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Integral Extensions

Let $R \subseteq S$ be rings, $1_R = 1_S$. $\iota: R \to S$ inclusion gives S an R-algebra/R-module structure

Def) We say that S is an <u>integral extension</u> of R if every $s \in S$ satisfies a monic polynomial equation with coefficients in R; ie., $\exists n \geq 1$ & $r_0, \dots, r_{n-1} \in R$ such that $s^n + r_{n-1}s^{n-1} + \dots + r_0 = 0$.

Eg. $\mathbb{Q}$ is not an integral extension of $\mathbb{Z}$. Look at $s = \frac{1}{2}$. If s satisfies a monic polynomial equation with coefficients in $\mathbb{Z}$

$\Rightarrow s^n + r_{n-1}s^{n-1} + \dots + r_0 = 0$, $r_i \in \mathbb{Z}$

$\Rightarrow \frac{1}{2} + \text{integer} = 0$ (multiply by $\frac{1}{2^{n-1}}$) $\times$

An element $s \in S$ is called <u>integral</u> over R if it satisfies a monic polynomial equation with coefficients in R.

Ex $\mathbb{Z}[\sqrt{2}]$ is integral over $\mathbb{Z}$

$s = a + b\sqrt{2}$ has $(x - (a+b\sqrt{2}))(x - (a - b\sqrt{2}))$

Remark: If $R \subseteq S$ & S is a finitely generated R-module. Then S is an integral extension of R.

Proof: Write $S = Ra_1 + Ra_2 + \dots + Ra_d$; $a_1, \dots, a_d \in S$. Let $s \in S$. We need to find a polynomial. Notice $\exists r_{ij} \in R$ $1 \leq i, j \leq d$ such that

$$s \cdot a_i = \sum_{j=1}^{d} r_{ij} a_j$$

Then

$$s \begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1d} \\ r_{21} & & & \vdots \\ \vdots & & & \\ r_{d1} & \cdots & & r_{dd} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix}$$

$$(sI - A)\begin{bmatrix} a_1 \\ \vdots \\ \vdots \\ a_d \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ \vdots \\ a_d \end{bmatrix}$$

where $A = (r_{ij}) \in M_d(R)$, $I: S^d \xrightarrow{id} S^d$.

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Adjugate: $C \in M_d(R)$, $C^{adj} = \left((-1)^{i+j} \det(C(j|i))\right)$
Then $CC^{adj} = C^{adj}C = \det(C)\, I$

If we multiply by $(sI-A)^{adj}$:
$$\det(sI-A)a_i = 0 \quad \forall i$$
$\Rightarrow \det(sI-A)(Ra_1 + \cdots + Ra_d) = 0$
$\Rightarrow \det(sI-A) = 0$
$\Rightarrow p_A(s) = 0$ monic with coefficients in R

We'll now show that if $S \supseteq R$ then $T := \{s \in S;\ s \text{ is integral over } R\}$ forms a ring with $R \subseteq T \subseteq S$.

<u>Ex</u> Let $R = \mathbb{Z}$; $S = \overline{\mathbb{Q}}$. Then $T = \mathbb{A}$ = algebraic integers

Proposition: Let $R \subseteq S$ be rings & let $s \in S$. Then TFAE:
1) s is integral over R
2) there exists a finitely-generated R-submodule M of S such that $sM \subseteq M$ & $\cancel{sM \neq (0)}$ $1 \in M$

Proof: $1 \Rightarrow 2$: $\exists\, n \geq 1$, $r_i \in R$ st $s^n + r_{n-1}s^{n-1} + \cdots + r_1 s + r_0 = 0$.
Let $M = R + Rs + \cdots + Rs^{n-1}$. Then M is a f.g R-module, $1 \in M$, $\cancel{sM \neq (0)}$, & $sM = s(R + \cdots + Rs^{n-1})$
$= Rs + \cdots + Rs^n$
$= Rs + \cdots + Rs^{n-1} + R(-r_{n-1}s^{n-1} - \cdots - r_0) \subseteq R + \cdots + Rs^{n-1}$

$2 \Rightarrow 1$: Suppose $M = Ra_1 + \cdots + Ra_d$, $sM \neq (0)$, $sM \subseteq M$. As before
$$sa_i = \sum_{j=1}^{d} r_{ij}a_j, \quad r_{ij} \in R$$
$A = (r_{ij})$; $(sI-A)[a_i] = [0]\ \forall i \Rightarrow \det(sI-A)a_i = 0\ \forall i \Rightarrow \det(sI-A)1 = 0$
$\Rightarrow \det(sI-A) = 0 \Rightarrow s$ is integral over R.

Corollary: If $T = \{s \in S;\ s \text{ is integral over } R\}$. Then T is a ring with $R \subseteq T \subseteq S$.

Proof: Let $x, y \in T$. So we have $x^n + r_{n-1}x^{n-1} + \cdots + r_0 = 0$ and
$y^m + r'_{m-1}y^{m-1} + \cdots + r'_0 = 0$.

Now let
$$M = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} R x^i y^j \subseteq S$$
Note M is f.g. and $1 \in M$. Notice $xM \subseteq M$. Why? $x(x^i y^j) \in M$ for each generator ✓. Similarly $yM \subseteq M$. So
$$(x+y)M \subseteq xM + yM \subseteq M + M \subseteq M$$
$$xyM \subseteq xM \subseteq M$$
So $x+y$ & xy are integral & hence are in T.
Also $T \supseteq R$: if $r \in R$ then r satisfies the monic polynomial $x - r = 0$. ∎

Given $R \subseteq S$, the ring
$$T = \{s \in S : s \text{ is integral over } R\}$$
is called the <u>integral closure</u> of R in S.

