2015 01 29 (2)

~~T~~The Jacobson Radical

Def Let R be a ring. We define the Jacobson Radical of R to be ~~$J(R)$~~

$$J(R) := \bigcap_{M \text{ maximal ideal}} M \supseteq \bigcap_{P \text{ prime ideal}} P = \sqrt{(0)}$$

Ex: What is $J(\mathbb{Z})$? $\bigcap p\mathbb{Z} = (0) = \sqrt{(0)}$

Ex $R = \left\{ \frac{a}{b} ; a, b \in \mathbb{Z}, b \text{ odd} \right\}$ This is a ring. What is $J(R)$?

What is a maximal ideal of R?

$$P = \left\{ \frac{a}{b} ; a \text{ even}, b \text{ odd} \right\}$$

Note we have a homomorphism $\varphi : R \to \mathbb{Z}/2\mathbb{Z}$

$$\varphi\left(\frac{a}{b}\right) = a + 2\mathbb{Z} = \frac{a}{b} + 2\mathbb{Z}$$

$$\ker(\varphi) = P$$

In fact, P is the only maximal ideal of R.
Why? Suppose $Q \subsetneq R$ is a proper ideal.
We claim that $Q \subseteq P$. To see this, assume $Q \not\subseteq P$. Then $\exists \frac{a}{b} \in R \setminus P$. We know a must be odd. But then $\frac{b}{a} \in R$, so $\frac{a}{b}$ is a unit $\Rightarrow 1 \in Q$ ※
So P is the only maximal ideal.

Characterization of $J(R)$:

Proposition: $x \in J(R) \iff 1 + ax$ is a unit of R $\forall a \in R$

Proof ($\Rightarrow$): $ax \in J(R)$. Suppose $1+ax$ is not a unit. Then $R(1+ax) \subsetneq R$ so $\exists$ max ideal M st $1+ax \in M$. So $ax \in J(R) \subseteq M \Rightarrow 1 \in M$ ※
($\Leftarrow$) suppose $x \notin J(R)$. So $\exists$ maximal ideal M st $x \notin M$. So $x + M \in R/M$ is non-zero. So $\exists a \in R$ st $-ax + M = 1 + M$
So $1 + ax \in M \Rightarrow 1 \in M$ ※ ∎

Def] A ring R is called a Jacobson ring if for every prime ideal P of R we have $J(R/P) = (0)$.

Ex A field is a Jacobson ring
Ex $\mathbb{Z}$ is a Jacobson ring

2015 01 30

By correspondence, being a Jacobson ring means
$$\bigcap_{\substack{M \supseteq P \\ \text{max}}} M = P \quad \forall \text{prime ideal } P, \text{ so } \bigcap_{M \text{ max}} M = \bigcap_{P \text{ prime}} P.$$

Nakayama's Lemma: Let R be a ring & let M be a finitely generated R-module. If $J(R)M = M$ then $M = (0)$.

E.g. $R = \{\frac{f(x)}{g(x)} ; f(x), g(x) \in \mathbb{C}[x], g(0) \neq 0\}$
$J(R) = xR$, xR is a max ideal of R & xR is the unique maximal ideal.

$$R \xrightarrow[\text{onto}]{\varphi} \mathbb{C} \qquad \ker \varphi = xR$$
$$f(x)/g(x) \longmapsto f(0)/g(0) \qquad R/xR \cong \mathbb{C} \text{ field}$$

xR is the unique maximal ideal. So $J(R) = xR$
$M = \mathbb{C}(x)$, $r \in R$, $f(x)/g(x) \in M$
$$r \frac{f(x)}{g(x)} \in M$$

$J(R)M = M$ why?

