Nullstellensatz (zero locus theorem)

Recall: R is Jacobson if for every $P \in \mathrm{Spec}(R)$ we have $J(R/P) = (0)$.

eg. $R = k, \mathbb{Z}, k[x]$.

Recall that if R is a ring & S is a ring, we say that S is an R-algebra if $\exists \alpha: R \to S$, $1_R \mapsto 1_S$ homomorphism.

S is finitely generated as an R-alg if $\exists n \geq 1$, $s_1, \dots, s_n \in S$ st every $x \in S$ can be expressed as a polynomial $p(s_1, \dots, s_n)$ with coefficients in R.

Equivalently, if S is f.g. by $s_1, \dots, s_n$ as an R-alg. Then $\exists$ homomorphism $\varphi: R[x_1, \dots, x_n] \to S$ onto

$$p(x_1, \dots, x_n) \longmapsto p(s_1, \dots, s_n)$$

Thus if $I \trianglelefteq R[x_1, \dots, x_n]$ is the ker of φ,

$$S \cong R[x_1, \dots, x_n]/I.$$

General Nullstellensatz: Let R be a Jacobson ring & let S be a finitely generated R-algebra. Then

1) S is also a Jacobson ring
2) if $\mathfrak{M} \subseteq S$ is a maximal ideal of S, then
$$\alpha(R) \cap \mathfrak{M} =: \mathfrak{N}$$
is a maximal ideal of $\alpha(R)$ & $S/\mathfrak{M} =: F$ is a finite extension of

probably α $\longrightarrow$ $\varphi(R)/\mathfrak{N}$.

Special case: Let $R = k = \bar{k}$, an algebraically closed field (eg. $k = \mathbb{C}$)
$S = k[x_1, \dots, x_n]/I$, I a proper ideal.
Notice a maximal ideal of $S \underset{\text{corresp.}}{\longleftrightarrow} \mathfrak{M} \subseteq k[x_1, \dots, x_n]$, $\mathfrak{M} \supseteq I$
$S/\text{max ideal} \cong k[x_1, \dots, x_n]/\mathfrak{M} \longleftarrow$ field, Then $\mathfrak{N} := \mathfrak{M} \cap k$ is a maximal ideal of k, so $\mathfrak{N} = (0)$, $k/\mathfrak{N} = k$.

finite extension

If k is alg. closed, a finite extension of k must be k itself
So

$$k[x_1,\dots,x_n]/\mathcal{M} \cong k$$

So $\exists \lambda_1,\dots,\lambda_n \in k$ st $\mathcal{M} = (x_1-\lambda_1,\dots,x_n-\lambda_n)$,
with isom given by

$$\varphi : k[x_1,\dots,x_n] \xrightarrow{\text{onto}} k \quad ; \quad \ker(\varphi) = \mathcal{M}$$
$$p(x_1,\dots,x_n) \longmapsto p(\lambda_1,\dots,\lambda_n)$$

Zero locus?
Let $f_1(x_1,\dots,x_n),\dots,f_d(x_1,\dots,x_n) \in k[x_1,\dots,x_n]$, $k=\bar{k}$.
If $I=(f_1,\dots,f_d) \trianglelefteq k[x_1,\dots,x_n]$ is a proper ideal $\Rightarrow \exists \mathcal{M} = (x_1-\lambda_1,\dots,x_n-\lambda_n) \supseteq I$.
This means $f_1,\dots,f_d$ ~~[illegible]~~ are in the kernel of

$$\varphi : k[x_1,\dots,x_n] \to k$$
$$p(x_1,\dots,x_n) \mapsto p(\lambda_1,\dots,\lambda_n)$$

$\Rightarrow f_1(\lambda_1,\dots,\lambda_n) = \dots = f_d(\lambda_1,\dots,\lambda_n) = 0$

To do the general Nullstellensatz, we'll use the so-called Rabinowitch trick. This gives a useful characterization of Jacobson rings.

