[illegible]

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Proof: Let $P \trianglelefteq R$ be maximal, $F = R/P$ field.

$$R^X \otimes_R R/P \cong (R/P)^X = F^X$$

$R^X \cong R^Y \Rightarrow F^X \cong F^Y$ as R-modules.

Fact: If M & N are isomorphic R-modules then they are iso. as R/I-modules, where $I = \mathrm{Ann}(M) = \mathrm{Ann}(N) \trianglelefteq R$.

Proof: If $\phi: M \to N$ is an R-module isomorphism. Create R/I-module isomorphism $\Phi(m) = \phi(m)$. Check this works.

Question: What is annihilator of F^X?

Answer: $F^X = (R/P)^X$, $\mathrm{Ann}(F^X) = P = \mathrm{Ann}(F^Y)$

And so $F^X \cong F^Y$ as R-modules implies $F^X \cong F^Y$ as R/P-modules implies (A1) $|X| = |Y|$. $\square$

Algebras

Let R be a ring. ~~An R-algebra~~

An R-algebra S is just a ring equipped with a ring homomorphism

$$\alpha: R \to S$$
$$\alpha(1_R) = 1_S$$

Ex Every ring is a $\mathbb{Z}$-algebra

$$\alpha: \mathbb{Z} \to R$$
$$\alpha(n) = n \cdot 1_R$$

Ex $\mathbb{C}[x,y]$ is a $\mathbb{C}$-algebra

$$\alpha: \mathbb{C} \to \mathbb{C}[x,y]$$
$$\alpha(c) = c$$

Ex $\mathbb{C}$ is an $\mathbb{R}$-algebra

$$\alpha: \mathbb{R} \to \mathbb{C}$$
$$\alpha(c) = c$$

Base Change / Extension of Scalars

Notice that if S is an R-algebra then S inherits an R-module structure as well.
Given $r \in R$, $s \in S$, define $r \cdot s = \alpha(r)s$, $\alpha: R \to S$.
Notice that if S is an R-algebra & M is an R-module then we can form the tensor product $S \otimes_R M$ R-module.
But notice we can endow $S \otimes_R M$ with an S-module structure via the rule

$$s_1 \cdot (s_2 \otimes m) = s_1 s_2 \otimes m$$

(extend linearly). Check it works.

What have we done?

R	M	S	$S \otimes_R M$
ring	R-module	R-algebra	S-module

This process is called either base change or extension of scalars.

Reasons why this is used.
Imagine V is a $\mathbb{Q}$-v.s.
$\mathbb{C}$ is a $\mathbb{Q}$-alg.
$V_{\mathbb{C}} := V \otimes_{\mathbb{Q}} \mathbb{C}$ is a $\mathbb{C}$-vector space
If $V \cong \mathbb{Q}^n$, $V \otimes_{\mathbb{Q}} \mathbb{C} \cong (\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$
$\cong \oplus_{i=1}^{n} (\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{C}) \cong \mathbb{C}^n$

Useful construction: If A & B are R-algebras. Then we can form $A \otimes_R B$ R-module.
But it actually has the structure of an ~~R-module~~ R-algebra

Ex2 $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$

$$\alpha(1) \otimes 1 = 1 \otimes \beta(r)$$
$$r(1 \otimes 1) = r(1 \otimes 1)$$

$$\gamma: R \longrightarrow A \otimes_R B$$
$$\gamma(r) = \alpha(1) \otimes 1 = 1 \otimes \beta(r)$$

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Noetherian Rings

Let R be a ring. We say that R is Noetherian if every ascending chain of ideals in R

$$I_1 \trianglelefteq I_2 \trianglelefteq \dots$$

terminates, ie $\exists n$ st $I_n = I_{n+1} = \cdots$

Ex a PID is Noetherian (in particular, fields)

Ex $\mathbb{C}[x_1, x_2, \dots]$, $(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \cdots$

Def Let R be a ring & let M be an R-module. Then M is Noetherian if every ascending chain of submodules of M terminates.

R Noetherian as a ring $\Leftrightarrow$ R Noetherian as an R-module

Ex If $R = \mathbb{Z}$ and $M = \mathbb{Q}$ then $\mathbb{Q}$ is not Noetherian

Why? $\mathbb{Z}\cdot 1 \subsetneq \mathbb{Z}\cdot\frac{1}{2} \subsetneq \mathbb{Z}\frac{1}{4} \subsetneq \cdots$

Proposition: Let R be a ring. Then the following are equivalent:

1) R is Noetherian
2) Every $I \trianglelefteq R$ is finitely generated
3) Every collection S of ideals has a maximal element, wrt $\subseteq$.

~~Pf~~ Similarly for modules:

Proposition:

M Noetherian $\Leftrightarrow$ every submodule is fg $\Leftrightarrow$ every non-empty collection of submodules has a maximal element.