If $\frac{a(x)}{b(x)} \in M$ then $\frac{a(x)}{b(x)} = \underbrace{x}_{\in J(R)} \underbrace{\frac{a(x)}{b(x)x}}_{\in M}$ So $J(R)M = M$ & $M \neq (0)$

Why is Nakayama's lemma useful?
If R is a local ring with unique maximal ideal P & if M is a f.g. R-module, then
$$M/PM$$
is an R/P-module = F-vector space

2015 01 30

If $m_1, \dots, m_d \in M$ have the property that
$\overline{m}_1, \dots, \overline{m}_d \in M/PM$
form a basis for M/PM as an F-vector space then
$M = Rm_1 + \cdots + Rm_e$

Proof: Let $N = Rm_1 + \cdots + Rm_e \subseteq M$. Let $A = M/N$. Then
$J(R) \cdot A = PA = P(M/N) = (PM+N)/N = A$
But $PM + N = M$ because $M/PM = \langle \overline{m}_1, \dots, \overline{m}_d \rangle$ so $M/PM = N \Rightarrow PM + N = M$
So by Nakayama's lemma, $A = 0 \Leftrightarrow M = N$

Proof (of lemma)
Suppose $M \neq (0)$. Then because M is finitely generated and non-zero, $\exists d \geq 1$ & $m_1, \dots, m_d \in M$ st $M = Rm_1 + \cdots + Rm_d$. Moreover, we may assume that d is minimal wrt there being a gen set of size d.
Since $M = J(R)M = \{j_1 m_1 + \cdots + j_d m_d \,;\, j_i \in J(R)\}$
We have $m_d = j_1 m_1 + \cdots + j_d m_d$ for some $j_i \in J(R)$. So
$(1 - j_d) m_d = j_1 m_1 + \cdots + j_{d-1} m_{d-1}$
$\Rightarrow m_d = (1-j_d)^{-1} j_1 m_1 + \cdots + (1-j_d)^{-1} j_{d-1} m_{d-1} \in Rm_1 + \cdots + Rm_{d-1}$
as $1 - j_d$ is unit by characterization of $j_d \in J(R)$.
So $M = Rm_1 + \cdots + Rm_{d-1}$ $\#$ ☒

Localization

Let R be a ring. A subset $S \subseteq R \setminus \{0\}$ is called a multiplicatively closed subset of R if $1 \in S$ and $s_1, s_2 \in S \Rightarrow s_1 s_2 \in S$.
Given a ring R & a multiplicatively closed subset $S \subseteq R$, we can define the localization of R wrt S, which we denote
$S^{-1}R$

As a set
$$S^{-1}R = R \times S / \sim$$
where
$$(r_1, s_1) \sim (r_2, s_2) \Leftrightarrow \exists s' \in S \text{ st } s'(s_1 r_2 - s_2 r_1) = 0.$$

Claim: $\sim$ is an equivalence relation on $R \times S$

Transitivity is the only challenge. Suppose $(r_1,s_1)\sim(r_2,s_2)$ & $(r_2,s_2)\sim(r_3,s_3)$. So $\exists s', s'' \in S$ st

① $s'(s_2r_1 - s_1r_2) = 0$, $s''(s_3r_2 - s_2r_3) = 0$. ②

We want $s''' \in S$ st

$$s'''(s_3r_1 - s_1r_3) = 0$$

$0 = s''s_3① = s''s_3^{s'}s_2r_1 - s''s_3^{s'}s_1r_2$

$0 = s_1s''② =$

add $0 = \ldots$

take $s''' = s's_2s''$

$R\times S/\sim$ is a ring. We write $s^{-1}r$ for $[(r,s)]$

$$s_1^{-1}r_1 \cdot s_2^{-1}r_2 = (s_1s_2)^{-1}r_1r_2$$

$$s_1^{-1}r_1 + s_2^{-1}r_2 = (s_1s_2)^{-1}(s_2r_1 + s_1r_2)$$

Check well-defined + ring axioms.

2015 02 03 (TA lect)

We know $\mathbb{Z} \hookrightarrow \mathbb{Q}$, but it is not true in general that R embeds into $S^{-1}R$.

eg $R = \mathbb{Z}\times\mathbb{Z}$, $S = ((\mathbb{Z}\setminus\{0\})\times\{0\}) \cup \{(1,1)\}$

Then S is multiplicatively closed, $0 \notin S$.

