 03 10

Given a ring R & $P \in \mathrm{Spec}(R)$, we define the <u>height</u> of P

$$ht(P) = \mathrm{Kdim}(R_P)$$
$$= \sup\{n \,;\, \exists Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n \,;\, Q_n \subseteq P,\ Q_i \in \mathrm{Spec}(R)\}$$

Spec & M-Spec

Given a ring R,

$$\mathrm{Spec}(R) = \{P \,;\, P \text{ a prime ideal of } R\},$$
$$\text{M-Spec}(R) = \{M \,;\, M \text{ a maximal ideal of } R\}.$$

We'll put a topology on Spec(R). Then M-Spec will inherit the subspace topology.

Zariski topology

Given an ideal $I \trianglelefteq R$. We define

$$C_I = \{P \in \mathrm{Spec}(R) \,;\, P \supseteq I\}.$$

These will be our closed sets. Let's see that this is a topology.

$\quad \emptyset = C_R, \quad \mathrm{Spec}(R) = C_{(0)}$.

If $I_1, \ldots, I_n \trianglelefteq R$,

$$C_{I_1} \cup \cdots \cup C_{I_n} = C_{I_1 \cap \cdots \cap I_n}$$

Let's check it!

$P \in C_{I_1} \cup \cdots \cup C_{I_n} \iff P \supseteq I_1$ or $\cdots$ or $P \supseteq I_n$

$\qquad\qquad \iff P \supseteq I_1 \cap \cdots \cap I_n$

Clearly $\Rightarrow$. $\Leftarrow$: if not, $\exists a_i \in I_i \setminus P$, so $a_1 \cdots a_n \in \bigcap I_j \subseteq P \Rightarrow \exists j\ a_j \in P$.

Notice an arbitrary union of C_I's need not be closed

$\underline{\text{Ex}}$: $R = \mathbb{C}[x]$, Kdim $= 1$

$$\begin{array}{ccc} \mathrm{Spec}(R) & (x-\lambda),\ \lambda\in\mathbb{C} & \text{M-Spec}(R) \quad (x-\lambda),\ \lambda\in\mathbb{C} \\ & | & \\ & (0) & \end{array}$$

$(0) \neq I = (p(x))$ $\quad C_I = \{(x-\lambda) ; (x-\lambda) \supseteq (p(x))\} = \{\lambda \in \mathbb{C} ; p(\lambda) = 0\}$ finite set

So in this case,

$$C_I = \begin{cases} \mathrm{Spec}(R) & \text{if } I = (0) \\ \text{finite set} & \text{if } I \neq (0) \end{cases}$$

$\hookrightarrow \{(x-\lambda) ; p(\lambda) = 0\}$ if $I = (p(x))$, $p(x) \neq 0$.

Notice that

$$\bigcup_{n\in\mathbb{Z}} C_{(x-n)}$$

is not closed (countably infinite, Spec is infinite)

Finally, we must show

$$\bigcap_\alpha C_{I_\alpha} \text{ is } C_{\sum I_\alpha} \leftarrow \text{finite sums from } I_\alpha\text{'s}$$

Proof: $P \in \bigcap_\alpha C_{I_\alpha} \iff P \in C_{I_\alpha}\ \forall \alpha \iff P \supseteq I_\alpha\ \forall\alpha$
$\iff P \supseteq \sum I_\alpha$ $\square$

Remark: We need only consider C_I with $I = \sqrt{I} = \{x \in R ; x^n \in I \text{ some } n \geq 1\}$.
Why? $I \subseteq \sqrt{I}$ So $P \supseteq \sqrt{I} \Rightarrow P \supseteq I$. But $P \supseteq I \Rightarrow P \supseteq \bigcap_{Q \supseteq I \text{ prime}} Q = \sqrt{I}$ as P prime
So $P \supseteq I \iff P \supseteq \sqrt{I}$. So $C_I = C_{\sqrt{I}}$.

Remark: If $I \supseteq J$ then $C_I \subseteq C_J$.

$\underline{\text{Ex}}$. What is $\mathrm{Spec}(\mathbb{Q})$? $\{(0)\}$
What is $\mathrm{Spec}(\mathbb{Z})$? $\quad (2\mathbb{Z})\ (3\mathbb{Z})\ (5\mathbb{Z})\ \cdots$ $\leftarrow$ closed pt

(0) $\leftarrow$ dense pt

Let $P \in \mathrm{Spec}(R)$. What is $\overline{\{P\}}$? $\quad C_P = \{Q \in \mathrm{Spec}(R) ; Q \supseteq P\}$
When is $\overline{\{P\}} = \{P\}$? $\iff P$ maximal
When is $\{P\}$ dense? $\iff \overline{\{P\}} = \underset{\mathrm{Spec}(R)}{\cancel{R}} \iff P = \sqrt{(0)}$

2015 03 10

Closed sets in $\mathrm{Spec}(\mathbb{Z})$: $\mathrm{Spec}(\mathbb{Z})$, $\{p_1\mathbb{Z}, \dots, p_d\mathbb{Z}\}$ $d \geq 0$.