Proposition: Let R be a ring & let M be an R-module & let $N \leq M$ be a submodule.

Then M is Noetherian $\Leftrightarrow$ N & M/N are Noetherian.

In other words, $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$

Noeth.

$\Updownarrow$

Noeth.

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Proof: We have $\pi : M \to M/N$ canonical surjection.

<u>Correspondence</u> $\begin{array}{c} M \\ \text{IU} \\ P \end{array} \longmapsto \begin{array}{c} M/N \\ \text{IU} \\ \pi(P) \end{array}$ gives an inclusion-preserving bijection between submodules of M/N & submodules of M that contain N.

($\Rightarrow$) If $N_1 \subseteq N_2 \subseteq \cdots$ is a chain in N then it is also a chain in $M \Rightarrow$ it terminates

If $A_1 \subseteq A_2 \subseteq \cdots$ is a chain in M/N then $\pi^{-1}(A_1) \subseteq \pi^{-1}(A_2) \subseteq \cdots$ is a chain in $M \Rightarrow$ terminates

($\Leftarrow$) Suppose that N & M/N are Noetherian

Let $M_1 \subseteq M_2 \subseteq \cdots$ be a chain in M.

Then $M_1 \cap N \subseteq M_2 \cap N \subseteq \cdots$ is a chain in $N \Rightarrow \exists m$ st $M_m \cap N = M_{m+1} \cap N = \cdots$.

Also $\pi_1(M_1) \subseteq \pi_2(M_2) \subseteq \cdots$ is a chain in M/N, so $\exists p$ st $\pi(M_p) = \pi(M_{p+1}) = \cdots$

Let $n = \max\{m, p\}$. Claim: $M_n = M_{n+1} = \cdots$

Pf: Let $x \in M_{n+1}$. We show $x \in M_n$. So $\pi(x) \in M_{n+1} = M_n$.

So $\exists y \in M_n$ st $\pi(x) = \pi(y)$

Then $\pi(x-y) = 0$ so $x-y \in N$. And $x-y \in M_{n+1}$, so $x-y \in N \cap M_{n+1} = N \cap M_n$ So $x-y = z \in M_n$. So $y \in M_n$. ∎

Corollary: If M & N are Noetherian then $M \oplus N$ is Noetherian

Proof: $M \oplus N / M \oplus (0) \cong N$. ∎

Not true for infinite direct sum.

<u>Ex</u> $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$

Corollary: If R is Noetherian & M is a f.g. R-module then M is Noetherian.

(all of a ...?) Proof. Induction on # generators d.

$d = 1$: $M = Rm$, $R \xrightarrow{\text{onto}} M$, $r \mapsto rm$ $\Rightarrow M \cong R/I$

Then $M \cong R/I$ is Noetherian (because R is Noetherian as an R-module & R/I is a quotient.)

it does make sense but it's up to you to make sense of the sense

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Assume true for $d < n$. Let $M = \langle m_1, \dots, m_n \rangle$. Then $N = \langle m_1, \dots, m_{n-1} \rangle$ is Noetherian by ind. hyp. and $M/N = R(m_n + N)$ so M/N is Noetherian $\Rightarrow$ M Noetherian.

Maximality Principle

Metatheorem: If R is a Noetherian ring & you choose an ideal I in R that is maximal wrt having some "nice" property. Then I is a prime ideal.

Where did we define ideal + module

Theorem: (Noether) Let R be a Noetherian ring & let $I \subsetneq R$ be a proper ideal. Then $\exists m \geq 1$ & prime ideals $P_1, \dots, P_m$ such that

$$P_1 \cdots P_m \subseteq I.$$

Proof: Suppose not. Let $\mathcal{S}$ be the collection of proper ideals $I \subsetneq R$ that do not contain a finite product of prime ideals. By assumption, $\mathcal{S} \neq \emptyset$. Pick $I \in \mathcal{S}$ maximal. We'll show that I must be prime. This would be a CONTRADICTION.

Where did we use R is Noetherian?

If not, $\exists a, b \in R \setminus I$ st $ab \in I$.

Now $J_1 = I + Ra \supsetneq I$, $J_2 = I + Rb \supsetneq I$. Rmk: $J_1 J_2 \subseteq I + Rab \subseteq I$ so J_i are proper

By maximality of I, $J_1, J_2 \notin \mathcal{S}$. So $\exists P_1, \dots, P_m, Q_1, \dots, Q_n$ prime ideals st $P_1 \cdots P_m \subseteq J_1$, $Q_1 \cdots Q_n \subseteq J_2$. So $P_1 \cdots P_m Q_1 \cdots Q_n \subseteq J_1 J_2 \subseteq I$. ∎

Remark: (Important)

If $I \subsetneq R$, R Noetherian, & $P_1 \cdots P_m \subseteq I$, $P_1, \dots, P_m \supseteq I$

Then if Q is a prime ideal & $Q \supseteq I \Rightarrow Q \supseteq P_i$ some i

(WLOG we may ~~assume~~ choose the P_i st they contain I (look at R/I))

Proof: Suppose not. Then for $1 \leq i \leq m$ $\exists a_i \in P_i \setminus Q$

So $a_1 \cdots a_m \in P_1 \cdots P_m \subseteq I \subseteq Q \Rightarrow a_j \in Q$ some j $\#$ ∎

So there are only finitely many primes in R that are minimal wrt ~~containg~~ containing I.