Let $(a,b), (c,d) \in R$, $(s,t), (s',t') \in S$.

Then $((a,b),(s,t)) \sim ((c,d),(s',t')) \iff \exists (u,0) \in S$ such that

$$(u,0)\cdot(a,b)\cdot(s',t') = (u,0)\cdot(c,d)\cdot(s,t)$$

This happens if and only if $as' = cs$. Therefore

$$S^{-1}R \longrightarrow \mathbb{Q}$$
$$[(a,b),(s,t)] \longmapsto \frac{a}{s}$$

In particular, $R \to S^{-1}R$, $r \mapsto 1^{-1}r$ is not an embedding ($(0,b)$, where $b\in\mathbb{Z}$, is in the kernel).

We say that S is <u>regular</u> if S does not contain zero divisors.

If S is regular then $R \to S^{-1}R : r \mapsto 1^{-1}r$ is an embedding.

If $1^{-1}r = 1^{-1}r'$ then $\exists s\in S$ st $s(r-r') = 0$, so $r = r'$.

2015 02 03

Note: If $I \subseteq R$ is an ideal and $S \subseteq R$ is multiplicatively closed, then
$$S^{-1}I = \{s^{-1}r \,;\, (s,r) \in S \times I\}$$
is an ideal of $S^{-1}R$.

Universal Property of Localizations

Proposition: Let R be a ring and S be a multiplicatively closed ~~subset~~ regular subset of R. Then if T is a ring and $\varphi : R \to T$ is a ring homomorphism such that $\varphi(S) \subseteq T^*$ (units of T) then φ extends to a homomorphism from $S^{-1}R$ into T, and this extension is unique. If φ is 1-1, then the extension is 1-1 as well.

Proof: Existence: We define $\psi : S^{-1}R \to T$ by
$$\psi(s^{-1}r) = \varphi(s)^{-1}\varphi(r).$$
Note $\varphi(s)$ is invertible as $\varphi(S) \subseteq T^*$. Next we show that ψ is well-defined. If $s_1^{-1}r_1 = s_2^{-1}r_2$ then $\exists s_3 \in S$ such that
$$s_3(s_1r_2 - s_2r_1) = 0.$$
Since s_3 is not a zero divisor, $s_1r_2 = s_2r_1$. Then $\varphi(s_1)\varphi(r_2) = \varphi(s_2)\varphi(r_1)$. Rearranging shows ψ is well-defined. If $r \in R$ then
$$\psi(1^{-1}r) = \varphi(1)^{-1}\varphi(r) = \varphi(r)$$
so ψ extends φ. Let's show that φ is a homomorphism:
$$\begin{aligned}
\psi(s_1^{-1}r_1 s_2^{-1}r_2) &= \psi((s_1s_2)^{-1}(r_1r_2)) \\
&= \varphi(s_1s_2)^{-1}\varphi(r_1r_2) \\
&= \varphi(s_1)^{-1}\varphi(s_2)^{-1}\varphi(r_1)\varphi(r_2) \\
&= \varphi(s_1)^{-1}\varphi(r_1)\varphi(s_2)^{-1}\varphi(r_2) \\
&= \psi(s_1^{-1}r_1)\psi(s_2^{-1}r_2)
\end{aligned}$$
$$\begin{aligned}
\psi(s_1^{-1}r_1 + s_2^{-1}r_2) &= \psi((s_1s_2)^{-1}(s_2r_1 + s_1r_2)) \\
&= \varphi(s_1s_2)^{-1}\varphi(s_2r_1 + s_1r_2) \\
&= \varphi(s_1)^{-1}\varphi(s_2)^{-1}\left(\varphi(s_2)\varphi(r_1) + \varphi(s_1)\varphi(r_2)\right) \\
&= \varphi(s_1)^{-1}\varphi(r_1) + \varphi(s_2)^{-1}\varphi(r_2) \\
&= \psi(s_1^{-1}r_1) + \psi(s_2^{-1}r_2).
\end{aligned}$$
Uniqueness: If $f : S^{-1}R \to T$ is an extension of φ to a homomorphism, then
$$1 = f(1) = f(s^{-1}s) = f(s^{-1})f(s) = f(s^{-1})\varphi(s) \Rightarrow f(s^{-1}) = \varphi(s)^{-1}.$$
So