Notice if $I \trianglelefteq \mathbb{Z}$ then either

$I = (0) \Rightarrow C_I = \mathrm{Spec}(\mathbb{Z})$

or $\underset{n\geq 2}{I = n\mathbb{Z}} \Rightarrow C_I = \{p\mathbb{Z};\ p\mathbb{Z} \supseteq n\mathbb{Z}\} = \{p\mathbb{Z};\ p \mid n\}$ finite set of primes

$I = \mathbb{Z} \Rightarrow C_I = \emptyset$

What is

$$\mathrm{Spec}(\mathbb{Z}_{(2)}) = \mathrm{Spec}(\{a/b;\ a,b \in \mathbb{Z},\ b \text{ odd}\})?$$

$$\mathrm{Spec}(\mathbb{Z}_{(2)}) \xleftrightarrow{\text{bijection}} \{P \in \mathrm{Spec}(\mathbb{Z});\ P \subseteq 2\mathbb{Z}\}$$

$(2\mathbb{Z}_{(2)})$ closed

|

(0) dense

$\mathrm{Spec}(\mathbb{Z})$ $\qquad\qquad$ $\mathrm{Spec}(\mathbb{Z}_{(2)})$

$2\mathbb{Z}$ $3\mathbb{Z}$ $5\mathbb{Z}$ $\cdot$ $\qquad\qquad$ $2\mathbb{Z}_{(2)}$

(0) $\qquad\qquad$ (0)

Theorem: Let $P \in \mathrm{Spec}(R)$, let

$$X = \{Q \in \mathrm{Spec}(R);\ Q \subseteq P\}$$

with the subspace topology. We showed

$$X \xleftrightarrow{\text{1-1 bij}} \mathrm{Spec}(R_P)$$
$$Q \xmapsto{f} QR_P$$
$$J \cap R \xleftarrow{g} J$$

Then f & g are continuous bijections and so X is homeomorphic to $\mathrm{Spec}(R_P)$.

Proof: Let C be a closed subset of $\mathrm{Spec}(R_P)$. Then $C = C_I$, $I \trianglelefteq R_P$.

$$C = \{Q \in \mathrm{Spec}(R_P);\ Q \supseteq I\}.$$

$$f^{-1}(C) = \{L \in \mathrm{Spec}(R);\ L \subseteq P \ \&\ f(L) = LR_P \supseteq I\}$$

$LR_P \supseteq I \Leftrightarrow L \supseteq I \cap R$ think about this $(L = LR_P \cap R)$ $\quad = \{L \in \mathrm{Spec}(R);\ L \supseteq I \cap R\} \cap X$

— closed in X

Conversely, if $C \subseteq X$ closed

$C = X \cap \{L \in \mathrm{Spec}(R);\ L \supseteq I\}$ for some radical ideal I.

I think it does
make sense but
that depends on
one's definition of sense.

So $g^{-1}(C) \subseteq \operatorname{Spec}(R_P)$

$$= \{LR_P \,;\, L \supseteq I,\ L \subseteq P\} = \{LR_P \,;\, L \supseteq I\} = \{LR_P \,;\, LR_P \supseteq IR_P\} = C_{IR_P}$$

2015 03 12

Remarks

1) C_I with the subspace topology on $\operatorname{Spec}(R)$ is homeomorphic to $\operatorname{Spec}(R/I)$ (correspondence)

Proof Define $f : C_I \to \operatorname{Spec}(R/I)$

$f(P) = \overline{P}$ where $\overline{J}$ is the image of J under the natural map $R \to R/I$.

f is a bijection; $C_I = \{P \,;\, P \supseteq I\} \longleftrightarrow \operatorname{Spec}(R/I)$

$$P \longleftrightarrow \overline{P}$$

To see that f is a homeomorphism, notice that if J is an ideal of R/J Then $\exists$ an ideal $L \supseteq I$ of R st $\overline{L} = J$ (correspondence)

$$f^{-1}(C_J) = f^{-1}(\{Q \in \operatorname{Spec}(R/I) \,;\, Q \supseteq J\})$$
$$= \{P \in \operatorname{Spec}(R) \,;\, P \supseteq I \ \&\ P \supseteq L\} = C_L \subseteq C_I \quad \text{closed}$$

so f is continuous.

Conversely, if C is a closed subset of $C_I \Rightarrow C = C_L$ for some $L \supseteq I$

$$\Rightarrow f(C_L) = \{Q \in \operatorname{Spec}(R/I) \,;\, Q \supseteq \overline{L}\} = C_{\overline{L}}.$$

So f^{-1} is continuous Thus f is a homeomorphism. ∎

This example fits more generally into the following framework:

A4: If R, S are rings & $\varphi^* : R \to S$ is a homeomorphism. Then we get a map $\varphi : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$

$$\varphi(P) \longmapsto (\varphi^*)^{-1}(P) =: Q$$

In this setting, if $\varphi^* : R \to R/I$ is $r \mapsto r + I$ Gives a cts map $\varphi : \operatorname{Spec}(R/I) \to \operatorname{Spec}(R)$ image of $\varphi = C_I$. This $\varphi = f^{-1}$.

