

Why does it span M ?

$$cn_1' - bn_2' = \dots = n_1$$

$$dn_1' + an_2' = \dots = n_2$$

Now what?

$$s(an_1 + bn_2) + r_3n_3 + \dots + r_dn_d = 0$$

$$\Rightarrow sn_1' + r_3n_3' + \dots = 0 \Rightarrow J(n_1', \dots, n_d') \geq (s) \not\equiv (r) \quad \square$$

Tensor Products

$$m: R \times R \rightarrow R$$

$$m(r+s, u) = m(r, u) + m(s, u)$$

$$m(ar, u) = am(r, u)$$

Let R be a ring & let M & N be two R -modules. We will create a module $M \otimes_R N$ called the tensor product of M & N (over R).

How do we build this?

Start by building a free module F with basis $\{e_{(m,n)}; (m,n) \in M \times N\}$.

We'll take a submodule $G \subseteq F$, G will be the R -submodule of F spanned by all elements of the following form:

$$e_{(rm,n)} - re_{(m,n)}, \quad e_{(m_1+m_2,n)} - e_{(m_1,n)} - e_{(m_2,n)},$$

$$e_{(m,n)} - re_{(m,n)}, \quad e_{(m,n_1+n_2)} - e_{(m,n_1)} - e_{(m,n_2)}$$

Define $M \otimes_R N = F/G$. We write $m \otimes n := e_{(m,n)} + G$.

Warning: ~~Not~~ Not every element of $M \otimes_R N$ need be expressible as $m \otimes n$.

Ex What is $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/3\mathbb{Z})$?

Solution: Write $\mathbb{Z}/2\mathbb{Z} = \{[0]_2, [1]_2\}$, $\mathbb{Z}/3\mathbb{Z} = \{[0]_3, [1]_3, [2]_3\}$

$$F = \mathbb{Z}e_{([0]_2, [0]_3)} \oplus \dots \oplus \mathbb{Z}e_{([1]_2, [2]_3)}$$

Claim $G = F$, i.e. $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = (0)$

$$e_{([0]_2, [1]_3)} = ie_{([1]_2, [1]_3)} = ie_{([0]_2, [0]_3)} \pmod{G}$$

So $e_{([0]_2, [0]_3)}$ generates. But this is $= e_{[1]_2, [1]_3} = 3e_{[1]_2, [1]_3} = e_{[1]_2, [1]_3}$

$$= 0 \quad e_{[1]_2, [1]_3} = 0 \pmod{G}$$

Universal Property

We have a ^{bilinear!} map

$$\begin{aligned}\phi: M \times N &\rightarrow M \otimes_R N \\ (m, n) &\mapsto m \otimes n = e_{(m,n)} + G \in F/G\end{aligned}$$

Warning: ϕ is not in general onto.

Notice that ϕ is bilinear

$$\begin{aligned}\phi(rm_1 + m_2, n) &= (rm_1 + m_2) \otimes n \\ &= r(m_1 \otimes n) + m_2 \otimes n \\ &= r\phi(m_1, n) + \phi(m_2, n) \\ \phi(m, r_1n + r_2n) &= r_1\phi(m, n_1) + \phi(m, n_2)\end{aligned}$$

Universal Property: Let M, N, P be R -modules and suppose that $f: M \times N \rightarrow P$ is R -bilinear. Then $\exists! \hat{f} \in \text{Hom}(M \otimes_R N, P)$ such that $\hat{f} \circ \phi = f$.

$$\begin{array}{ccc} M \times N & \xrightarrow{f} & P \\ \phi \downarrow & \nearrow \hat{f} & \\ M \otimes_R N & & \end{array}$$

Why does this work?

Remark: If F is a free R -module with basis $\{e_a\}$ & if P is an R -module, then any map $\phi: \{e_a\} \rightarrow P$ extends uniquely to an element $\hat{\phi} \in \text{Hom}(F, P)$.

So what do we do?

We have a unique homomorphism

$$\begin{aligned}\psi: F &\rightarrow P \\ e_{(m,n)} &\mapsto f(m, n)\end{aligned}$$

Remark: $\psi|_G = 0$.

$$\begin{aligned}\psi(e_{(rm_1, n)} - r e_{(m_1, n)}) &= f(rm_1, n) - r f(m_1, n) = 0 \\ \Downarrow \psi(e_{(rm_1, n)}) - r \psi(e_{(m_1, n)}) &= 0\end{aligned}$$

$\swarrow f$ bilinear

This means we can define

$$\begin{aligned}\tilde{f}: F/G &\rightarrow P && R\text{-module homomorphism} \\ \tilde{f}(x + G) &= \psi(x)\end{aligned}$$

If $x + G = y + G$ then $x - y \in G \Rightarrow \psi(x - y) = 0$, hence $\psi(x) = \psi(y)$.