$$f(s^{-1}r) = f(s^{-1})f(r) = \varphi(s)^{-1}\varphi(r) = \psi(s^{-1}r).$$

Finally suppose that φ is 1-1, and let $s^{-1}r \in \ker(\psi)$. Then

$$\varphi(s)^{-1}\varphi(r) = 0,$$

so $\varphi(r) = 0$, hence $r = 0$, by injectivity of φ. So ψ is 1-1. ■

Uniqueness in the proposition gives that $S^{-1}R$ is the unique ring (up to isomorphism) with the universal property.

Suppose B_1, B_2 have the universal property.

$$\begin{tikzcd}
 & B_1 \arrow[d, dashed, "\varphi"] \\
R \arrow[ur, hook] \arrow[r, hook] \arrow[d, hook] & B_2 \arrow[dl, dashed, "\exists\psi"] \\
B_1 &
\end{tikzcd}$$

By uniqueness, $\psi \circ \varphi$ is the identity on B_1. Similarly $\varphi \circ \psi$.
Hence $B_1 \cong B_2$.

<u>Eg</u>

1) Suppose $f \in R$ is not nilpotent, let $S = \{1, f, f^2, \ldots\}$. We define

$$R_f := S^{-1}R.$$

2) Let P be a prime ideal in R and let $S = R \setminus P$. Then S is mult. closed. Write

$$R_P := S^{-1}R$$

Note that R_P has an ideal

$$PR_P = \{s^{-1}r \,;\, s \notin P,\ r \in P\}.$$

We'll show that PR_P is a prime ideal.

Let $T = \mathrm{Frac}(R/P)$. Define $\varphi : R \to T$,

$$r \longmapsto \frac{r+P}{1}$$

If $s \in S = R/P$ then $\varphi(s) \in T^*$. Let $\psi : R_P \to T$ be the unique extension of φ (by universal property).

Claim: φ is onto.

If $x = \frac{a+P}{b+P}$, $b \notin P$. So $b \in S = R \setminus P$, and $\psi(b^{-1}a) = x$.

Hence $\ker(\psi)$ is a maximal ideal of R_P, since T is a field. But $\ker(\psi) = \{s^{-1}r \,;\, s \in S,\ r \in P\} = PR_P$, so PR_P is a maximal ideal of R_P

2015 02 03

Recall that a ring R is a <u>local ring</u> if it has a unique maximal ideal.

Eg. fields are local rings, $\mathbb{Z}$ is not a local ring

2015 02 05

Criterion to check (R,M) is local: M is the unique maximal ideal of R if and only if $1+x$ is a unit $\forall x\in M$.

Proof: If M is unique,

$$J(R) = \bigcap_{P \text{ prime}} P = M$$

hence $1+x$ is a unit $\forall x\in J(R)=M$.
If M is not unique, there exists a maximal ideal $Q\neq M$. Then $Q+M=R$. So $\exists q\in Q, +x\in M$ such that

$$q-x = q+(-x) = 1 \Rightarrow q = 1+x \text{ not a unit}$$

But $x\in M$ with $1+x=q$ not a unit (as $Q\subsetneq R$). ∎

<u>Ex</u> $PR_P \trianglelefteq R_P$ is the unique maximal ideal of R_P.