<u>Open sets</u>

Given $f \in R$, we define

$$U(f) = \operatorname{Spec}(R) \setminus C_{(f)} = \{P \,;\, P \in \operatorname{Spec}(R)\} \setminus \{P \,;\, \underset{f \in P}{P \supseteq (f)}\}$$

$$= \{P \in \operatorname{Spec}(R) \,;\, f \notin P\} \cong \operatorname{Spec}(R_f)$$

The rest of the proof is all downhill ↳ in every sense of the word except literally

2015 03 12 (2)

So $U(f)$ is open and it is called a <u>principal open set</u>. The sets $U(f)$ form a basis for the Zariski topology. Notice that if U is an open set $U = \mathrm{Spec}(R) \setminus C_I$. If $f \in I$: $U(f) \subseteq U$. Why?

$U(f) \subseteq U \iff \mathrm{Spec}(R) \setminus C_{(f)} \subseteq \mathrm{Spec}(R) \setminus C_I \iff C_I \subseteq C_{(f)} \iff \sqrt{(f)} \subseteq \sqrt{I}$ $\because f \in I$

Exercise: Show

$$\mathrm{Spec}(R) \setminus C_I = \bigcup_{f \in I} U(f).$$

<u>Connectedness of Spec(R)</u>

Recall that X is disconnected if $X = C_1 \dot\cup C_2$, C_1, C_2 closed, non-empty

Theorem: Let R be a ring. Then the following are equivalent:
1) $\mathrm{Spec}(R)$ is disconnected;
2) R has an idempotent $e \neq 0, 1$ $(e^2 = e)$;
3) $R \cong R_1 \times R_2$, R_1, R_2 non-trivial

Proof:
(1)⇒(2): Suppose that $\mathrm{Spec}(R)$ is disconnected. Then $\mathrm{Spec}(R) = C_I \dot\cup C_J$. $C_I \cap C_J = \emptyset \iff C_{I+J} = \emptyset$ (as they are the same) $\iff I+J = R$

$C_I \cup C_J = \mathrm{Spec}(R) \iff \underset{\substack{\parallel \\ C_{IJ}}}{C_{I \cap J}} = \mathrm{Spec}(R) \iff$ every prime $P \supseteq IJ$

$\iff \bigcap P = \sqrt{0} \supseteq IJ$ (2)

Since $I+J = R$, $\exists x \in I$ & $y \in J$ s.t. $x+y = 1$. & $xy \in IJ \subseteq \sqrt{0} \Rightarrow$ $\exists n \geq 1$ st $(xy)^n = 0$. Now

$$1 = (x+y)^{2n} = \underbrace{x^{2n} + \binom{2n}{1} x^{2n-1} y + \cdots + \binom{2n}{n} x^n y^n}_{e} + \underbrace{\binom{2n}{n+1} x^{n-1} y^{n+1} + \cdots + \binom{2n}{2n-1} x y^{2n-1} + y^{2n}}_{1-e}$$

Note $e \in x^n R$, $1-e \in y^n R$, so $e(1-e) \in (xy)^n R = (0)$ i.e. $e^2 = e$. Notice $e \in x^n R \subseteq I \subset R$ so $e \neq 1$. Similarly $1-e \in J \subset R$ so $e \neq 0$

(2)⇒(3) Let $R_1 = Re \neq (0)$, $R_2 = (1-e)R \neq (0)$.

Fact: If f is an idempotent in R then Rf is a ring with unit f.

$1f \in Rf \Rightarrow (1f)f = 1f^2 = 1f$ & $f(1f) = 1f^2 = 1f$

$(1f)(sf) = 1sf^2 = 1sf$, $1f + sf = (1+s)f$.

If we define

$$\varphi : R \to R_1 \times R_2$$
$$\varphi(r) = (re, r(1-e))$$

φ is a homeomorphism:

$$\varphi(r_1 + r_2) = \quad = \varphi(r_1) + \varphi(r_2)$$
$$\varphi(r_1 r_2) = \quad = \varphi(r_1)\varphi(r_2)$$

$\varphi(r) = (0,0) \iff re = 0$ & $r(1-e) = 0 \Rightarrow re + r(1-e) = 0 \Rightarrow r = 0$

So injective. To see φ is onto, given $ae \in Re$ & $b(1-e) \in R(1-e)$, we have $\varphi(ae + b(1-e)) = (ae, b(1-e))$. So $R \cong R_1 \times R_2$ as claimed.

(3)$\Rightarrow$(1) Suppose that $R = R_1 \times R_2$, ~~as claimed~~ non-zero rings.

