Why does it span M?

$$cn_1' - bn_2' = \cdots = n_1$$
$$dn_1' + an_2' = \cancel{n_2} \cdots = n_2$$

Now what?

$$s(an_1 + bn_2) + r_3 n_3 + \cdots + r_d n_d = 0$$
$$\Rightarrow sn_1' + r_3 n_3' + \cdots = 0 \Rightarrow J(n_1', \ldots, n_d') \supseteq (s) \supsetneq (r) \quad \cancel{\#} \quad \blacksquare$$

Tensor Products

$$m : R \times R \to R$$
$$m(r+s, u) = m(r,u) + m(s,u)$$
$$m(ar, u) = am(r,u)$$

Let R be a ring & let M & N be two R-modules. We will create a module $M \otimes_R N$ called the tensor product of M & N (over R).
How do we build this?
Start by building a free module F with basis $\{e_{(m,n)} ; (m,n) \in M \times N\}$.
We'll take a submodule $G \subseteq F$, G will be the R-submodule of F spanned by all elements of the following form:

$$e_{(rm,n)} - re_{(m,n)}, \quad e_{(m_1+m_2,n)} - e_{(m_1,n)} - e_{(m_2,n)},$$
$$e_{(m,rn)} - re_{(m,n)}, \quad e_{(m,n_1+n_2)} - e_{(m,n_1)} - e_{(m,n_2)}$$

Define $M \otimes_R N := F/G$. We write $m \otimes n := e_{(m,n)} + G$.
Warning: ~~Not~~ Not every element of $M \otimes_R N$ need be expressible as $m \otimes n$.

Ex What is $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/3\mathbb{Z})$?
Solution: Write $\mathbb{Z}/2\mathbb{Z} = \{[0]_2, [1]_2\}$, $\mathbb{Z}/3\mathbb{Z} = \{[0]_3, [1]_3, [2]_3\}$

$$F = \mathbb{Z}e_{([0]_2,[0]_3)} \oplus \cdots \oplus \mathbb{Z}e_{([1]_2,[2]_3)}.$$

Claim $G = F$, ie $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = (0)$

$$e_{([i]_2,[j]_3)} = ie_{([1]_2,[j]_3)} = ije_{([1]_2,[1]_3)} \mod G$$

So $e_{([1]_2,[1]_3)}$ generates. But this is $= e_{[3]_2,[1]_3} = 3e_{[1]_2,[1]_3} = e_{[1]_2,[0]_3}$
$= 0\, e_{[1]_2,[1]_3} = 0 \mod$

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Universal Property

We have a bilinear! map
$$\phi: M \times N \longrightarrow M \otimes_R N$$
$$(m,n) \longmapsto m \otimes n = e_{(m,n)} + G \in F/G$$

Warning: ϕ is not in general onto.
Notice that ϕ is bilinear
$$\phi(rm_1+m_2,n) = (rm_1+m_2)\otimes n$$
$$= r(m_1 \otimes n) + m_2 \otimes n$$
$$= r\phi(m_1,n)+\phi(m_2,n)$$
$$\phi(m,n_1+n_2) = r\phi(n,m_1)+\phi(n,m_2)$$

Universal Property: Let M, N, P be R-modules and suppose that $f: M \times N \to P$ is R-bilinear. Then $\exists!\ \hat{f} \in Hom(M \otimes_R N, P)$ such that $\hat{f} \circ \phi = f$.

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "\phi"'] & P \\ M \otimes_R N \arrow[ur, dotted, "\exists! \hat{f}"'] & \end{tikzcd}$$

Why does this work?
Remark: If F is a free R-module with basis $\{e_\alpha\}$ & if P is an R-module, then any map $\phi: \{e_\alpha\} \to P$ extends uniquely to an element $\hat{\phi} \in Hom(F,P)$.
So what do we do?
We have a unique homomorphism
$$\psi: F \to P$$
$$e_{(m,n)} \to f(m,n)$$

Remark: $\psi|_G = 0$.
$$\underbrace{\psi(e_{(rm,n)} - re_{(m,n)})}_{=\psi(e_{(rm,n)}) - r\psi(e_{(m,n)})} = f(rm,n) - rf(m,n) \overset{f \text{ bilinear}}{=} 0$$

This means we can define
$$\tilde{f}: F/G \to P \qquad R\text{-module homomorphism}$$
$$\tilde{f}(x+G) = \psi(x)$$

If $x+G = y+G$ then $x-y \in G \Rightarrow \psi(x-y)=0$, hence $\psi(x)=\psi(y)$.