Why? Let $x\in PR_P$. Then $x=s^{-1}a$ with $s\in S=R\setminus P$, $a\in P$. So

$$1+x = s^{-1}s+s^{-1}a = s^{-1}(s+a) = s^{-1}t, \quad t=s+a$$

So $(s^{-1}t)^{-1} = t^{-1}s$ as $t\in S$.

Let R be a ring and let $S\subseteq R$ be multiplicatively closed and regular. Given an ideal $J\trianglelefteq R$, we say that J is <u>S-saturated</u> if whenever $s\in S$ & $x\in R$ are such that $sx\in J$ we necessarily have $x\in J$.

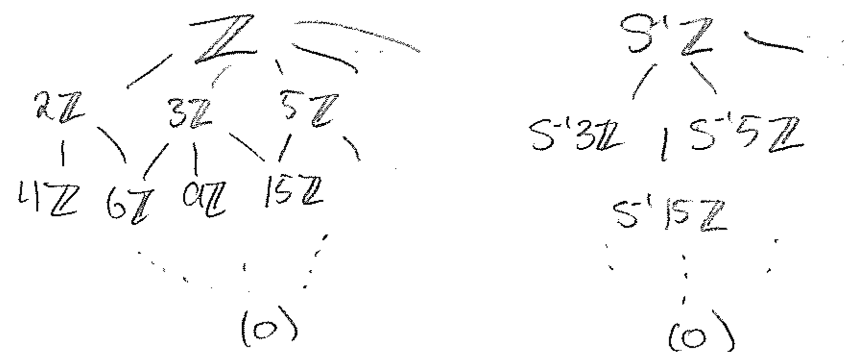
Eg $R=\mathbb{Z}$, $S=\{1,2,4,8,\dots\}$. Then $3\mathbb{Z}$ is S-saturated, $4\mathbb{Z}$ is not.

Let R be a ring and let S be a multiplicatively closed set of regular elements. Then there is an inclusion-preserving bijection between the poset of proper ideals of $S^{-1}R$ and the poset of S-saturated ideals of R that intersect S trivially

$R \subseteq S^{-1}R$ $\qquad I \trianglelefteq S^{-1}R \xmapsto{f} I \cap R$

$\qquad\qquad S^{-1}J \xleftarrow{g} J \trianglelefteq R,\ J \cap S = \emptyset,\ J$ S-saturated

Ex $R = \mathbb{Z}$, $S = \{1, 2, 4, 8, \dots\}$
Then $S^{-1}R$ has proper ideals $S^{-1}(nR)$, $n > 1$, n odd



First, if $I \underset{\neq}{\trianglelefteq} S^{-1}R$, then claim $g \circ f(I) = I$.
$\qquad = S^{-1}(I \cap R)$

Notice $I \cap R \subseteq I$ & since I is an ideal, $S^{-1}(I \cap R) \subseteq I$.
Conversely, let $x \in I$. Then $x = s^{-1}a$ for some $s \in S$, $a \in R$. Since $I \trianglelefteq S^{-1}R$, $sx = a \in I$ & so $a \in I \cap R \Rightarrow x = s^{-1}a \in S^{-1}(I \cap R)$.

If $J \trianglelefteq R$, $J \cap S$, J S-saturated, then claim $f \circ g(J) = J$.
$\qquad = S^{-1}J \cap R$.
Proof: $S^{-1}J \cap R \supseteq 1^{-1}J \cap R = J$.
To show converse, let $x \in S^{-1}J \cap R$. Then $x \in S^{-1}J$ so $\exists s \in S$, $j \in J$ st $x = s^{-1}j$. So $sx = j \in J \Rightarrow x \in J$ since $x \in R$ and J is S-saturated.

Notice that if $I \underset{\neq}{\trianglelefteq} S^{-1}R$, then $f(I)$ is S-saturated.
? Why? If $x \in R$ and $sx \in f(I) = I \cap R \Rightarrow x \in S^{-1}I \cap R = I$. So I is S-saturated.