Important case:
When R is an integral domain, we say that the <u>integral closure</u> of R is the integral closure of R in its field of fractions.

Ex. Let $R = \mathbb{C}[t^2, t^3] \subseteq \mathbb{C}[t]$. What is the integral closure of R? It is $\mathbb{C}[t]$. First notice t is a root of $x^2 - t^2 \in R[x]$. So t is integral over $R \Rightarrow$ integral closure of R contains $\mathbb{C}[t]$. Field of fractions of R is $\mathbb{C}(t)$. Notice if $s \in \mathbb{C}(t)$ is integral over R, it is integral over $\mathbb{C}[t]$ because $\mathbb{C}[t] \supseteq R$. So it suffices to show that $\mathbb{C}[t]$ is integrally closed, ie it is its own integral closure.

Theorem: Let R be a UFD. Then R is integrally closed.

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Proof: Let $\frac{a}{b} \in \mathrm{Frac}(R)$, with $a, b \in R$, $\gcd(a,b) = 1$, $b \neq 0$ & $\frac{a}{b}$ integral over R. Then $\exists\, n \geq 1$ & $r_1, \dots, r_n \in R$ such that
$$\left(\frac{a}{b}\right)^n + r_{n-1}\left(\frac{a}{b}\right)^{n-1} + \dots + r_1 \frac{a}{b} + r_0 = 0.$$
Multiply by b^n. So $b \mid a^n$. But $\gcd(a,b) = 1$ so $\gcd(a^n, b) = 1$. So b is a unit. Hence $\frac{a}{b} \in R$. So the integral elements of $\mathrm{Frac}(R)$ are precisely the elements of R. ∎

Laying over & Going up

These theorems tell us about prime ideals in S in terms of prime ideals of R when S is integral over R.

Theorem: Let $R \subseteq S$ be an integral extension. If $P \in \operatorname{Spec}(R)$ then there exists a (in general not unique) $Q \in \operatorname{Spec}(S)$ such that $Q \cap R = P$. Moreover, if $Q_1 \in \operatorname{Spec}(S)$ is such that $Q_1 \cap R \subsetneq P$ then we can find $Q \in \operatorname{Spec}(S)$ such that $Q \supseteq Q_1$ & $Q \cap R = P$.

some examples

Proof: Let $P_1 = Q_1 \cap R \trianglelefteq R$, a prime ideal of R.
Reduction 1: We may replace S by S/Q_1 and R by $R/P_1 = R/R\cap Q_1$ & S is still integral over R.
Thus we may assume that $Q_1 = (0)$, and it suffices to show there exists a Q such that $Q \cap R = P$.
Now let $U = R \setminus P \subseteq S$, multiplicatively closed.
Reduction 2: Replace S by $U^{-1}S$. Then $U^{-1}S$ is integral over $U^{-1}R$ (A3) & R by $U^{-1}R = R_P$.
We can reduce to this case by results in localization.
Now R is a local ring with unique maximal ideal P (really PR_P from before). Consider the ideal $PS \subseteq S$. If $PS \subsetneq S$ then there is a ~~unique~~ maximal ideal $L \supseteq PS$ (so L is prime) & $L \cap R = P$ (Why?) $L \cap R \subsetneq R$ as $1 \notin L \Rightarrow L \cap R \subseteq P$ because P is unique maximal.
But $L \cap R \supseteq PS \cap R \supseteq P \Rightarrow L \cap R = P$. So we may take $Q = L$ and we're done.
So we may assume that $PS = S$. In particular, $1 \in PS$. So there exist $p_1, \dots, p_d \in P$ ~~such that~~, $s_1, \dots, s_d \in S$ such that
$$p_1 s_1 + \cdots + p_d s_d = 1.$$
Let S' be the R-algebra generated by $s_1, \dots, s_d$. Then $PS' = S'$.
Why? $p_1 s_1 + \cdots + p_d s_d = 1 \in PS'$ so $S' \subseteq (p_1 s_1 + \cdots + p_d s_d) S' \subseteq PS'S' \subseteq PS'$.
Also S' over R is a finitely generated R-module. Why?
$$S' = \sum_{i_1, \dots, i_d \geq 0} R s_1^{i_1} \cdots s_d^{i_d} \subseteq \sum_{0 \leq i_j \leq n-1} R s_1^{i_1} \cdots s_d^{i_d} =: M \subseteq S' \text{ f.g. } R\text{-module}$$

it made sense in my head,
but you know, my head is wrong

But S is integral over R so $\exists\, n_i \geq 1$ st $s_i^{n_i} \in R s_i^{n_i - 1} + \cdots + R s_i + R$.

So $PM = M$ & $P = J(R) \overset{\text{Nakayama}}{\Longrightarrow} M = (0)$

$\times$ $\because 1 \in S' = M$ so $S' \supseteq R \Rightarrow M \neq (0)$. So we're done. ☐

$$\begin{array}{ccccc} Q_1 & \subseteq & Q & \subseteq & S \\ | & & \cup\!\!| & & \\ P_1 & - & P & \subseteq & R \end{array}$$

We also have incomparability:

If $Q, Q' \in \operatorname{Spec}(S)$ & $Q \cap R = Q' \cap R = P$
$Q \neq Q'$

Then Q & Q' are incomparable, ie $Q \not\subseteq Q'$ & $Q' \not\subseteq Q$???

Why is this?