Theorem (Rabinowitch trick): Let R be a ring. Then R is a Jacobson ring if and only if whenever $P \in \operatorname{Spec}(R)$ & $T := R/P$ has the property that $\exists\, 0 \neq b \in T$ st T_b is a field, then T is already a field.

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Proof: ($\Rightarrow$): Let $P \in \operatorname{Spec}(R)$ & $T = R/P$. Suppose $\exists b \in T \setminus \{0\}$ st T_b is a field. Goal: must show that T is a field
Recall that the prime ideals of $T_b \xleftrightarrow{1-1}$ prime ideals of T that don't ~~[illegible]~~ contain b.
Since $\operatorname{spec}(T_b) = \{(0)\} \Rightarrow$ every nonzero prime ideal of T contains b.
Now if T is not a field
$\Rightarrow$ every maximal ideal of T contains b
$\Rightarrow J(T) = \bigcap\{\mathcal{M} ;\ \text{maximal}\} \ni b \Rightarrow J(T) \neq \emptyset$
$\times$ R is Jacobson and $T = R/P$

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($\Leftarrow$) We must show that $J(R/Q) = (0)$ $\forall Q \in \mathrm{Spec}(R)$.
Towards a contradiction, suppose that $J(R/P) \neq (0)$ for some prime ideal P. Let $S = R/P$.
Pick $0 \neq b \in J(S)$. Then every maximal ideal of S contains b.
Consider the ring S_b. Then $\exists$ a maximal ideal Q of S_b.
Recall

$$\begin{array}{ccc} \{\text{prime ideals of } S_b\} & \xleftrightarrow{1-1} & \{\text{prime ideals of } S \text{ that do not contain } b\} \\ Q & \longmapsto & Q \cap S \\ S_b Q' & \longleftarrow & Q' \end{array}$$

So $\exists Q' \in \mathrm{Spec}(S)$ st $S_b Q' = Q$ & $b \notin Q'$.
Notice Q' is not maximal as $b \notin Q'$. So S/Q' is not a field.
Let $\bar{b}$ be the image of b in S/Q', ie $\bar{b} = b + Q' \neq 0$.
Thus

$$S/Q' \left[\tfrac{1}{\bar{b}}\right] \cong \underbrace{S[\tfrac{1}{b}]}_{S_b} \Big/ \underbrace{S[\tfrac{1}{b}]Q'}_{Q} = S_b/\underset{\substack{\uparrow \\ \text{maximal}}}{Q} \quad \text{a field}$$

but S/Q' is not (a field)

$S = R/P$

$$R \twoheadrightarrow S \longrightarrow S/Q$$

So $S/Q \cong R/I$ for some prime ideal I of R
※ because $(R/I)[\frac{1}{\bar{b}}]$ is a field but R/I is not. ☒

Nullstellensatz

- R Jacobson
- S is a f.g. R-algebra $\Leftrightarrow S \cong R[x_1, \dots, x_n]/I$ (Let $\alpha(R) = R/R \cap I$)

Then ① S is Jacobson
② If $\mathfrak{M}$ is a maximal ideal of S
Then $\mathfrak{N} := \mathfrak{M} \cap \alpha(R)$ is a maximal ideal of $\alpha(R)$
& $S/\mathfrak{M}$ is a finite field extension of $\alpha(R)/\mathfrak{N}$.

Reduction: WLOG $\alpha(R)=R$

Why? First if R is Jacobson, so is R/J for any proper ideal J. If Q is a prime ideal of R/J.

Then $\exists P \supseteq J$ in R st

$$R/P = (R/J)/Q \quad \text{because } J(R/P)=(0) \Rightarrow J((R/J)/Q)=(0)$$

$$\Longrightarrow R/J \text{ is Jacobson}$$

Notice that S is a f.g. $\alpha(R)$-algebra

$$S \cong \underline{\alpha(R)}[x_1,\dots,x_n]/I, \quad I \cap \alpha(R) = (0))$$

So WLOG we may take $R=\alpha(R)$.