Next, if $J \trianglelefteq R$, $J \cap S = \emptyset$, J S-saturated $\Rightarrow g(J) = S^{-1}J$ is proper. If not, $1 \in S^{-1}J \Rightarrow 1 = s^{-1}j$ for some $s \in S$, $j \in J$.
$\Rightarrow j = s \Rightarrow J \cap S \neq \emptyset$

2015 02 05

Corollary: Let R be a ring & let $S \subseteq R$ be a multiplicatively closed subset of regular elements. If R is Noetherian $\Rightarrow S^{-1}R$ is Noetherian.

Converse is false: $R = \mathbb{C}[x_1, \ldots]$, $S = R \setminus \{0\}$, $S^{-1}R = \mathbb{C}(x_1, \ldots)$ field.

Proof: Let $J_1 \trianglelefteq J_2 \trianglelefteq \cdots$ be a chain of ideals in $S^{-1}R$. If $J_n = S^{-1}R$ for some $n \Rightarrow J_n = J_{n+1} = \cdots$.
Otherwise, we can apply the map f to get a chain $f(J_1) \trianglelefteq f(J_2) \cdots$ of ideals in R. Since R is Noetherian $\Rightarrow \exists n$ st $f(J_n) = f(J_{n+1}) \cdots$
$\Rightarrow J_n = g(f(J_n)) = J_{n+1} = g(f(J_{n+1})) = \cdots$
So $S^{-1}R$ is Noetherian. ∎

One thing to point out: These bijections (f and g) restrict to bijections between prime ideals of $S^{-1}(J)$ and $\{P \subseteq R;\ P \text{ prime}, P \cap S = \emptyset\}$.
R?

2015 02 06

Given a ring R, we let $\mathrm{Spec}(R)$ denote the set of prime ideals of R.

Remark: $\mathrm{Spec}(R)$ is a poset with respect to $\subseteq$.

ex $\mathrm{Spec}(\mathbb{Z})$ $2\mathbb{Z}$ $3\mathbb{Z}$ $5\mathbb{Z}$ $7\mathbb{Z}$ . . . .
$\mathbb{Z}$

Last time, R, $S \subseteq R$ multiplicatively closed & regular.
bijection (inclusion-preserving)
$\{\text{proper ideals of } S^{-1}R\} \longleftrightarrow \{S\text{-saturated ideals of } R \text{ that intersect } S \text{ trivially}\}$
$I \leq S^{-1}R \xmapsto{f} f(I) = I \cap R$
$S^{-1}J = g(J) \xleftarrow{g} J \leq R$
These bijections send prime ideals to prime ideals
If $P \leq S^{-1}R$ is prime $\Rightarrow f(P) = P \cap R \subseteq R$ is prime in R
Why? Suppose that $a, b \in R$, $ab \in f(P)$. Then $ab \in P$ so $a \in P$ or $b \in P$.

Conversely, if $Q \subseteq R$, Q prime, $Q \cap S = \emptyset$. (Notice this implies Q is S-saturated)
Why? Suppose $s \in S$, $x \in R$, & $sx \in Q \Rightarrow \cancel{s \in Q}$ or $x \in Q$

Then $g(Q) \subseteq S^{-1}R$ is prime. Suppose $s_1^{-1}a, s_2^{-1}b \in S^{-1}R$ & $s_1^{-1}a \, s_2^{-1}b \in g(Q)$. So $(s_1s_2)^{-1}ab \in g(Q) \Rightarrow ab \in g(Q) \cap R = Q$
$\Rightarrow a \in Q$ or $b \in Q$.