So we have produced ~~map~~

$$\begin{array}{ccc} M \times N & \xrightarrow{f} & P \\ \phi \downarrow & \nearrow \tilde{f} & \\ M \otimes_R N & & \end{array}$$

$$\tilde{f} \circ \phi(m, n) = \tilde{f}(e_{(m,n)} + G) = \psi(e_{(m,n)}) = f(m, n)$$

Remark: ~~$M \otimes_R N$~~ is spanned as an R -module by $\{m \otimes n; (m, n) \in M \times N\}$.
 Why? (image of a spanning set)

ex What is $(\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$?

Compute F/G : $F = \mathbb{Z}e_{(1,0)} \oplus \mathbb{Z}e_{(0,1)} \oplus \mathbb{Z}e_{(1,1)} + \mathbb{Z}e_{(1,1)}$

As in last time, $e_{(i,j)} \equiv ie_{(1,0)} + je_{(0,1)} \equiv ij e_{(1,1)} \pmod{G}$

So $F/G = \mathbb{Z}(e_{(1,1)} + G) = \mathbb{Z}(1 \otimes 1)$

$$2(1 \otimes 1) = 2 \otimes 1 = 0 \otimes 1 = 0(1 \otimes 1) = 0$$

So F/G is either $\mathbb{Z}/2\mathbb{Z}$ or (0) .

Let's show $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.

Let

$$f: \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$$

$$f(i, j) = ij.$$

$$f(1, 1) = 1 \neq 0.$$

So $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.

Uniqueness of the tensor product

with bilinear $\phi: M \times N \rightarrow M \otimes_R N$

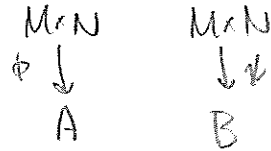
Theorem: \exists a unique R -module $M \otimes_R N$ w/ having the universal property that $\forall R$ -module P & bilinear $f: M \times N \rightarrow P$

$$\begin{array}{ccc} M \times N & \xrightarrow{f} & P \\ \phi \downarrow & \nearrow \tilde{f} & \\ M \otimes_R N & & \end{array}$$

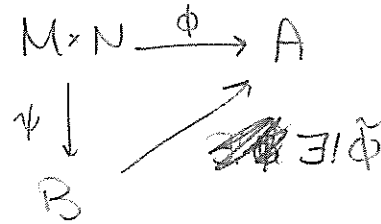
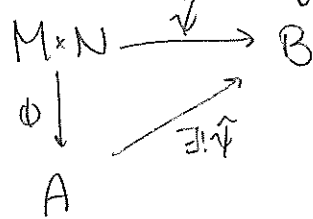
It's hard to do let's
 in some cases let's compute
 by first a tensor product

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Proof: Suppose that we have R -modules A & B with bilinear maps



with the universal property.



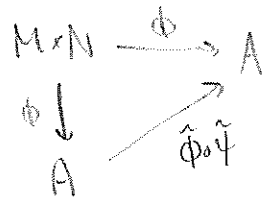
Claim: $\tilde{\phi} \circ \tilde{\psi} = \text{id}_A$, $\tilde{\psi} \circ \tilde{\phi} = \text{id}_B$.

Well $\phi = \tilde{\phi} \circ \tilde{\psi} \circ \phi$.

Recall: We know the image of ϕ must ~~span~~ extend uniquely

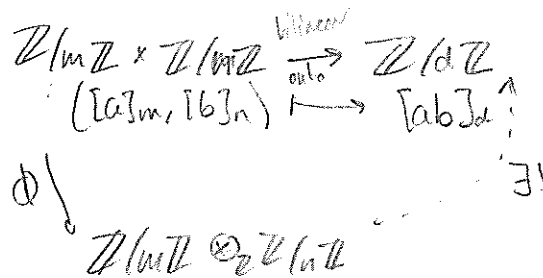
So $\tilde{\phi} \circ \tilde{\psi} (\phi(m,n)) = \phi(m,n) \quad \forall m,n \Rightarrow \tilde{\phi} \circ \tilde{\psi} = \text{id}_A$

Another way: Think about



What is $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$? $\cong \mathbb{Z}/d\mathbb{Z}$, $d = \text{gcd}(m,n)$

Why?