Suppose that $Q' \subsetneq Q$. Then $Q' \cap R = P$. So if we mod out by Q' we can replace S by S/Q' & R by $R/R\cap Q' = R/P$.
Now this reduces to the case where $P = (0)$, $Q' = (0)$, $Q \neq (0)$, $Q \cap R = (0)$. Pick $x \in Q \setminus \{0\}$. Then x is integral over R, so $\exists\, n \geq 1$ and $r_1, \ldots, r_n \in R$ st $x^n + \cdots + r_0 = 0$.
WLOG $r_0 \neq 0$, as S is integral dom and $x \neq 0$.
But now $R \ni r_0 = \cdots \in Q \Rightarrow r_0 \in R \cap Q = P = (0)$ $\times$
So it is possible to have $(0) \subsetneq Q \subseteq S$ st $Q \cap R = (0)$ with R an integral domain. So we obtain the incomparability result.

Krull dimension

Given a ring R, we ~~let~~ define the Krull dimension of R to be
$\sup\{n \,;\, \exists$ a chain $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ of prime ideals in $R\}$.

ex F a field, $\operatorname{Kdim}(F) = (0)$
$\operatorname{Kdim}(\mathbb{Z}) = 1$
F a field, $\operatorname{Kdim}(F[x]) = 1$.

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Theorem: If $R \subseteq S$, S an integral extension of R. Then
$$\mathrm{Kdim}(R) = \mathrm{Kdim}(S).$$

Ex Let K be a finite field extension of $\mathbb{Q}$, ie $[K:\mathbb{Q}] < \infty$.
Then if we take $R = \mathbb{Z}$, $S = \{s \in K;\ s \text{ is integral over } \mathbb{Z}\} = \mathbb{A} \cap K =: \mathcal{O}_K$
$\because \mathrm{Kdim}(\mathbb{Z}) = 1 \Rightarrow \mathrm{Kdim}(\mathcal{O}_K)$.

Proof: First suppose that $\mathrm{Kdim}(R) \geq n$. So there exists a chain
$$P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_n \quad \text{in } \mathrm{Spec}(R)$$
Now we use lying over and going up
$$Q_0 \cap R = P_0, \quad Q_1 \cap R = P_1$$

? $\longrightarrow$
$$\begin{array}{lccccc} S. & Q_0 & \subseteq & Q_1 & \subseteq Q_2 \subseteq \cdots & P_n \\ & \cup\!| & & \cup\!| & \shortparallel & | \\ & P_0 & \subsetneq & P_1 & \subsetneq P_2 \subseteq \cdots & Q_n \end{array} \; ?$$

$\Rightarrow \exists$ a chain $Q_0 \subsetneq Q_1 \subsetneq Q_2 \subsetneq \cdots \subsetneq Q_n$ in $\mathrm{Spec}(S)$ with $Q_i \cap R = P_i$.
So $\mathrm{Kdim}(S) \geq n$. So $\mathrm{Kdim}(R) \geq n \Rightarrow \mathrm{Kdim}(S) \geq n$ ... ???
If $\mathrm{Kdim}(S) \geq n$ then $\exists$ a chain
$$Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n \quad \text{in } \mathrm{Spec}(S).$$
Take $P_i = Q_i \cap R$. Then we have $P_0 \subsetneq P_1 \subsetneq \cdots \subseteq P_n$.
Why is $P_i \neq P_{i+1}$? The incomparibility thing he said. $\boxtimes$

A Noetherian ring has Kdim = 0 $\Leftrightarrow$ We let $N = \sqrt{(0)} = \{x;\ x \text{ is nilpotent}\}$
Then $R/N \cong F_1 \times \cdots \times F_s$, $s \geq 1$, F_i fields

$\Updownarrow$
all prime ideals are maximal
Why? Take $P \in \mathrm{Spec}(R)$
If P not maximal $\exists M \supsetneq P$
maximal with $P \subsetneq M$ so have ≥ 1 Kdim

Proof: $\Leftarrow$ ✓
$\mathrm{Kdim}(R) = \mathrm{Kdim}(R/N)$
Claim: $F_1 \times \cdots \times F_s$ has Kdim = 0.
First, if P is a prime ideal, we claim $\exists j$ st $P = F_1 \times \cdots \times F_{j-1} \times (0) \times F_{j+1} \times \cdots \times F_s$
Why? Let $e_j = (0, \ldots, 1, \ldots, 0) \in F_1 \times \cdots \times F_s$
Then $e_k e_\ell = 0$ for $k \neq \ell$ $\Rightarrow e_k e_\ell \in P \Rightarrow e_k$ OR $e_\ell \in P \Rightarrow \exists j$ st $e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_s \in P$.
$$\Rightarrow P \supseteq F_1 \times \cdots \times F_{j-1} \times (0) \times F_{j+1} \times \cdots \times F_s = I_j.$$

$F_1 \times \cdots \times F_s / I_j \cong F_j$

$(a_1, \ldots, a_s) \longmapsto a_j \qquad \ker = I_j$

So I_j is maximal $\because F_j$ is a field. $\Rightarrow P = I_j$.

So all prime ideals are maximal $\Rightarrow$ Kdim $R = 0$

Converse ($\Rightarrow$)

Suppose R has Kdim$=0$. By Noether's theorem $\exists P_1, \ldots, P_s \in \mathrm{Spec}(R)$ st

$$\begin{array}{ccc} P_1 & \ldots & P_s \\ & (0) & \end{array}$$

& if Q is another prime in R with $Q \supseteq (0)$ $\Rightarrow Q \supseteq P_i$ for some i

This means $\mathrm{Spec}(R) = \{P_1, \ldots, P_s\}$.