So the maps f and g restrict:

$$\mathrm{Spec}(S^{-1}R) \underset{g}{\overset{f}{\rightleftarrows}} \{Q \in \mathrm{Spec}(R);\ Q \cap S = \emptyset\}$$

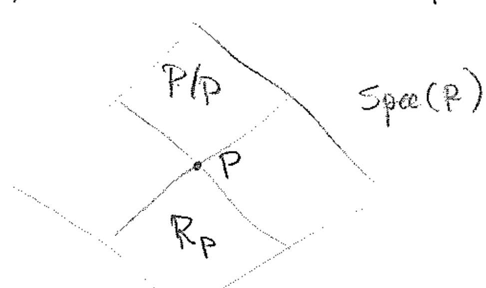
Special cases: Let R be a ~~prime ideal~~ ring, x be a non-zero divisor.
$S = \{1, x, x^2, \ldots\}$.
Then $\mathrm{Spec}(R_x) \underset{g}{\overset{f}{\rightleftarrows}} \{Q \in \mathrm{Spec}(R);\ x \notin Q\}$.

<u>Ex2</u> Let R be an integral domain. Let P be a prime ideal, let $S = R \setminus P$.
Then $S^{-1}R = R_P$. Then

$$\mathrm{Spec}(R_P) \overset{1-1}{\longleftrightarrow} \{Q \in \mathrm{Spec}(R);\ Q \cap S = \emptyset\}$$
$$= \{Q \in \mathrm{Spec}(R);\ Q \subseteq P\}$$

This gives another proof that R_P is a local ring.

$\mathrm{Spec}(R)$	$\mathrm{Spec}(R_P)$	$\mathrm{Spec}(R/P)$
$\parallel$	$\updownarrow$ 1-1	$\updownarrow$ 1-1
all prime ideals of R	$\{$all $Q \in \mathrm{Spec}(R);\ Q \subseteq P\}$	$\{Q \in \mathrm{Spec}(R);\ Q \supseteq P\}$



On the second assignment, you'll show that if K is a field extension of a field F. Then K is finitely generated as an extension of F.
Can be deduced by showing that $K \otimes_F K$ is Noetherian.
In fact, in the assignment, you'll show

$K \otimes_F K$ is Noetherian $\Rightarrow$ any field L with $F \subseteq L \subseteq K$ is finitely generated over F "it's like, you know, I don't even know what it's like"

"I was going to say it's like wrestling your grandfather, but what does that even mean?"

Theorem: Let K/F be finitely generated extension. If $F \subseteq L \subseteq K$ then L/K is finitely generated too!

Proof: It suffices, by A2, to show that $K \otimes_F K$ is Noetherian.

Step 1: Since K/F is finitely generated, $\exists a_1, .., a_d \in K$ such that

$$K = F(a_1, .., a_d).$$

Let

$$A = F[a_1, .., a_d]$$

Then A is a finitely generated F-algebra. By HBT, A is Noetherian.

Step 2 On A2, you'll show that if A & B are f.g. F-algebras Then $A \otimes_F B$ is also f.g.

So since $A = F[a_1, .., a_d]$ is f.g. as an F-algebra, so is $A \otimes_F A$. So by HBT, $A \otimes_F A$ is Noetherian.

Step 3: Let $S = A \setminus \{0\}$, multiplicatively closed and regular. Let

$$T = \{ s_1 \otimes s_2 ; s_1, s_2 \in S\}.$$

We'll show that

$$T^{-1}(A \otimes_F A) \cong S^{-1}A \otimes_F S^{-1}A = K \otimes_F K$$

& T is regular.

So because $A \otimes A$ is Noetherian $\Rightarrow T^{-1}(A \otimes_F A)$ is Noetherian $\Rightarrow$ $K \otimes_F K$ is Noetherian ✓

Proposition: Let A, B be F-algebras & let S, T be multiplicatively closed, regular subsets of A & B respectively.
Then if $U = \{ s \otimes t ; s \in S, t \in T\}$, then $U^{-1}(A \otimes_F B) \cong S^{-1}A \otimes_F T^{-1}B$.