So we have produced ~~map~~

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "\phi"'] & P \\ M \otimes_R N \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

$$\tilde{f} \circ \phi(m,n) = \tilde{f}(e_{(m,n)} + G) = \psi(e_{(m,n)}) = f(m,n)$$

Remark: $M \otimes_R N$ is spanned as an R-module by $\{m \otimes n ; (m,n) \in M \times N\}$.
Why? (image of a spanning set)

ex. What is $(\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$?
Compute F/G: $F = \mathbb{Z} e_{(0,0)} \oplus \mathbb{Z} e_{(0,1)} \oplus \mathbb{Z} e_{(1,0)} + \mathbb{Z} e_{(1,1)}$
As in last time, $e_{(i,j)} \equiv i e_{(1,j)} \equiv ij e_{(1,1)} \mod G$
So $F/G = \mathbb{Z}(e_{(1,1)} + G) = \mathbb{Z}(1 \otimes 1)$

$$2(1 \otimes 1) = 2 \otimes 1 = 0 \otimes 1 = 0(1 \otimes 1) = 0$$

So F/G is either $\mathbb{Z}/2\mathbb{Z}$ or (0).
Let's show $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.
Let

$$f : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$$
$$f(i,j) = ij.$$
$$f(1,1) = 1 \neq 0.$$

So $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.

<u>Uniqueness of the tensor product</u>

Theorem: $\exists$ a unique R-module $M \otimes_R N$ with bilinear $\phi : M \times N \to M \otimes_R N$ w/ having the universal property that $\forall$ R-module P & bilinear $f : M \times N \to P$

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "\phi"'] & P \\ M \otimes_R N \arrow[ur, "\exists ! \tilde{f}"'] & \end{tikzcd}$$

It's friday so let's have some fun where by fun I mean let's compute a tensor product

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Proof: Suppose that we have R-modules A & B with bilinear maps

$$\begin{tikzcd} M\times N \arrow[d, "\phi"'] & M\times N \arrow[d, "\psi"] \\ A & B \end{tikzcd}$$

with the universal property.

$$\begin{tikzcd} M\times N \arrow[r, "\psi"] \arrow[d, "\phi"'] & B \\ A \arrow[ur, "\exists!\hat{\psi}"'] & \end{tikzcd} \qquad \begin{tikzcd} M\times N \arrow[r, "\phi"] \arrow[d, "\psi"'] & A \\ B \arrow[ur, "\exists!\tilde{\phi}"'] & \end{tikzcd}$$

Claim: $\tilde{\phi}\circ\hat{\psi} = id_A$, $\hat{\psi}\circ\tilde{\phi} = id_B$.

Well $\phi = \tilde{\phi}\circ\hat{\psi}\circ\phi$.

Recall: We know the image of ϕ must ~~span A~~. extend uniquely

So $\tilde{\phi}\circ\hat{\psi}(\phi(m,n)) = \phi(m,n)$ $\forall m,n \Rightarrow \tilde{\phi}\circ\hat{\psi} = id_A$

Another way: Think about

$$\begin{tikzcd} M\times N \arrow[r, "\phi"] \arrow[d, "\phi"'] & A \\ A \arrow[ur, "\tilde{\phi}\circ\hat{\psi}"'] & \end{tikzcd}$$

■

What is $\mathbb{Z}/m\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z}$? $\cong \mathbb{Z}/d\mathbb{Z}$, $d = \gcd(m,n)$

Why?

$$\begin{tikzcd} \mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z} \arrow[r, "\text{bilinear}", "\text{onto}"'] \arrow[d, "\phi"'] & \mathbb{Z}/d\mathbb{Z} \\ \mathbb{Z}/m\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

$$([a]_m, [b]_n) \longmapsto [ab]_d$$

$\mathbb{Z}/m\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z}$ spanned by $[i]_m\otimes[j]_n = ij(1\otimes 1)$

$\quad = \mathbb{Z}(1\otimes 1)$

Notice $d(1\otimes 1) = 0$

$= (am+bn)(1\otimes 1) = am\otimes 1 + 1\otimes bn = 0+0 = 0$

Proposition: The following hold
1) $M \otimes_R N \cong N \otimes_R M$
2) $(M \otimes_R N) \otimes_R P \cong M \otimes_R (N \otimes_R P)$ *(associativity)*
In particular, we can write $M \otimes N \otimes P$ without ambiguity

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Commutativity

$$\begin{tikzcd}
(m,n) \arrow[r, mapsto, "\tau"] & (n,m) & \\
M \times N \arrow[dr, "\phi"'] & N \times M \arrow[d, "\psi"] \arrow[r, "f", "\text{bilinear}"'] & P \\
& M \otimes_R N \arrow[ur, dotted, "\widehat{f \circ \tau}"'] &
\end{tikzcd}$$

Check: $\widehat{f \circ \tau} \circ \psi = f$.