$\because$ if $Q \notin \{P_1, \ldots, P_s\}$ then $Q \supseteq P_i$ for some $i \Rightarrow Q = P_i$, Kdim$=0$ $\lightning$

Now each P_i is maximal $\Rightarrow P_i + P_j = R$ for $i \neq j$

ie $P_1, \ldots, P_s$ are pairwise comaximal. By CRT

$$R/\textstyle\bigcap_i P_i \cong \prod_i R/P_i = \prod_i F_i$$

$$R/N, \quad N = \sqrt{(0)}.$$

Krull dimension of Polynomial rings

Lemma: Let R be a ring with Kdim$(R) = d$. Then

$$d+1 \leq \mathrm{Kdim}(R[x]) \leq 2d+1.$$

Proof: Let $P_0 \subsetneq \cdots \subsetneq P_d$ be a chain in $\mathrm{Spec}(R)$. Let $Q_i = P_i R[x]$ $= \{a_0 + a_1 x + \cdots + a_m x^m \,;\, m \geq 0,\ a_0, \ldots, a_m \in P_i\}$. Then

$$R[x]/Q_i \cong (R/P_i)[x] \leftarrow \text{integral domain}$$

So $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_d$ is a chain in $\mathrm{Spec}(R[x])$. Then

$$R[x]/Q_d \cong (R/P_d)[x] \leftarrow \text{\& } (x) \text{ is a prime ideal in this ring.}$$

Corresp. $\Rightarrow Q_0 \subsetneq \cdots \subsetneq Q_d \subsetneq (Q_d, x)$ is a chain of len $d+1$.

For the other bound, suppose towards a contradiction that $\exists$ a chain

$$Q_0 \subsetneq \cdots \subsetneq Q_{2d+1} \subsetneq Q_{2d+2} \text{ in } \mathrm{Spec}(R[x]).$$

Then if we let $P_i = Q_i \cap R$. Then

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$P_0 \subseteq P_1 \cdots \subseteq P_{2d+2}$ is a chain in $\operatorname{Spec}(R)$.
We can't have more than $d+1$ distinct guys in this chain.
By Pigeonhole principle $\exists\, i$ s.t. $P_i = P_{i+1} = P_{i+2} = P$. Then

$$\begin{tikzcd}[column sep=small] Q_i \arrow[dr, dash] & \subsetneq \; Q_{i+1} \arrow[d, dash] \; \subsetneq & Q_{i+2} \arrow[dl, dash] \\ & P & \end{tikzcd} \quad \text{in } \operatorname{Spec}(R)$$

By corresp. $\exists$ a chain $\tilde{Q}_i \subsetneq \tilde{Q}_{i+1} \subsetneq \tilde{Q}_{i+2}$ in $\operatorname{Spec}((R/P)[x]) = \operatorname{Spec}(R[x]/PR$
Let $S = R/P$ So we have a chain $\mathcal{J}$ in $\operatorname{Spec}(S[x])$.
Moreover $\tilde{Q}_i \cap S = \tilde{Q}_{i+1} \cap S = \tilde{Q}_{i+2} \cap S = (0)$. Let $T = S \setminus \{0\}$
mult. closed since S is an integral domain.

$$T^{-1}S = \operatorname{Frac}(S) =: K.$$

So $\tilde{Q}_i \cap S = (0) \Rightarrow \tilde{Q}_i \cap T = (0)$ (and other 2)
So by results from localization, $\exists$ primes $\tilde{\tilde{Q}}_i \subsetneq \tilde{\tilde{Q}}_{i+1} \subsetneq \tilde{\tilde{Q}}_{i+2}$
in $T^{-1}(S[x]) = (T^{-1}S)[x] = K[x]$. But now $\operatorname{Kdim} K[x] = 1$
// we have chain of 3 things

R [illegible] $\operatorname{Kdim} R = d$

$$\begin{array}{c} R \\ \big| \; \text{finite} \\ k[y_1,\dots,y_d] \end{array}$$

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Noether Normalization Theorem

Theorem (Noether): Let R be a finitely generated k-algebra. Then there exists a k-subalgebra S of R such that
1) $S \cong k[x_1,\dots,x_d]$, where $d = \operatorname{Kdim}(R)$;
2) R is finitely generated S-module.

Remark: If R is finitely-generated as a k-algebra then $\operatorname{Kdim}(R) < \infty$.

Proof: $R \cong k[x_1,\dots,x_n]/I$, so $\operatorname{Kdim}(R) = \operatorname{Kdim}(\quad) \le \operatorname{Kdim}(k[x_1,\dots,x_n])$
(correspondence).

We showed If $\operatorname{Kdim}(R) = s \Rightarrow s+1 \le \operatorname{Kdim} R[x] \le 2s+1$. By induction, $\operatorname{Kdim} k[x_1,\dots,x_n] \le 2^n - 1$.

Proof (of Noether Normalization Theorem):

Let m be the number of generators for the k-algebra R. We'll do this by induction on m.

Let's say $R = k[a_1, \dots, a_m]$, ~~a polynomial ring~~ not nec. $\{a_1, \dots, a_m\}$ a set of generators.

<u>Base case $m=1$:</u>

$R = k[a_1]$. So $R \cong k[x]/I$, where $I = (0)$ or $I = (p(x))$ where $p(x)$ is the minimal polynomial of a_1.

If $I = (0)$, $R \cong k[x] \Rightarrow S = k[a_1] \cong k[x]$ & $R = S$ so R is a finitely generated S-module.