Notice $I = R_1 \times \{0\}$ is an ideal, $J = \{0\} \times R_2$ is an ideal.

& $IJ = (0,0)$, $I + J = R \Rightarrow C_I \cup C_J = \mathrm{Spec}(R)$.

proper

<u>Def'l</u> A topological space X is <u>reducible</u> if $X = C_1 \cup C_2$, C_1 & C_2 are proper closed sets. If X is not reducible, X is <u>irreducible</u>.

Notice: X disconnected $\Rightarrow$ X reducible

2015 03 1

Can we find a ring R such that $\mathrm{Spec}(R)$ is connected but reducible?

ex $R = \mathbb{C}[x,y]/(xy)$. Let $\bar{x}, \bar{y}$ denote the images of x, y in R.

$\mathrm{Spec}(R)$ is reducible:

$$C_{(\bar{x})} \cup C_{(\bar{y})} = \mathrm{Spec}(R)$$

$\bar{x}\bar{y} = 0$ so if $P \in \mathrm{Spec}(R)$, $\bar{x} \in P$ or $\bar{y} \in P$.

Why is it connected?

$$R = \{c + \bar{x}p(\bar{x}) + \bar{y}p(\bar{y});\ c \in \mathbb{C},\ p(t), q(t) \in \mathbb{C}[t]\}$$

We showed that $\mathrm{Spec}(R)$ is disconnected $\iff \exists e^2 = e \neq 0, 1$.

If $e = c + \bar{x}p(\bar{x}) + \bar{y}q(\bar{y})$ is an idempotent then

$$e^2 = e \Rightarrow \deg(\bar{x}p(\bar{x})) = \deg(\bar{y}q(\bar{y})) = 0 \quad \text{so } e = c \in \mathbb{C} \Rightarrow e \in \{0,1\}$$

What is $M\text{-}\mathrm{Spec}(R)$? $\{(x-\alpha, y-\beta);\ (\alpha,\beta) \in \mathbb{C}^2\} = M\text{-}\mathrm{Spec}(\mathbb{C}[x,y])$

so $M\text{-}\mathrm{Spec}(R) = \{(x-\alpha, y-\beta) \supseteq (xy)\}$

$$= \{(x, y-\beta);\ \beta \in \mathbb{C}\} \cup \{(x-\alpha, y);\ \alpha \in \mathbb{C}\}$$

2015 03 1?

Theorem: Spec(R) is irreducible $\iff N = \sqrt{(0)}$ is a prime ideal
$\iff R/N$ is an integral domain.

why?

Proof: $\mathrm{Spec}(R) \cong \mathrm{Spec}(R/N)$, so we may assume WLOG that R is reduced, ie $\sqrt{(0)} = (0)$.
Now if R is not an integral domain, $\exists a, b \in R \setminus \{0\}$ st $ab = 0$.
Then $\mathrm{Spec}(R) = \underset{(\text{proper})}{C_{(a)} \cup C_{(b)}} = C_{(a)\cap(b)} = C_{(ab)} = C_{(0)} = \mathrm{Spec}(R)$

So R not an integral domain $\Rightarrow$ Spec(R) is reducible. If Spec(R) is reducible $\Rightarrow \mathrm{Spec}(R) = C_I \cup C_J = C_{IJ}$. So $IJ = (0)$, & $I, J \neq (0)$. Pick $a \in I \setminus \{0\}$ (proper), $b \in J \setminus \{0\} \Rightarrow ab = 0$. $\rightarrow$ not int dom. ∎

Corollary: $C_I \subseteq \mathrm{Spec}(R)$ is irreducible $\iff \sqrt{I}$ is a prime ideal.

Proof: $C_I \cong \mathrm{Spec}(R/I)$
$C_{\sqrt{I}} \cong \mathrm{Spec}(R/\sqrt{I})$, $S = R/\sqrt{I}$ is reduced
So C_I is irreducible $\iff S$ is an integral domain $\iff \sqrt{I}$ is a prime ideal. ∎

Theorem: Let R be a ring. Then Spec(R) is quasi-compact ie if
$$\mathrm{Spec}(R) = \bigcup_\alpha U_\alpha,$$
U_α open $\Rightarrow \exists$ a finite subset $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ st
$$\mathrm{Spec}(R) = \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Proof: Suppose $U_\alpha = \mathrm{Spec}(R) \setminus C_{I_\alpha}$ for some $I_\alpha \trianglelefteq R$. Then

(*) — $$\bigcup_\alpha U_\alpha = \mathrm{Spec}(R) \iff \bigcap_\alpha C_{I_\alpha} = \emptyset \iff C_{\sum_\alpha I_\alpha} = \emptyset \iff \sum_\alpha I_\alpha = R.$$

So $\exists$ an expression $i_{\alpha_1} + \dots + i_{\alpha_s} = 1$, $i_{\alpha_i} \in I_{\alpha_i}$. So $\sum_{i=1}^{s} I_{\alpha_i} = R$.
Going reverse in (*) shows $\bigcup_{i=1}^{s} U_{\alpha_i} = \mathrm{Spec}(R)$. ∎

Let R be (a) Noetherian (ring). When is Spec(R) Hausdorff?
Spec(R) Hausdorff $\Rightarrow \{P\}$ is closed $\forall P \in \mathrm{Spec}(R)$
$\Rightarrow P$ is maximal "
$\Rightarrow R$ has Krull dimension 0.