Proof: Let

$$f : A \xrightarrow{1-1} S^{-1}A \qquad g : B \xrightarrow{1-1} T^{-1}B$$
$$a \longmapsto 1^{-1}a \qquad b \longmapsto 1^{-1}b$$

So we have a homomorphism

$$f \otimes g : A \otimes_F B \longrightarrow S^{-1}A \otimes_F T^{-1}B$$

Notice that if $s \otimes t \in U \longmapsto f \otimes g(s \otimes t) = f(s) \otimes g(t)$
$= s \otimes t$ ← unit in $S^{-1}A \otimes T^{-1}B$
inverse $s^{-1} \otimes t^{-1}$

So $f \otimes g$ extends to a homomorphism

$$f \otimes g : U^{-1}(A \otimes_F B) \longrightarrow S^{-1}A \otimes_F T^{-1}B$$

by UP of localization

All that remains is to show $f \otimes g$ is an isomorphism.

1) $f \otimes g$ is onto: $S^{-1}A \otimes_F T^{-1}B$ is generated by things of the form

$$s^{-1}a \otimes t^{-1}b = f \otimes g\left((s \otimes t)^{-1}(a \otimes b)\right)$$

Warning: $f: A \xrightarrow{1-1} C$, $g: B \xrightarrow{1-1} D \not\Rightarrow f \otimes g : A \otimes_R B \to C \otimes_R D$ is 1-1

eg $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ $[a] \mapsto [2a]$ is 1-1,

but $f \otimes f(1 \otimes 1) = 2 \otimes 2 = (4 \otimes 1) = 0 \otimes 1 = 0$ and $1 \otimes 1 \neq 0$ by A1

we saw $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2$

2015 02 10

Claim: If A, B, C, D are F-v.s. & $f: A \xrightarrow{1-1} C$, $g: B \xrightarrow{1-1} D$ $\Rightarrow f \otimes g : A \otimes_F B \xrightarrow{1-1} C \otimes_F D$.

Proof: Let $\{x_\alpha\}_{\alpha \in I}$ be an F-basis for A.

Let $\{y_\beta\}_{\beta \in J}$ " B.

Since f & g are 1-1, $\{f(x_\alpha)\}_{\alpha \in I} \subseteq C$ is lin. indep. and $\{g(y_\beta)\}_{\beta \in J} \subseteq D$ "

Since we can extend $\{f(x_\alpha)\}_{\alpha \in I}$ to a basis $\{z_\gamma\}_{\gamma \in I'}$ for C, we " $\{g(y_\beta)\}_{\beta \in J}$ " $\{w_g\}_{g \in J'}$ for D.

Recall that $\{x_\alpha \otimes y_\beta\}_{(\alpha,\beta) \in I \times J}$ forms an F-basis for $A \otimes_F B$.

So if $f \otimes g$ is not 1-1 $\Rightarrow \exists\, c_{\alpha,\beta} \in F$, not all zero, st only finitely many non-zero, st

$$f \otimes g\left(\sum c_{\alpha\beta}\, x_\alpha \otimes y_\beta\right) = 0$$

Notice we may write this ~~of the form~~ as

$$\sum_{(\gamma,g) \in I' \times J'} d_{\gamma,g}\, z_\gamma \otimes w_g = 0$$

where $d_{\gamma,g} = 0$ for all but finitely many (γ, g), and not all zero.

$\times$ as $\{z_\gamma \otimes w_g\}_{(\gamma,g) \in I' \times J'}$ form a basis for $C \otimes_F D$. □

Corollary: $S^{-1}A \otimes_F T^{-1}B \cong (S \otimes T)^{-1}(A \otimes_F B)$

Remark: If S, T regular in A, B resp.

Then $S \otimes T = \{s \otimes t ;\ s \in S, t \in T\} \subseteq A \otimes_F B$ is regular

Why: Let $s \in S$, $t \in T$. Define $f: A \to A$, $a \xmapsto{1-1} sa$, $g: B \to B$, $b \xmapsto{1-1} tb$

so $f \otimes g : A \otimes_F B \xrightarrow{1-1} A \otimes_F R$ $\Rightarrow s \otimes t$ is not a zero divisor.

$a \otimes b \longmapsto (s \otimes t)(a \otimes b)$