If $I = (p(x))$, $p(x) \neq 0 \Rightarrow R \cong k[x]/(p(x))$ is e-dimensional as a k-vector space, where $e = \deg(p(x))$. In this case, take $S = k$. So $\dim_k R < \infty \Rightarrow R$ is a finitely generated S-module.

<u>Inductive case</u>

Let's assume that the claim holds whenever R is generated by $<m$ generators. Consider the case when $R = k[a_1, \dots, a_m]$.

<u>Case 1:</u> $a_1, \dots, a_m$ are algebraically independent over k. That is, there is no $q(x_1, \dots, x_m) \in k[x_1, \dots, x_m] \setminus \{0\}$ such that $q(a_1, \dots, a_m) = 0$.

In this case $R \cong k[x_1, \dots, x_m]$. Why? Consider

$$\varphi : k[x_1, \dots, x_m] \to R$$

$$x_i \mapsto a_i$$

$$q(x_1, \dots, x_m) \mapsto q(a_1, \dots, a_m)$$

$\ker(\varphi) = (0)$. So in this case, we take $S = R$ & S is clearly a finitely generated R-module.

<u>Case 2:</u> $a_1, \dots, a_m$ are not algebraically independent over k. So there is a non-trivial polynomial relation

$$q(a_1, \dots, a_m) = 0, \quad q(x_1, \dots, x_m) \neq 0.$$

Exercise: $\exists$ natural numbers $A_1, \dots, A_{m-1} > 0$ st if

$$q(x_1 + x_m^{A_1}, \dots, x_{m-1} + x_m^{A_{m-1}}, x_m) = \underset{\substack{\uparrow \\ c \neq 0 \\ \text{constant}}}{c}\, x_m^{D} + \underset{\substack{\uparrow \\ x_m \text{ only appears to deg} < D}}{p(x_1, \dots, x_m)}$$

There exist $u_1, \dots, u_m$ in R such that

$$a_i = u_i + u_m^{A_i} \quad \text{for } i = 1, \dots, m-1$$

$$a_m = u_m$$

Why? $a_i = u_i + u_m^{A_i} \iff u_i = a_i - u_m^{A_i} = a_i - a_m^{A_i} \in R$ for $i < m$

Sometimes I worry I don't make sense. {Pause} Sometimes I don't make sense.

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Notice that
$$k[u_1, \dots, u_m] = k[x_1, \dots, x_m] = R$$
as each a_i is a polynomial in $u_1, \dots, u_m$ and vice-versa.
Notice that
$$0 = q(a_1, \dots, a_m) = q(u_1 + u_m^{A_1}, \dots, u_{m-1} + u_m^{A_{m-1}}, u_m)$$
$$= C \cdot u_m^D + \sum_{i=0}^{D-1} p_i(u_1, \dots, u_{m-1})\, u_m^i. \qquad (*)$$
Now let $T = k[u_1, \dots, u_{m-1}]$. Then $R = T[u_m] = T + Tu_m + \cdots + Tu_m^{D-1}$ by $(*)$.
So R is a finite T-module (?). But T is generated by $<m$ elements, so by inductive hypothesis, there exists $T \supseteq S \cong k[x_1, \dots, x_d]$ such that T is a finite S-module. So now
$$\begin{array}{c} R \\ \mid^{<\infty} \\ T \\ \mid^{<\infty} \\ S \end{array}$$
Ex: If R is a finitely generated T-module and T is a finitely generated S-module $\Rightarrow$ R is a finitely-generated S-module. (A4). So we're done. ∎

To show that $\operatorname{Kdim} k[x_1, \dots, x_d] = d$ & $R \setminus^{<\infty}_{k[x_1, \dots, x_d]} \Rightarrow \operatorname{Kdim} R = d$, we'll use Gelfand-Kirillov (?) dimension and integrality.
How does Gelfand-Kirillov dimension work?
Let k be a field, let A be a finitely-generated k-algebra & let V be a finite-dimensional k-vector space with $V \subseteq A$ such that $1 \in V$ and V contains a set of generators for A. } we call suc a V a <u>generating subspace</u> for A
We define
$$V^2 = \operatorname{span}\{vw \,;\, v, w \in V\} \supseteq V \ (\text{as } 1 \in V)$$
$\dim V^2 < \infty$ because if $x_1, \dots, x_m$ is a basis for V,
$$\{x_i x_j \,;\, i, j \in \{1, \dots, m\}\}$$
span V^2. Similarly, we get a chain
$$V \subseteq V^2 \subseteq V^3 \subseteq$$
and $\dim_k V^n < \infty$ $\forall n \geq 1$.
We now define

$$\operatorname{GKdim}(A) = \limsup_{n\to\infty} \frac{\log(\dim V^n)}{\log n}$$

Remarks:

1) We'll see that this is independent of V.
2) Intuitively, $\dim V^n \sim Cn^d \Rightarrow d = \operatorname{GKdim} A$, $(C>0)$

Ex 1. $\operatorname{GKdim} k[x] = 1$ Take $V = k + kx = \operatorname{span}\{1, x\}$.