In fact, we have:

Theorem: Let R be Noetherian. Then the following are equivalent:
1) $\mathrm{Kdim}(R)=0$;
2) $\mathrm{Spec}(R)$ is Hausdorff;
3) $\mathrm{Spec}(R)$ is finite & discrete;
4) $R/N \cong F_1 \times \cdots \times F_s$, $N=\sqrt{(0)}$, $s \geq 1$, F_i fields.

Proof: Earlier we showed $(1) \Leftrightarrow (4)$ and we showed $(2) \Rightarrow (1)$. Clearly $(3) \Rightarrow (1)$. So it suffices to show $(4) \Rightarrow (3)$. But if (4),

$$\mathrm{Spec}(R) \cong \mathrm{Spec}(R/N) \cong \mathrm{Spec}(F_1 \times \cdots \times F_s)$$

$F_1 \times \cdots \times F_s$ is Noetherian & Kdim 0, so it has only finitely many prime ideals, all of which are maximal. So $|\mathrm{Spec}(R)| < \infty$ & pts are closed $\Rightarrow$ discrete. ∎

Noetherian Topological subspaces

Let X be a topological space. We say that X is Noetherian if whenever $C_1 \supseteq C_2 \supseteq \cdots$ is a descending chain of closed subsets of X, $\exists n$ st $C_n = C_{n+1} = C_{n+2} = \cdots$.
If X is a topological space, we can define the Krull dimension of X to be

$$\mathrm{Kdim}(X) = \sup_n \{\exists \text{ a chain } C_0 \supsetneq \cdots \supsetneq C_n \,;\, C_i \text{ irreducible closed sets}\}.$$

Ex: If R is a ring then $\mathrm{Kdim}(R) = \mathrm{Kdim}(\mathrm{Spec}(R))$.

Artinian Rings

A ring R is Artinian if every descending chain of ideals terminates; ie if $I_1 \supseteq I_2 \supseteq \cdots$ then $\exists n$ st $I_n = I_{n+1} = \cdots$.
An R-module M is Artinian if it satisfies the descending chain condition on submodules.

In particular, R is Artinian $\Leftrightarrow$ R is Artinian as an R-module

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Just as with the Noetherian case, the same proof shows that R is Artinian if and only if every non-empty subset of ideals has a minimal element with respect to $\supseteq$.

Theorem: Let R be a ring. Then R is Artinian $\Leftrightarrow$ R is Noetherian and has Kdim 0 $\Leftrightarrow$ R is Noetherian, $R/N \cong F_1 \times \cdots \times F_s$, $N = \sqrt{(0)}$.

2015 03 17

Lemma: If R is a ring & $0 \to M_1 \to M \xrightarrow{\pi} M_2 \to 0$ is a short exact sequence of R-modules then M is Artinian $\Leftrightarrow$ M_1, M_2 are Artinian.

Proof ($\Rightarrow$) $M_1 \subseteq M \Rightarrow M_1$ is Artinian
& M_2 is Artinian because $M_2 \cong M/M_1$.
($\Leftarrow$): Suppose that M_1 & M_2 are Artinian & let
$N_1 \supseteq N_2 \supseteq \cdots$ be a descending chain of submodules in M.
Then (1) $N_1 \cap M_1 \supseteq N_2 \cap M_1 \supseteq \cdots$ is a " in M_1
so $\exists i$ st $N_i \cap M_1 = N_{i+1} \cap M_1 = \cdots$
(2) $\pi(N_1) \supseteq \pi(N_2) \supseteq \cdots$ " in M_2
so $\exists j$ st $\pi(N_j) = \pi(N_{j+1}) = \cdots$
Let $n \geq \max(i,j)$. Claim $N_n = N_{n+1}$.
Well $\pi(N_n) = \pi(N_{n+1})$. Suppose $x \in N_n$. Then $\pi(x) \in \pi(N_n) = \pi(N_{n+1})$
so $\exists y \in N_{n+1}$ st $\pi(x) = \pi(y) \Rightarrow x - y \in M_1$. So
$$x - y \in N_n \cap M_1 = N_{n+1} \cap M_1$$
so $x - y \in N_{n+1} \Rightarrow x \in y + N_{n+1} \subseteq N_{n+1}$. So $N_n = N_{n+1}$. ☒

Proposition: If R is a ring in which $(0) = M_1 \cdots M_s$, $M_1, \ldots, M_s$ maximal ideals of R. Then R is Artinian if and only if R is Noetherian.