$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ spanned by $[i]_m \otimes [j]_n = ij(1 \otimes 1)$
 $\cong \mathbb{Z}(1 \otimes 1)$

Notice $d(1 \otimes 1) = 0$

$$(am + bn)(1 \otimes 1) = am \otimes 1 + 1 \otimes bn = 0 + 0 = 0$$

Proposition: The following hold

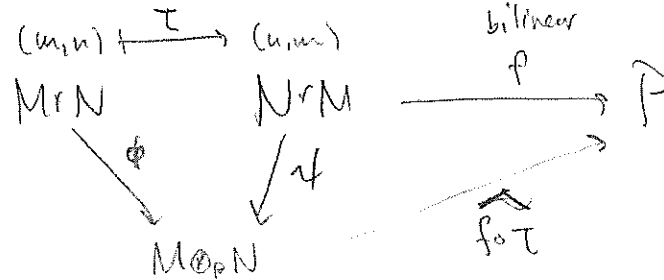
1) $M \otimes_R N \cong N \otimes_R M$

2) $(M \otimes_R N) \otimes_R P \cong M \otimes_R (N \otimes_R P)$ (associativity)

In particular, we can write $M \otimes N \otimes P$ without ambiguity

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Commutativity



Check: $\hat{f} \circ \tau \circ \psi = f$

So $M \otimes_P N$ satisfies the universal property for $N \times M$ with $\psi: N \times M \rightarrow M \otimes_P N$ bilinear.

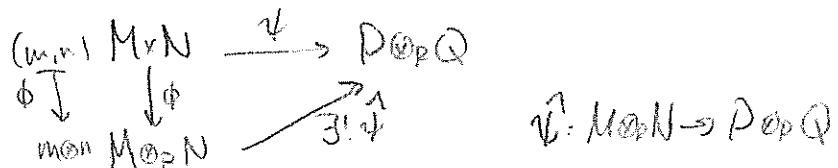
So $M \otimes_P N \cong N \otimes_P M$.

Tensor products of maps

If $f: M \rightarrow P, g: N \rightarrow Q$ homomorphisms. Then we can make a bilinear map $M \times N \xrightarrow{\psi} P \otimes_P Q$ $(m, n) \mapsto f(m) \otimes g(n)$

Bilinear: $(r m_1 + m_2, n) \mapsto f(r m_1 + m_2) \otimes g(n)$

$= r \psi(m_1, n) + \psi(m_2, n)$



& $\hat{\psi} \circ \phi = \psi$. In particular, $\psi(m, n) = \hat{\psi} \circ \phi(m, n) = \hat{\psi}(m \otimes n) = f(m) \otimes g(n)$

It is customary to let $f \otimes g$ denote $\hat{\psi}$.

Summary: If $f \in \text{Hom}(M, P)$ & $g \in \text{Hom}(N, Q)$ then

$\exists f \otimes g \in \text{Hom}(M \otimes N, P \otimes Q)$ $f \otimes g(m \otimes n) = f(m) \otimes g(n)$

An important special case: If $f: M \rightarrow N$, $f \otimes \text{id}_C: M \otimes C \rightarrow N \otimes C$
 $f \otimes \text{id}_C (m \otimes c) = f(m) \otimes c.$

Right Exactness

Theorem: If $M \xrightarrow{f} N \xrightarrow{g} P \rightarrow 0$ is exact & C is an R -module. Then $M \otimes_R C \xrightarrow{f \otimes \text{id}_C} N \otimes_R C \xrightarrow{g \otimes \text{id}_C} P \otimes_R C \rightarrow 0$ is exact.

Let's check that $g \otimes \text{id}_C$ is onto. $P \otimes_R C$ is generated by $p \otimes c$

In general, if $0 \rightarrow M \xrightarrow{f} N$ is exact $\nRightarrow 0 \rightarrow M \otimes C \xrightarrow{f \otimes \text{id}_C} N \otimes C$ is exact.

Ex $M=N=\mathbb{Z}, R=\mathbb{Z}, C=\mathbb{Z}/2\mathbb{Z}$

$$0 \rightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z}$$

$m \mapsto 2m$

$$0 \rightarrow \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \text{ is not exact}$$

$f \otimes \text{id}_C$ is the zero map

$$f \otimes \text{id}_C (n \otimes \epsilon) = 2n \otimes \epsilon = n \otimes 2\epsilon = n \otimes 0 = 0$$

Notice $\psi: \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$

$$(n, \epsilon) \mapsto n\epsilon \text{ is bilinear \& onto } \Rightarrow \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \neq (0)$$

An R -module M is flat if whenever $0 \rightarrow A \rightarrow B$ is exact $\Rightarrow 0 \rightarrow A \otimes M \rightarrow B \otimes M$ is exact
 " is faithfully flat if M is flat & $A \neq (0) \Rightarrow A \otimes_R M \neq (0).$

Ex \mathbb{Q} is faithful and flat but not faithfully flat = $\mathbb{Q} \otimes \mathbb{Z}/2\mathbb{Z} = (0).$

Proposition: Let R be a ring and let M be an R -module. Then $R \otimes_R M \cong M.$

Proof: Let $\phi: R \times M \rightarrow M, \phi(r, m) = rm.$ Bilinear.