$V^n = \operatorname{span}\{1, \dots x^n\}$, $\dim V^n = n+1$

$$\limsup_{n\to\infty} \frac{\log(n+1)}{\log(n)} = 1$$

Ex 2. $\operatorname{GKdim} k[x,y] =$

$V = k + kx + ky = \operatorname{span}\{1, x, y\}$

$V^n = \operatorname{span}\{x^i y^j ;\ i+j \le n\}$

So $\dim V^n = \binom{n+2}{2} \sim \frac{1}{2} n^{2}$

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Proof Suppose that V & W are two generating sub spaces. Then

$$V \subseteq V^2 \subseteq V^3 \subseteq \cdots \subseteq \bigcup_{n=1}^{\infty} V^n = A$$
$$W \subseteq W^2 \subseteq W^3 \subseteq \cdots \subseteq \bigcup_{n=1}^{\infty} W^n = A$$

It follows that there exist $p, q \ge 1$ such that $V \subseteq W^p$ & $W \subseteq V^q$. This means that $V^n \subseteq W^{pn}$ $\forall n \ge 1$. So

$$\limsup_{n\to\infty} \frac{\log(\dim V^n)}{\log(n)} \le \limsup_{n\to\infty} \frac{\log(\dim(W^{pn}))}{\log(n)}$$
$$= \limsup_{n\to\infty} \frac{\log(\dim(W^{pn}))}{\log(pn)} \quad \text{as } \frac{\log(pn)}{\log(n)} \longrightarrow 1$$
$$\le \limsup_{n\to\infty} \frac{\log(\dim W^n)}{\log(n)}$$

Similarly, since $W \subseteq V^q$, we have

$$\limsup_{n\to\infty} \frac{\log(\dim V^n)}{\log(n)} \ge \limsup_{n\to\infty} \frac{\log(\dim W^n)}{\log(n)}.$$

So GKdimension is independent of choice of generating subspace.

Remark 2: If A is a finitely generated k-algebra then
$$\mathrm{GKdim}\, A = 0 \iff \dim_k A < \infty.$$
$$\text{If } \dim_k A > \infty \Rightarrow \mathrm{GKdim}\, A \geq 1. \qquad \leftarrow ???$$

Proof: If $\dim_k A < \infty \Rightarrow V = A$ is a generating subspace. But now $V^n = A$ $\forall n \geq 1$. So $\mathrm{GKdim}\, A = \limsup_{n\to\infty} \log(\dim_k(A))/\log(n) \longrightarrow 0$.
Conversely, suppose that $\dim_k A = \infty$, and let V be a generating subspace for A. So $V \subseteq V^2 \subseteq V^3 \subseteq \cdots$. Note $V^i \subsetneqq V^{i+1}$ $\forall i \geq 1$. Otherwise, $V^n = V^i$ $\forall n \geq i$ so $A = \bigcup_{n=1}^{\infty} V^n = V^i$ so $\dim_k A < \infty$. So $V \subsetneqq V^2 \subsetneqq V^3 \subsetneqq \cdots$. Hence $\dim V^n \geq n$ and so
$$\limsup_{n\to\infty} \frac{\log \dim V^n}{\log n} \geq \limsup_{n\to\infty} \frac{\log n}{\log n} = 1. \qquad \blacksquare$$

Theorem: Let $d \geq 1$. Then $\mathrm{GKdim}(k[x_1,\ldots,x_d]) = d$.

Proof: Let $V = k + kx_1 + \cdots + kx_d$. Then
$$V^n = \mathrm{span}\{x_1^{i_1}\cdots x_d^{i_d} ; i_1 + \cdots + i_d \leq n\}.$$
So
$$\dim(V^n) = \binom{n+d}{d},$$
by bijection
$$x_1^{i_1}\cdots x_d^{i_d} \longmapsto \underbrace{\_\,\_\,\_\,\_}_{i_1}\, X\, \underbrace{\_\,\_\,\_\,\_\,\_\,\_}_{i_2}\, \underbrace{X}_{i_3}\, X\, \_\,\_ \quad \cdots \quad \underbrace{\_\,\_}_{i_d}\, X\, \_\,\_\,\_ .$$
$$\binom{n+d}{d} = \frac{(n+d)\cdots(n+1)}{d!} = \frac{n^d}{d!}\left(1 + O\left(\tfrac{1}{n}\right)\right)$$
So
$$\frac{\log \dim V^n}{\log n} = \frac{\log \frac{n^d}{d!}\left(1+O\left(\frac{1}{n}\right)\right)}{\log n}$$
$$= \frac{d\log n - \log d! + \log\left(1+O\left(\frac{1}{n}\right)\right)}{\log n}$$
$$= \frac{d\cancel{\log n}}{\cancel{\log n}} - \frac{\log d!}{\log n} + \frac{O\left(\frac{1}{n}\right)}{\log n} \longrightarrow d \text{ as } n \to \infty \qquad \blacksquare$$

Our goal

Theorem: Let k be a field and let A be a finitely generated k-algebra. Then
$$\mathrm{GKdim}(A) = \mathrm{Kdim}(A).$$

Corollary: $\mathrm{Kdim}\, k[x_1,\dots,x_d] = d$.

The proof will use two facts:
1) (A3) If $S \supseteq R$ & S is a finitely generated R-module $\Rightarrow$ S is integral over R
$\overset{\text{cor}}{\Rightarrow}$ $\mathrm{Kdim}(S) = \mathrm{Kdim}(R)$ If S is a finite R-module.
2) (A4) If $S \supseteq R$ & S is a finitely generated R-module then $\mathrm{GKdim}\, R = \mathrm{GKdim}\, S$.