Proof: Let $A_1 = M_1$, $A_2 = M_1 M_2, \ldots, A_s = M_1 \cdots M_s = (0)$. Also $A_0 = R$
Suppose R is Noetherian but not Artinian. Then A_0 is not Artinian as an R-module. But $A_s = (0)$ is Artinian as an R-module. So there exists some largest $i < s$ such that A_i is not Artinian.
$$0 \to \underset{\text{Artinian}}{A_{i+1}} \to A_i \to \underset{\text{Goal: Artinian}}{A_i/A_{i+1}} \to 0$$

(*) So it suffices to show that $A_i/A_{i+1} = M_1 \cdots M_i / M_1 \cdots M_{i+1}$ is Artinian. Notice M_{i+1} annihilates A_i/A_{i+1} so A_i/A_{i+1} inherits the structure of an $R/M_{i+1} = F$-module (ie v.s.) R is Noetherian $\Rightarrow A_i$ is Noetherian $\Rightarrow A_i/A_{i+1}$ is Noetherian as an R-module $\Rightarrow M_i/M_{i+1}$ is Noetherian as an F-module

(**) Fact: If F is a field & V is an F-vs $\Rightarrow V$ is Noetherian F-module $\Leftrightarrow \dim_F V < \infty$.

So $\dim_F A_i/A_{i+1} < \infty$.

(***) Fact: If F is a field & V is an F-v.s. $\Rightarrow V$ is an Artinian F-module $\Leftrightarrow \dim_F V < \infty$.

(****) So A_i/A_{i+1} is an Artinian F-module $\Rightarrow A_i/A_{i+1}$ is Artinian as an R-module $\because M_{i+1}$ annihilates it.

Conversely, swap (**) and (***) and ~~replace~~ swap Artinian and Noetherian in (*) and (****). ∎

Theorem: R is Artinian $\Leftrightarrow R$ is Noetherian & $\operatorname{Kdim}(R) = 0$.

Proof: ($\Leftarrow$) If R is Noetherian & Kdim 0. By Noether's result, $\exists$ prime ideals $P_1, \dots, P_s$ st $P_1 \cdots P_s = (0)$. $\operatorname{Kdim} R = 0 \Rightarrow$ each P_i is maximal, so (0) is a product of maximal ideals & R is Noetherian $\Rightarrow R$ is Artinian.

($\Rightarrow$): To do the other direction, we'll prove a few claims.

<u>Claim</u>: Let R be Artinian and let $P \in \operatorname{Spec}(R)$. Then P is maximal.

In particular, $\operatorname{Kdim} R = 0$.

Verification: Let $S = R/P$. By correspondence, S is Artinian & S is an integral domain. We'll show that S is a field. Let $x \in S \setminus \{0\}$. Consider the chain

$$xS \supseteq x^2S \supseteq x^3S \supseteq \cdots$$

As S is Artinian, $\exists n$ st $x^nS = x^{n+1}S$. So $x^n \in x^{n+1}S$, so $\exists y \in S$ st $x^n = x^{n+1}y$. $\Rightarrow 1 = xy$. □

<u>Claim</u>: Let R be Artinian. Then $\operatorname{Spec}(R)$ is finite.

Verification: Suppose we have distinct prime ideals $P_1, \dots$ By earlier claim, they are all maximal. Consider the chain

$$P_1 \supseteq P_1 \cap P_2 \supseteq P_1 \cap P_2 \cap P_3 \supseteq \cdots$$

This terminates so $\exists n$ st $P_1 \cap \cdots \cap P_n = P_1 \cap \cdots \cap P_{n+1}$

$P_1 \cdots P_n \subseteq P_1 \cap \cdots \cap P_n$

2015 03 19

Claim 3. If R is Artinian then $J(R)$ is nilpotent.

Assuming the claims:

Claim 1 gives $\mathrm{Kdim}(R) = 0$. Claim 2 says $\mathrm{Spec}(R) = \{P_1, \ldots, P_k\}$, $k \geq 1$.

Moreover, by Claim 1, $P_1, \ldots, P_k$ are maximal so

$$J(R) = \bigcap_{i=1}^{k} P_i \supseteq P_1 \cdots P_k$$

Claim 3 says $\exists m \geq 1$ s.t. $J(R)^m = (0)$

$$\Rightarrow (0) = J(R)^m \supseteq (P_1 \cdots P_k)^n \quad \leftarrow \text{finite product of maximal ideals}$$

So (0) is a product of maximal ideals, and so because R is Artinian, key lemma gives R is Noetherian.

Verification of claim 2 (cont.): so $\underset{P_1 \cdots P_n}{\overset{\cup\!\shortmid}{P_1 \cap \cdots \cap P_n}} \subseteq P_{n+1}$

Because $P_1, \ldots, P_{n+1}$ are distinct and maximal (claim 1) $\exists\, a_i \in P_i \setminus P_{n+1}$.