Key steps in Proof:

1) Show true ~~for~~ when $S=R[x]$
2) Use induction to show true for $S=R[x_1,\dots,x_d]$
3) Use correspondence to show true for $S=R[x_1,\dots,x_d]/I$, $I\cap R=(0)$.

Proofs of 2 and 3 assuming 1:

(2): Assuming 1, we have R Jacobson $\Rightarrow R[x]$ Jacobson $\Rightarrow$ $R[x_1,x_2]$ is Jacobson $\Rightarrow R[x_1,\dots,x_n]$ is Jacobson

So $S=R[x_1,\dots,x_d]$ is Jacobson.

Next, let $\mathfrak{M}$ be a maximal ideal of $S=R[x_1,\dots,x_d]$. Then $S=T[x_d]$, $T=R[x_1,\dots,x_{d-1}]$. So $S/\mathfrak{M}$ is a finite extension of $T/T\cap\mathfrak{M}$ by (1). & $\mathfrak{N}$ is maximal ideal of T.

By arguing by induction on d, we then have $T/\mathfrak{N}$ is a finite extension of $R/R\cap\mathfrak{N}$.

Now

$$\begin{tikzcd}
S/\mathfrak{M} \arrow[d, dash, "<\infty"] \arrow[dd, dash, bend left=60, "<\infty"] \\
T/\mathfrak{N} \arrow[d, dash, "<\infty"'] \\
R/R\cap\mathfrak{N} = R/R\cap\mathfrak{M}
\end{tikzcd}$$

induction step

For step 3: We have by (2) $R[x_1,\dots,x_d]$ is Jacobson

$$\Rightarrow S=R[x_1,\dots,x_d]/I \text{ is Jacobson}; \ I\cap R=(0)$$

Let $\mathfrak{M}$ be a maximal ideal of S. Correspondence $\Rightarrow$

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$$\begin{array}{ccc} R[x_1,\dots,x_d] & \trianglerighteq & \mathfrak{M}' \\ | & & | \\ S = R[x_1,\dots,x_d]/I & \trianglerighteq & \mathfrak{M} \\ | & & \\ R & & \end{array}$$

$\mathfrak{M} \longleftrightarrow \mathfrak{M}' \trianglelefteq R[x_1,\dots,x_d]$, $\mathfrak{M}' \supseteq I$.

Then $\mathfrak{n} : \mathfrak{M}' \cap R$ is maximal in R by ② & $R[x_1,\dots,x_d]/\mathfrak{M}' \cong S/\mathfrak{M}$

$$\begin{array}{ccc} R[x_1,\dots,x_d]/\mathfrak{M}' & \cong & S/\mathfrak{M} \\ \Big| {\scriptstyle <\infty} & \diagup {\scriptstyle <\infty} & \\ R/\mathfrak{n} & & \end{array}$$

& $\underbrace{\mathfrak{M} \cap R = \mathfrak{M}' \cap R}_{\mathfrak{n}}$

So it remains to prove the Nullstellensatz when $S = R[x]$.
Let's look at ~~special case~~ $R = k$, $S = k[X]$. Look at $|k| = \infty$.
Why is S a Jacobson ring?

$\mathrm{Spec}(S) = \mathrm{Spec}(k[x])$

$$\begin{array}{c} (f(x)) \quad \dots \\ \diagdown \ | \ \dots \\ (0) \end{array}$$

If $P \neq (0)$. Then $P = (f(x))$, $f(x)$ irreducible, P maximal

So $J(\underbrace{k[x]/P}_{\text{field}}) = (0)$; & $k[x]/P \cong k[x]/(f(x))$

$$\begin{array}{c} k[x]/(f(x)) \\ \Big| {\scriptstyle <\infty} \Big) \ \deg = \deg \text{ of } f(x) \\ k = k/k \cap P \end{array}$$