Not true in general.

Eg. $R = \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \cdots$ infinitely many primes above (0) (each copy of $\mathbb{C}$)

Let R be a ring, & let an $I \trianglelefteq R$.
We define the <u>radical</u> of I

$$\sqrt{I} := \bigcap \{P \,;\, P \supseteq I \text{ prime}\}$$

<u>Ex</u> Let $R = \mathbb{Z}_4$. What is $\sqrt{(0)}$? $2\mathbb{Z}_4$.

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Theorem: Let R be a ring. Then $\sqrt{(0)}$ is a nil ideal, ie if $x \in \sqrt{(0)}$ then $\exists n \geq 1$ st $x^n = 0$.
Proof: Suppose not. Then $\exists x \in I = \sqrt{(0)}$ st x is not nilpotent.
Let $T = \{1, x, x^2, \ldots\}$. Then $0 \notin T$. Let

$$\mathcal{S} = \{J \trianglelefteq R \,;\, J \cap T = \emptyset\}$$

Then $\mathcal{S} \neq \emptyset$ because $(0) \cap T = \emptyset$ means $(0) \in \mathcal{S}$.
If R is Noetherian, we just let J be a maximal element of $\mathcal{S}$.
If R is not Noetherian, we use Zorn's lemma to produce a maximal element. (easy)
Let J be a maximal element of $\mathcal{S}$. We claim that J is prime.
To see this, observe that if J is not prime then $\exists a, b \in R \setminus J$ st $ab \in J$. Then $J_1 = J + Ra \supsetneq J$, $J_2 = J + Rb \supsetneq J$.
Since J is maximal in $\mathcal{S}$, $J_1, J_2 \notin \mathcal{S}$. So $\exists n_1, n_2 \geq 1$ st $x^{n_1} \in J_1$, $x^{n_2} \in J_2$. Then $x^{n_1+n_2} \in J_1 J_2 \underset{\uparrow}{\subseteq} J$ $\times$
distribute the product

So $J \in \mathcal{S}$ is prime. So $x \in J$ because

$$J \supseteq \bigcap_{P \supseteq (0),\, P \text{ prime}} P = \sqrt{(0)} \ni x.$$

Contradiction. ∎

Corollary: If $I \trianglelefteq R$ & $x \in \sqrt{I}$ then $\exists n \geq 1$ st $x^n \in I$.
(This is why $\sqrt{I}$ is called the radical.)
Proof: Let $S = R/I$. By correspondence $\sqrt{0}$ in $S \longleftrightarrow \sqrt{I}$ in R.
If $x \in \sqrt{I}$ then $(x+I)^m = 0$ in $S \Rightarrow x^m \in I$.

Exercise: Let R be a ring. Then $a_0 + a_1 x + \cdots + a_n x^n \in R[x]$ is a unit in $R[x]$ if and only if a_0 is a unit & $a_1, \ldots, a_n$ are nilpotent.

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Proposition. Let R be a Noetherian ring & let $I \trianglelefteq R$ be a nil ideal. Then I is nilpotent. ie $\exists n \geq 1$ st $I^n = (0)$.

here it begins

Proof. Suppose not. Let

$$\mathcal{S} = \{ J \trianglelefteq R \, ; \, \pi(I) \text{ is not nilpotent in } R/J \} \quad \pi(I) = (I+J)/J$$

where $\pi : R \to R/J$ is the canonical surjection. Then $\mathcal{S} \neq \emptyset$ as $(0) \in \mathcal{S}$. Let $J \in \mathcal{S}$ be maximal. We claim that J is prime. If not, $\exists a,b \in R \setminus J$ st $ab \in J$. Let $J_1 = J + Ra \supsetneq J$, $J_2 = J + Rb \supsetneq J$. By maximality of J, neither J_1 nor J_2 is in $\mathcal{S}$. So $\pi(J_i)$ is nilpotent for $i \in \{1,2\}$.

Aside: What does it mean to say $\pi(I)$ is nilpotent in R/I? $\pi : R \to R/L$. Answer: $\pi(I)$ nilpotent $\iff \pi(I)^n = (0) \iff (I+L)^n/L = L/L \iff I^n \subseteq L$.