So $a_1 \cdots a_n \in P_1 \cdots P_n \subseteq P_{n+1}$. $\square$

Verification of claim 3: Let $J = J(R)$. Consider the chain

$$J \supseteq J^2 \supseteq J^3 \supseteq \cdots$$

Since R is ~~Noetherian~~ Artinian, $\exists\, n$ st $J^n = J^{n+1} = \cdots$. So $J^n = J^{2n}$.

~~Trick~~. If $J^n = (0)$ we're done, so suppose $J^n \neq (0)$.

Trick: Let $\mathcal{S} = \{ I \trianglelefteq R \,;\, I \subseteq J^n,\ I J^n \neq (0)\}$. Note $\mathcal{S} \neq \emptyset$ as $J^n \in \mathcal{S}$. As R is Artinian, $\exists$ a maximal element $L \in \mathcal{S}$. So $LJ^n \neq (0)$ but if $L' \subset L$ then $L'J^n = (0)$. So $\exists\, x \in L$ st $xJ^n \neq (0)$. We claim that $L = Rx$. $Rx \subseteq L$ as $x \in L$. $J^n Rx \neq (0)$ as $xJ^n \neq (0)$. By minimality, $L = Rx$.

Now $J^n L \neq (0)$ & $J^n J^n L = J^n L \neq (0)$ so $J^n L = L$ by minimality. $J^n L \subseteq JL = L$. by Nakayama's lemma $L = (0)$. $\#$ $\blacksquare$

Corollary: If R is Artinian & $J(R) = (0)$ then $R \cong F_1 \times \cdots \times F_s$, $s \geq 1$, F fields.

Proof. $(0) = J(R) = P_1 \cap \cdots \cap P_k$, P_i maximal, so P_i are p.w. comaximal. By Chinese Remainder theorem,

$$R = R \Big/ \bigcap_{i=1}^{k} P_i \cong \prod_{i=1}^{k} R/P_i .$$

$\blacksquare$

If R is a non-commutative ring, then we say R is <u>Artinian</u> if every descending chain of <u>left</u> ideals $L_1 \supseteq L_2 \supseteq \cdots$ terminates. We can define

$$J(R) = \bigcap_{M \text{ max/left ideal}} M.$$

Artin-Wedderburn theorem: If R is an Artinian ring with $J(R) = (0)$ then

$$R = \prod_{i=1}^{k} M_{n_i}(D_i), \quad D_i \text{ division ring.}$$

Primary Decomposition

Motivation: $P_1 \cdots P_s$ maximal over $I \trianglelefteq R$, so $\sqrt{I} = P_1 \cap \cdots \cap P_s$.

In particular, if $I = \sqrt{I}$ then I is a finite intersection of prime ideals.

<u>Def</u> Let $I \trianglelefteq R$. We say that I is <u>primary</u> if whenever $xy \in I$ we have either $x \in I$ or $\exists n \geq 1$ st $y^n \in I$.

ex Let $R = \mathbb{Z}$.

is $6\mathbb{Z}$ primary? no, $2 \cdot 3 \in 6\mathbb{Z}$, $2 \notin 6\mathbb{Z}$, $3^n \notin 6\mathbb{Z}$ $\forall n$

is $8\mathbb{Z}$ primary? yes, if $xy \in 8\mathbb{Z}$, either $8|x$ or $(2|y \Rightarrow 8|y^3)$

is $3\mathbb{Z}$ primary? yes, $xy \in 3\mathbb{Z} \Rightarrow x \in 3\mathbb{Z}$ or $y^1 \in 3\mathbb{Z}$

Alternatively, I is primary if whenever $xy \in I$, at least one of the following outcomes holds:

1) $x \in I$
2) $y \in I$
3) $\exists n$ st $x^n, y^n \in I$

Remark: Let $n \geq 1$. Then $n\mathbb{Z}$ is primary $\Leftrightarrow$ $n = p^k$ for some prime p, $k \geq 1$.

Pf If $n \neq p^k$, then $n = ab$, $a > 1, b > 1$, $\gcd(a,b) = 1$. So $n \nmid a^k$, $n \nmid b^k$ $\forall k$.
Conversely, if $n = p^k$, $xy \in p^k\mathbb{Z} \Rightarrow p^k | x$ or $p^k | y$ or $(p|x$ or $p|y)$
$\Rightarrow p^k | x^k$ or $p^k | y^k$

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Proposition: Let Q be primary. Then $\sqrt{Q}$ is a prime ideal.

Proof: Suppose that $\sqrt{Q}$ is not prime. Then $\exists x, y$ with $x, y \notin \sqrt{Q}$ but $xy \in \sqrt{Q}$. So $\exists n \geq 1$ with $x^n y^n = (xy)^n \in Q$. Now no power of x can be in Q. Similarly for y $\#$ ◻

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Note: In general, $\sqrt{P}$ prime $\not\Rightarrow$ P primary

Example: Let $R = \mathbb{C}[x,y,z]/\langle xy - z^2\rangle$ and let $P = \langle \bar{x}, \bar{z}\rangle \subsetneq R$. Notice

$$R/P = (\mathbb{C}[x,y,z]/\langle xy - z^2\rangle)/\langle \bar{x}, \bar{z}\rangle \cong (\mathbb{C}[x,y,z]/\langle x,z\rangle)/\langle xy - z^2\rangle \cong \mathbb{C}[y].$$

So R/P is an integral domain and hence P is prime. Let $Q = P^2$. Then $\sqrt{Q} = P$ is prime, but we claim that Q is not primary.