Why is $J(k[x]) = (0)$?

$$J(k[x]) = \bigcap_{f(x) \text{ irred}} (f(x)) \subseteq \bigcap_{\lambda \in k} (x - \lambda) = (0) \quad \text{if } |k| = \infty$$

We'll use the following Black box.
If $R \subseteq S$ are integral domains then if S is finitely generated as an R-module then if S is a field, R is a field.
$1_S = 1_R$

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Proof of Nullstellensatz for $S = R[X]$, R Jacobson
To show S is Jacobson, let $T = S/p = R[X]/p$, $R' = R/R \cap P$
We must show that if T_b is a field, $b \in T \setminus \{0\} \Rightarrow T_{\cancel{b}}$ is a field.
Notice $R[X]/p \cong R'[X]/Q$ for some prime $Q \trianglelefteq R'[x]$, $Q \cap R' = (0)$.
Assume that T_b is a field, ie $T_b = \mathrm{Frac}(T)$
Claim If $Q = (0) \Rightarrow T = R[x]/Q = R'[X]$. Let K = field of fractions of R'. Then $T \subseteq K[X] \subseteq \mathrm{Frac}(T)$
$\Rightarrow T_b \subseteq K[X]_b \subseteq \mathrm{Frac}(T)_b$

$$\begin{array}{ccccc} T_b & \subseteq & K[x]_b & \subseteq & \mathrm{Frac}(T)_b \\ \| & & & & \| \\ \mathrm{Frac}(T) & \subseteq & K[x]_b & \subseteq & \mathrm{Frac}(T) \end{array} \Rightarrow K[x]_b = \mathrm{Frac}(T), \text{ a field.}$$

We showed that $K[x]$ is Jacobson, so by Rabinowitch, $K[x]_b$ a field $\Rightarrow K[x]$ a field. # if is not

So we know $Q \neq (0)$.

$T = R'[x]/Q$, $Q \cap R' = (0)$, $Q \neq (0)$, $T_b = \mathrm{Frac}(T)$, a field.

Goal: Show T is a field.

Since $Q \cap R' = (0)$ we have $\underset{R'[x]/Q}{\overset{\|}{T}} \hookrightarrow \underset{S^{-1}R'[x]/Q,\ S = R' \setminus \{0\}}{\overset{\|}{K[x]/K[x]Q}}$

image in $R'[x]/Q$

Now T_b is a field, so $(K[x]/K[x]Q)_b$ is a field too.

$\because$ it is a localization of T_b

Because $k[x]$ is Jacobson, $Q_1 := K[x]Q$ has the property $k[x]/Q_1$ is a field $\Rightarrow Q_1$ is maximal in $k[x]$.

Also $\begin{array}{c} k[x]/Q_1 \\ \big| {<}\infty \\ k \end{array}$ $\quad Q_1 = (x^d + \cdots + b_0 x^0)$ $\quad b_i \in K \quad b_i \in \mathrm{Frac}(R')$

Clearing denominators, we get $a_d x^d + \cdots + a_0 x^0 \in \overset{\subseteq R[x]}{Q} = k[x] \cap Q_1$ $\quad a_d \neq 0$.

Then $Q_0' := Q R'_{a_d}[x] \ni x^d + \frac{a_{d-1}}{a_d} x^{d-1} + \cdots + \frac{a_0}{a_d}$

Then the image $\bar{x}$ of x in $R'_{a_d}[x]/Q_0'$ satisfies a monic poly

$$(*) \text{ — } \quad \bar{x}^d + \cdots + \frac{a_0}{a_d}\bar{x}^0 = 0,$$

recall $R'[x]/Q = T$ so $\begin{array}{c} R'_{a_d}[x]/Q_0 = T_{a_d} \\ \cup\!\! \mid \\ R'_{a_d} \end{array}$