So $\exists n_1, n_2 \geq 1$ st $I^{n_i} \subseteq J_i$. So $I^{n_1+n_2} \subseteq J_1 J_2 \subseteq J$ #

So $J \in \mathcal{S}$ is prime. But this says $\pi(I)$ is a nil but not nilpotent ideal in R/J (integral domain). But the only nilpotent element of an integral domain is 0

why?

$\implies \pi(I) \subseteq (0)$ # ∎

Hilbert's Basis Theorem.

Theorem. If R is Noetherian then $R[x]$ is Noetherian

Aside: $R = \mathbb{C}(x_1, x_2 \ldots) \supseteq \mathbb{C}[x_1, x_2, \ldots]$

Fact: If S is Noetherian then S/I is Noetherian (correspondence)
$S = R[x]$ $I = (x)$; $S/I \cong R$ ie converse holds

Corollary: If R is Noetherian then $R[x_1, \ldots, x_n]$ is Noetherian, and also $R[x_1, \ldots, x_n]/I$ is Noetherian.

Remark: A ring of the form $R[x_1, \ldots, x_n]/I$ is called a finitely-generated R-algebra.

In particular, if k is a field (hence Noetherian) then a finitely-generated k-algebra is Noetherian.

Notation: Given $p(x) = p_0 + \cdots + p_n x^n \in R[x]$ with $p_n \neq 0$, we define $in(p(x)) = p_n \in R$.

Proof (of Hilbert basis theorem)
Let I be an ideal of $R[x]$. We'll show that I is finitely-generated. Pick $f_1(x) \in I \setminus \{0\}$ of smallest degree. ($I = (0)$ is trivial)
Let $d_1 = \deg(f_1(x))$; $a_1 = in(f_1(x))$; let $J_1 = a_1 R$; let $I_1 = f_1(x) R[x] \subseteq I$. If $I = I_1$, done.
If not, pick $f_2(x) \in I \setminus I_1$ with minimal degree.
Let $d_2 = \deg(f_2)$; $a_2 = in(f_2)$; $J_2 = a_1 R + a_2 R$;
$I_2 = f_1(x) R[x] + f_2(x) R[x] \subseteq I$.
Notice $d_2 \geq d_1$.
In general, having produced $f_1(x), \ldots, f_{n-1}(x)$, $a_1, \ldots, a_{n-1} \in R$, $d_1, \ldots, d_n$ 2015 01 2[illegible]
$\in \mathbb{N}$, let $I_{n-1} = f_1(x) R[x] + \cdots + f_{n-1}(x) R[x]$

$$J_{n-1} = a_1 R + \cdots + a_{n-1} R$$
$$I_1 \subseteq \cdots \subseteq I_{n-1} \subseteq I$$
$$J_1 \subseteq \cdots \subseteq J_{n-1} \subseteq R$$

If $I = I_{n-1}$, stop, I is finitely generated.
If not, pick $f_n(x) = a_n x^{d_n} + \text{lower} \in I \setminus I_{n-1}$ of minimal degree d_n. Let $I_n = I_{n-1} + f_n(x) R[x]$, $J_n = J_{n-1} + a_n R$.
If $\exists m$ st $I = I_m$, we have I is generated by $(f_1(x), \ldots, f_m(x))$, and we're done. Thus we may assume $I_1 \subsetneq I_2 \subsetneq \cdots$
But because R is Noetherian, the chain $J_1 \subseteq J_2 \subseteq \cdots$ must terminate in R. Thus $\exists m$ st $J_m = J_{m+1} = \cdots$. Claim: $I_m = I_{m+1}$.
This will give a contradiction.
Let's see why?
We picked $f_{m+1}(x) \in I \setminus I_m$ of minimal degree.

$$f_{m+1} = a_{m+1} x^{d_{m+1}} + \text{lower, with } a_{m+1} \in J_{m+1} = J_m = a_1 R + \cdots + a_m R.$$

So $\exists r_i \in R$ st $a_{m+1} = r_1 a_1 + \cdots + r_m a_m$

$$r_i f_i(x) = r_i a_i x^{d_i} + \text{lower} \in I_m \qquad i \in \{1, \ldots, m\}$$

Now $d_{m+1} \geq d_i$, so
$g(x) = f_{m+1}(x) - r_1 x^{d_{m+1} - d_1} f_1(x) - r_2 x^{d_{m+1} - d_2} f_2(x) - \cdots - r_m x^{d_{m+1} - d_m} f_m(x)$
has degree at most $d_{m+1} - 1$. $g(x) \in I$. So $g(x) \in I_m$ as $f_{m+1}(x) \in I \setminus I_m$ was chosen as an elt. of smallest degree in $I \setminus I_m$.
Thus $g(x) \in I_m$. So $f_{m+1}(x) = g(x) + \langle \text{stuff in } I_m \rangle \in I_m$. $\times$