Suppose that $\psi: R \times M \rightarrow P$ is bilinear.

Goal: find a unique hom $\tilde{\psi}: M \rightarrow P$ st $\tilde{\psi} \circ \phi = \psi.$

Since ψ is bilinear, $\psi(r, m) = \psi(r \cdot 1, m) = r \psi(1, m)$

We need $\hat{\psi}(m) = \hat{\psi} \circ \phi(1, m) \stackrel{\text{want}}{=} \psi(1, m)$

So if $\hat{\psi}$ exists, it is unique

Why is this $\hat{\psi}$ a homomorphism? $\hat{\psi}(r_1 m_1 + r_2 m_2) = \hat{\psi}(1, r_1 m_1 + r_2 m_2) = \dots$

Direct Sums

Theorem: If $\{M_\alpha\}_{\alpha \in I}$ is a collection of R -modules & C is an R -module then

$$\left(\bigoplus_{\alpha \in I} M_\alpha \right) \otimes_R C \cong \bigoplus_{\alpha \in I} (M_\alpha \otimes_R C)$$

Let us consider $I = \{1, 2\}$, $M_1 = M$, $M_2 = N$. We'll make a ~~linear~~ linear map

$$(M \oplus N) \otimes C \xrightarrow{\pi_1 \otimes \text{id}} M \otimes C$$

$$\xrightarrow{\pi_2 \otimes \text{id}} N \otimes C$$

$$(M \oplus N) \otimes C \xrightarrow{(\pi_1 \otimes \text{id}, \pi_2 \otimes \text{id})} (M \otimes C) \oplus (N \otimes C)$$

$$\begin{array}{ccc} & M \oplus N & \\ \iota_1 \nearrow & & \searrow \iota_2 \\ M & & N \end{array} \quad \begin{array}{l} \iota_1(m) = (m, 0) \\ \iota_2(n) = (0, n) \end{array}$$

The inclusions give maps $\iota_1 \otimes \text{id}: M \otimes C \rightarrow (M \oplus N) \otimes C$
 $\iota_2 \otimes \text{id}: N \otimes C \rightarrow \dots$

This extends to a map $(M \otimes C) \oplus (N \otimes C) \rightarrow (M \oplus N) \otimes C$

So we get a map $h: (M \otimes C) \oplus (N \otimes C)$

where $h = \iota_1 \otimes \text{id} + \iota_2 \otimes \text{id}$

Notice

$$(\pi_1 \otimes \text{id}, \pi_2 \otimes \text{id}) \circ h (m \otimes c, n \otimes c) = \dots = (m \otimes c, n \otimes c)$$

these generate

Check the other composition on pure tensors too. \square

Corollary: If $R^X \cong R^Y \Rightarrow |X| = |Y|$

Proof: Let P be a maximal ideal of R . Let $F = R/P$. Then $R^X \otimes_R F \cong R^Y \otimes_R F \cong F^X$

sketch

low energy state
ground state (0K) ... (check)

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Proof: Let $\mathfrak{p} \triangleleft R$ be maximal, $F = R/\mathfrak{p}$ field.

$$R^X \otimes_R R/\mathfrak{p} \cong (R/\mathfrak{p})^X = F^X$$

$R^X \cong R^Y \Rightarrow F^X \cong F^Y$ as R -modules.

Fact: If M & N are isomorphic R -modules then they are iso. as R/I -modules, where $I = \text{Ann}(M) = \text{Ann}(N) \triangleleft R$.

Proof: If $\phi: M \rightarrow N$ is an R -module isomorphism. Create R/I -module isomorphism $\Phi(m) = \phi(m)$. Check this works.

Question: What is annihilator of F^X ?

Answer: $F^X = (R/\mathfrak{p})^X$, $\text{Ann}(F^X) = \mathfrak{p} = \text{Ann}(F^Y)$

And so $F^X \cong F^Y$ as R -modules implies $F^X \cong F^Y$ as R/\mathfrak{p} -modules implies (A) $|X| = |Y|$. □

Algebras

Let R be a ring. ~~An R -algebra is~~

An R -algebra S is just a ring equipped with a ^{ring} homomorphism $\alpha: R \rightarrow S$
 $\alpha(1_R) = 1_S$

Ex Every ring is a \mathbb{Z} -algebra

$$\alpha: \mathbb{Z} \rightarrow R$$
$$\alpha(n) = n \cdot 1_R$$

Ex $\mathbb{C}[x, y]$ is a \mathbb{C} -algebra

$$\alpha: \mathbb{C} \rightarrow \mathbb{C}[x, y]$$
$$\alpha(c) = c$$

Ex \mathbb{C} is an \mathbb{R} -algebra

$$\alpha: \mathbb{R} \rightarrow \mathbb{C}$$
$$\alpha(c) = c$$