$\begin{array}{c} T_{a_d} \\ \cup\!\! \mid \\ R'_{a_d} \end{array}$ $\leftarrow$ is generated by $\bar{x}$ over R'_{a_d} So $T_{a_d} \subseteq R'_{a_d} + R'_{a_d}\bar{x} + \cdots + R'_{a_d}\bar{x}^{d-1}$ using $(*)$

By assumption, T_b is a field $\Rightarrow T_{a_d b}$ is a field $\Rightarrow (R'_{a_d}[x])_b$ is a field

$\begin{array}{c} T_{a_d} \\ | \\ R'_{a_d} \end{array}$ finite module over R'_{a_d}

Claim: $\exists\, m \geq 1$, $c_0, \ldots, c_m \in R'_{a_d}$ not all 0 st $c_0 + \cdots + c_m b^m = 0$

Why? $1 = 1'$

$b = a_{10} + a_{11}\bar{x} + \cdots + a_{1\,d-1}\bar{x}^{d-1}$ $\quad a_{ij} \in R'_{a_d}$

$b^2 = a_{20} + \cdots + a_{2\,d-1}\bar{x}^{d-1}$

$\vdots$

Think of

$b^i \rightsquigarrow (a_{i0}, \ldots, a_{i\,d-1}) \in (R'_{a_d})^{d-1} \subseteq K^{d-1}$

If $m > d$, ~~then~~ $1, b, \ldots, b^m$ are lin dep over K

So $\exists\, \gamma_1, \ldots, \gamma_m \in K$, not all zero, st $\gamma_0 + \gamma_1 b + \cdots + \gamma_m b^m = 0$.

Clear denom $\Rightarrow c_0 + c_1 b + \cdots + c_m b^m = 0$

WLOG $c_0, c_m \neq 0$. So

$$1 = b\left(\frac{-c_1}{c_0} + \cdots + \frac{-c_m}{c_0} b^{m-1}\right) \text{ in } (R'_{a_d})_{c_0}[X]/R'_{a_d c_0}Q$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{\tilde{Q}}$

This means b is a unit in

$$R'_{a_d c_0}[X]/\tilde{Q} \quad (R'[X]/Q)_{a_d c_0} = T_{a_d c_0}$$

Now T_b field $\Rightarrow (T_{a_d c_0})_b$ is a field. But b is a unit in $T_{a_d c_0} \Rightarrow T_{a_d c_0}$ is a field

Now

$$\begin{array}{c} T_{a_d} \\ \big| \text{ finite module} \\ R'_{a_d} \end{array} \Longrightarrow \begin{array}{c} T_{a_d c_0} \\ \big| \text{ finite module} \\ R'_{a_d} \end{array} \quad \text{(check)}$$

$(R' = R/R\cap P)$

By the black box from last time

$T_{a_d c_0}$ a field $\Rightarrow R'_{a_d c_0}$ a field $\underset{\text{Rabinowitch}}{\Longrightarrow} R'$ a field

$\Longrightarrow R' = K$

So $S = R'[X]/Q = k[x]/Q \leftarrow$ know Jacobson already ✓

Last step. Need to show $\mathfrak{M} \trianglelefteq S = R[x]$

$\mathfrak{M} \cap R = \mathfrak{N}$ is maximal in R & $\quad \begin{array}{c} S/\mathfrak{M} \\ \big|_{<\infty} \\ R/\mathfrak{N} \end{array}$

But we ~~we~~ just showed if $\mathfrak{M} \subseteq S = R[X]$ is maximal

$R' = R/R\cap\mathfrak{M}$

Then R' is a field

Why? $S/\mathfrak{M} = R'[x]/P$, ~~Prob~~ $\quad P \cap R' = (0)$

$\quad\quad\quad\quad\quad\quad\quad\quad$ field $\curvearrowright$ $R' \cap P = (0)$ every in R' is a unit $\Rightarrow R = K$

& $S/\mathfrak{M} = k[x]/P \leftarrow$ maximal

$\quad\quad\quad\quad\quad\quad\quad\quad \big|_{<\infty}$