Proof: Notice that if A is a finitely generated k-algebra then by NNT, there exists $A \supseteq B \cong k[y_1,\dots,y_d]$ such that A is a finite B-module.

$$\begin{array}{c|l|l} A & \because A \text{ is finite as } B\text{-module} & A \text{ is f.g. as a } B\text{-module} \\ \big| {<}\infty & \Rightarrow A \text{ is integral over } B & \Rightarrow \mathrm{GKdim}\, A = \mathrm{GKdim}\, B = d \\ B \cong k[y_1,\dots,y_d] & \Rightarrow \mathrm{Kdim}\, A = \mathrm{Kdim}\, B \geq d & \end{array}$$

So we at least know $\mathrm{Kdim}\, A \geq \mathrm{GKdim}\, A$.
Now suppose that there exists a finitely-generated k-algebra A such that $\mathrm{Kdim}\, A > \mathrm{GKdim}\, A$. Let
$$\alpha = \inf\{\mathrm{GKdim}\, A \,;\, A \text{ f.g. s.t. } \mathrm{Kdim}\, A > \mathrm{GKdim}\, A\}.$$
So there exists A such that $\mathrm{Kdim}\, A > \mathrm{GKdim}\, A$ & $\mathrm{GKdim}\, A < \alpha + \frac{1}{2}$.
So now by NNT, $\exists d$ & $B \cong k[y_1,\dots,y_d]$ such that $\underset{A}{\overset{B}{\cap}}$

$$\begin{array}{c} A \\ \big| {<}\infty \\ B \end{array}$$

So $\mathrm{Kdim}\, A = \mathrm{Kdim}\, B$ & $\mathrm{GKdim}\, A = \mathrm{GKdim}\, B = d$. Now $\mathrm{Kdim}\, A > \mathrm{GKdim}\, A$ $\Rightarrow \mathrm{Kdim}(A) \geq d+1$. So we are in the situation where we may assume $\mathrm{Kdim}(k[x_1,\dots,x_d]) \geq d+1$. That means there exists a chain
$$(0) \subsetneq P_1 \subsetneq \cdots \subsetneq P_{d+1} \quad \text{in } \mathrm{Spec}(k[x_1,\dots,x_d]).$$
Then we claim $\mathrm{Kdim}(k[x_1,\dots,x_d]/P_1) \geq d$.
Claim: If B is a finitely generated k-algebra that is an integral domain & $I \neq (0)$ is an ideal of B $\Rightarrow \mathrm{GKdim}(B/I) \leq \mathrm{GKdim}\, B - 1$.

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Once we have the claim, we see
$$\mathrm{Kdim}(k[x_1,\dots,x_d]/P_1) \geq d$$
& $\mathrm{GKdim}(k[x_1,\dots,x_d]/P_1) \leq d-1 < \alpha$. But ✗ choice α.

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To finish the proof, we need to prove the claim. (We replace B by A.)
Let $\pi: A \to A/I$ be the canonical surjection. Let V be a generating space st $V \cap I \neq (0)$ for A. Then $\bar{V} := \pi(V)$ is a generating space for A/I. For each $n \geq 1$ pick a subspace W_n of V^n such that $\pi(W_n) = \bar{V}^n$ & $\dim(W_n) = \dim(\bar{V}^n)$. Pick $f \in V \setminus \{0\}$ such that $f \in I$. Then we claim that the sum
$$W_n + fW_{n-1} + \cdots + f^{n-1}W_1 \subseteq V^n$$
is direct. To see this, suppose it is not. Then $\exists w_i \in W_i$, $i=1,\dots,n$ not all zero such that $w_n + fw_{n-1} + \cdots + f^{n-1}w_1 = 0$. Then there is a largest $m \leq n$ st $w_m \neq 0$.
So $f^{n-m}w_m + \cdots + f^{n-1}w_1 = 0$, $w_m \neq 0$
$$\Rightarrow w_m + \underbrace{fw_{m-1} + \cdots + f^{m-1}w_1}_{\in I} = 0 \quad \text{as } A \text{ is int. dom.}$$
$\Rightarrow \pi(w_m) = 0$ as $\pi(I) = (0)$. But $\dim W_m = \dim \bar{V}^m$ & $\pi(W_m) = \bar{V}^m$ so π is 1-1. So $w_m = 0$ ✗
Now we want to show $\mathrm{GKdim}(A/I) \leq d-1$ where $d = \mathrm{GKdim}(A)$. Suppose that $\mathrm{GKdim}(A/I) > d-1$. Then $\mathrm{GKdim}(A/I) \geq d$. Let $\varepsilon < 1$. So this means that $\dim(\bar{V}^n) \geq n^{d-\varepsilon}$ for infinitely many n.
Why? $\bar{V}$ is a generating space for A/I. If it is not true $\Rightarrow \dim \bar{V}^n < n^{d-\varepsilon}$ $\forall n \gg 0 \Rightarrow \dfrac{\log \dim \bar{V}^n}{\log n} < \dfrac{(d-\varepsilon)\log n}{\log n} = d-\varepsilon \quad \forall n \gg 0$
$$\Rightarrow \mathrm{GKdim}(A/I) = \limsup_{n\to\infty} \frac{\log \dim \bar{V}^n}{\log n} \leq d-\varepsilon < d \quad ✗$$
By assumption $\dim W_n = \dim \bar{V}^n$. So $\dim W_n \geq n^{d-\varepsilon}$ for ∞'ly many n.
Now $\bar{V}^n \subseteq \bar{V}^{n+1} \subseteq \cdots \subseteq \bar{V}^{2n} \Rightarrow \dim W_n \leq \cdots \leq \dim W_{2n}$.
So there are infinitely many n such that $\dim W_{2n}, \dots, \dim W_n \geq n^{d-\varepsilon}$
Now recall $W_{2n} \oplus \cdots \oplus f^{2n-1}W_1 \subseteq V^{2n}$
So $\dim V^{2n} \geq \dim(W_{2n}) + \dim(fW_{2n-1}) + \cdots$
$$\geq \dim(W_{2n}) + \cdots + \dim(W_n) \geq n^{d-\varepsilon} \cdot n = n^{d+1-\varepsilon}$$
So $\dfrac{\log \dim V^{2n}}{\log 2n} \geq \dfrac{(d+1-\varepsilon)\log n}{\log 2n} \longrightarrow d+1-\varepsilon \Rightarrow \mathrm{GKdim}\, A = \limsup_{n\to\infty} \leftarrow \geq d+1-\varepsilon > d$. ✗

Corollary: Let A be a finitely generated k-algebra. Then $\operatorname{Kdim}(A[x]) = \operatorname{Kdim}(A)+1$.