Note $\bar{x}\bar{y} = \bar{z}^2 \in P^2$. If Q is primary, either $\bar{x} \in Q$ or $\bar{y}^n \in Q$ for some n.

If $\bar{y}^n \in Q$ then $\bar{y}^n \in P$. But $R/P \cong \mathbb{C}[t]$, $\bar{y}^n \mapsto t^n \neq 0$.

Also $\bar{x} \notin P$ since things in P have degree at least two.

Remark: There is a partial converse.

Proposition: Let Q be an ideal of R and suppose $P := \sqrt{Q}$ is maximal. Then Q is primary.

Proof: Let $S = R/Q$. Then S is a local ring with unique maximal ideal $M = P/Q$.

Since S is ideal (?), if $x \in S \setminus M$ then x is a unit in S.

Our goal is to show Q is primary. So suppose $xy \in Q$. Then $\bar{x}\bar{y} = 0$ in S.

If $\bar{x} \notin M$, then $\bar{x}$ is a unit, so $\bar{y} = 0$ in S, so $y \in Q$.

If $\bar{y} \notin M$, $\bar{x} \in Q$ similarly.

If $\bar{x}, \bar{y} \in M$, then since $P = \sqrt{Q}$, $\exists n$ st $\bar{x}^n = \bar{y}^n = 0 \Rightarrow x^n, y^n \in Q$. ◻

Def: Let $I \subsetneq R$ be an ideal. We say

- I is reducible if $I = J \cap K$ for some ideals J, K with $J \supsetneq I$, $K \supsetneq I$;
- I is irreducible if whenever $I = J \cap K$, $J, K \supseteq I \Rightarrow J = I$ or $K = I$.

Proposition: Let R be Noetherian. Then every proper ideal is a finite intersection of irreducible ideals.

[illegible] example

Proof: Suppose not. Then $S = \{ I \subsetneq R ; I \text{ is not a finite intersection of irreducible ideals}\}$ is non-empty.
Since R is Noetherian, there exists a maximal element $J \in S$. Note J is reducible.
So there are $L, K \supsetneq J$ such that $J = I \cap K$. Now $L, K \notin S$ by maximality of $J \in S$, so

$$I = L_1 \cap \cdots \cap L_s, \quad K = N_1 \cap \cdots \cap N_t$$

with L_i, N_j irreducible.
But now

$$J = L_1 \cap \cdots \cap L_s \cap N_1 \cap \cdots \cap N_t.$$

Contradicting ~~Ea~~ $J \in S$. ■

Theorem: Let R be a Noetherian ring. Then every $I \subsetneq R$ has a decomposition $I = Q_1 \cap \cdots \cap Q_s$, where the Q_i are primary.

This theorem follows immediately from the following lemma:

Lemma: Let R be a Noetherian ring. Then every irreducible ideal is primary.

Proof: Let $I \subsetneq R$ be irreducible. Let $S = R/I$ and suppose $xy \in I$. In S we have $\bar{x}\bar{y} = 0$. We know (0) is irreducible in S. Why? Correspondence.
Suppose $\bar{x} \neq 0$. Let $J_m = \{a \in S ; a\bar{y}^m = 0\} \trianglelefteq S$. We have a chain

$$J_1 \subseteq J_2 \subseteq \cdots$$

which terminates since R is Noetherian. So for some n, $a\bar{y}^{n+1} \overset{=0}{} \Rightarrow a\bar{y} = 0$.
We claim $(0) = (\bar{x}) \cap (\bar{y}^n)$. Suppose $a \in (\bar{x}) \cap (\bar{y}^n)$. Then $a = b\bar{x}$ and $a = c\bar{y}^n$. So $a\bar{y} = b\bar{x}\bar{y} = 0 \Rightarrow 0 = a\bar{y} = c\bar{y}^n\bar{y} = c\bar{y}^{n+1} \Rightarrow c\bar{y}^n = 0$. Thus $a = 0$, and the claim holds.
But (0) is irreducible, so $(\bar{y}^n) = (0)$. Thus $\bar{y}^n = 0$. So (0) is primary in $S \Rightarrow I$ is primary in R. ■

Remark: This is quite nice for R a Noetherian integral domain of Kdim 1. Here Q is primary if and only if $Q = (0)$ or $\sqrt{Q}$ is maximal. Then every proper ideal is an intersection $Q_1 \cap \cdots \cap Q_s$ where $\sqrt{Q_i}$ are maximal.