Proof: We already ~~know~~ showed $\geq$. Now by NNT, there exists $B \subseteq A$ such that $B \cong k[x_1,\ldots,x_d]$, $d = \operatorname{Kdim}(A)$, and A is a f.g. B-module. Write $A = Ba_1 + \cdots + Ba_s$. Then $A[x] = B[x]a_1 + \cdots$ So

$$\begin{array}{c} A[x] \\ \Big| \text{ f.g. module} \\ B[x] \cong k[x_1,\ldots,x_d][x] \end{array}$$

So by A3, $\operatorname{Kdim} A[x] = \operatorname{Kdim} k[x_1,\ldots,x_{d+1}] = d+1$ ▫

Transcendence degree

Let K be a field extension of a field k. Recall a set $S \subseteq K$ is algebraically independent if ~~for~~ every finite subset $\{x_1,\ldots,x_d\}$ of S has the property that if $p(x_1,\ldots,x_d)=0$, $p(t_1,\ldots,t_d) \in k[t_1,\ldots,t_d]$, then $p(t_1,\ldots,t_d)=0$.
Then

$$\operatorname{trdeg}_k K = \sup\{|S| \,;\, S \text{ is an alg. indep. subset of } K\}.$$

We'll only worry about finite trdeg.

Theorem: Let k be a field and let A be a finitely generated k-algebra that is an ~~finite~~ integral domain and let $K = \operatorname{Frac}(A)$. Then

$$\operatorname{Kdim}(A) = \operatorname{trdeg}_k K.$$

Proof: Let $d = \operatorname{Kdim}(A)$. By NNT, there is $B \cong k[x_1,\ldots,x_d]$, $B \subseteq A$, A f.g. B-module. Let $b_1,\ldots,b_d$ be a set of generators for B. So

$$B = k[b_1,\ldots,b_d] \cong k[x_1,\ldots,x_d].$$

Then the set $\{b_1,\ldots,b_d\} \subseteq K$ is algebraically independent over ~~K~~ k. So $\operatorname{trdeg}_k K \geq d$.
Now suppose that $\operatorname{trdeg}_k K > d$. Then there exist $c_1,\ldots,c_{d+1} \in K$ alg. indep. over k. So there is $b \in A \setminus \{0\}$ & $a_1,\ldots,a_{d+1} \in A$ st $c_i = a_i/b$ (common denominator). So

$$k[c_1,\ldots,c_{d+1}] \subseteq A[\tfrac{1}{b}]. \quad \text{Now } \operatorname{Kdim} A[b] \leq \operatorname{Kdim} A = d$$

$$k[c_1,\ldots,c_{d+1}] \cong k[y_1,\ldots,y_{d+1}] \qquad \Rightarrow \operatorname{Kdim} A[\tfrac{1}{b}] \leq d.$$

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So $\underset{\cup}{A[\frac{1}{b}]} \leftarrow GK\dim \le d$
$k[c_1, \ldots, c_{d+1}] \cong k[x_1, \ldots, x_{d+1}] \leftarrow GK\dim = d+1$

But the following fact gives us ~~that~~ our contradiction.

Fact: If $R \subseteq S$, R, S f.g. Then $GK\dim R \le GK\dim S$. } Why? Pick V gen space for R. add a set of gens for S to obtain a gen space $W \supseteq V$ for S. Then $\dim V^n \le \dim W^n \ \forall n$.

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Given a ring R & $P \in \mathrm{Spec}(R)$, we define the <u>height</u> of P

$$ht(P) = K\dim(R_P)$$
$$= \sup\{n ;\ \exists Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n ;\ Q_n \subseteq P,\ Q_i \in \mathrm{Spec}(R)\}$$

Spec & M-Spec

Given a ring R,

$$\mathrm{Spec}(R) = \{P ;\ P \text{ a prime ideal of } R\},$$
$$\text{M-Spec}(R) = \{M ;\ M \text{ a maximal ideal of } R\}.$$

We'll put a topology on Spec(R). Then M-Spec will inherit the subspace topology.

Zariski topology

Given an ideal $I \trianglelefteq R$. We define

$$C_I = \{P \in \mathrm{Spec}(R) ;\ P \supseteq I\}.$$

These will be our closed sets. Let's see that this is a topology.

$\emptyset = C_R$, $\mathrm{Spec}(R) = C_{(0)}$.

If $I_1, \ldots, I_n \trianglelefteq R$,

$$C_{I_1} \cup \cdots \cup C_{I_n} = C_{I_1 \cap \cdots \cap I_n}$$

Let's check it!

$P \in C_{I_1} \cup \cdots \cup C_{I_n} \iff P \supseteq I_1$ or $\cdots$ or $P \supseteq I_n$
$\iff P \supseteq I_1 \cap \cdots \cap I_n$

Clearly $\Rightarrow$. $\Leftarrow$: if not, $\exists a_i \in I_i \setminus P$, so $a_1 \cdots a_n \in \bigcap I_j \subseteq P \Rightarrow \exists j\ a_j \in P$.

Notice an arbitrary union of C_I's need not be closed