So $M \otimes_R N$ satisfies the universal property for $N \times M$ with $\psi : N \times M \to M \otimes_R N$ bilinear.
So $M \otimes_R N \cong N \otimes_R M$.

Tensor products of maps

If $f : M \to P$, $g : N \to Q$ homomorphisms. Then we can make a bilinear map $M \times N \xrightarrow{\psi} P \otimes_R Q$ $(m,n) \longmapsto f(m) \otimes g(n)$
Bilinear: $(rm_1 + m_2, n) \overset{\psi}{\longmapsto} f(rm_1 + m_2) \otimes g(n)$

$$\vdots$$

$$= r\psi(m_1, n) + \psi(m_2, n).$$

$$\begin{tikzcd}
(m,n) \arrow[d, "\phi"'] & M \times N \arrow[d, "\phi"] \arrow[r, "\psi"] & P \otimes_R Q \\
m \otimes n & M \otimes_R N \arrow[ur, "\exists!\, \hat{\psi}"'] &
\end{tikzcd}$$

$\hat{\psi} : M \otimes N \to P \otimes_R Q$

& $\hat{\psi} \circ \phi = \psi$. In particular, $\underset{f(m) \otimes g(n)}{\overset{\|}{\psi(m,n)}} = \hat{\psi} \circ \phi(m,n) = \hat{\psi}(m \otimes n)$

It is customary to let $f \otimes g$ denote $\hat{\psi}$.

Summary: If $f \in \mathrm{Hom}(M,P)$ & $g \in \mathrm{Hom}(N,Q)$ then
$\exists\, f \otimes g \in \mathrm{Hom}(M \otimes N, P \otimes Q)$ $f \otimes g(m \otimes n) = f(m) \otimes g(n)$.

An important special case: If $f: M \to N$, $f \otimes id_c : M \otimes C \to N \otimes C$
$f \otimes id_c (m \otimes c) = f(m) \otimes c$.

Right Exactness

Theorem: If $M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ is exact & C is an R-module. Then $M \otimes_R C \xrightarrow{f \otimes id_c} N \otimes_R C \xrightarrow{g \otimes id_c} P \otimes_R C \longrightarrow 0$ is exact.

Let's check that $g \otimes id_c$ is onto. $P \otimes_R C$ is generated by $p \otimes c$

In general, if $0 \to M \xrightarrow{f} N$ is exact $\not\Rightarrow 0 \longrightarrow M \otimes C \xrightarrow{f \otimes id_c} N \otimes C$ is exact.

Ex $M = N = \mathbb{Z}$, $R = \mathbb{Z}$, $C = \cancel{\mathbb{Z}}\ \mathbb{Z}/2\mathbb{Z}$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z}$$
$$m \longmapsto 2m$$

$$0 \longrightarrow \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$$ is not exact

$f \otimes id_c$ is the zero map
$f \otimes id_c (n \otimes \bar{\varepsilon}) = 2n \otimes \bar{\varepsilon} = n \otimes 2\bar{\varepsilon} = n \otimes 0 = 0$

Notice $\psi : \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$
$(n, \bar{\varepsilon}) \longmapsto n\bar{\varepsilon}$ is bilinear & onto so $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \neq (0)$

An R-module M is flat if whenever $0 \to A \to B$ is exact $\to 0 \to A \otimes M \to B \otimes M$ is exc
" is faithfully flat if M is flat & $A \neq (0) \Rightarrow A \otimes_R M \neq (0)$.

Ex $\mathbb{Q}$ is faithful and flat but not faithfully flat: $\mathbb{Q} \otimes \mathbb{Z}/2\mathbb{Z} = (0)$.

Proposition: Let R be a ring and let M be an R-module. Then $R \otimes_R M \cong M$.
Proof. Let $\phi : R \times M \to M$, $\phi(r, m) = rm$. Bilinear.
Suppose that $\psi : R \times M \to P$ is bilinear.
Goal: find a unique hom $\Phi : M \to P$ $\ni$ $\Phi \circ \phi = \psi$.

Since ψ is bilinear, $\psi(r,m) = \psi(r\cdot 1, m) = r\psi(1,m)$

We need $\hat{\psi}(m) = \hat{\psi}\circ\Phi(1,m) \overset{\text{want}}{=} \psi(1,m)$

So if $\hat{\psi}$ exists, it is unique

Why is this $\hat{\psi}$ a homomorphism? $\hat{\psi}(rm_1 + m_2) = \psi(1, rm_1 + m_2) = \cdots$

<u>Direct Sums</u>

Theorem: If $\{M_\alpha\}_{\alpha\in I}$ is a collection of R-modules & C is an R-module then

$$\left(\bigoplus_{\alpha\in I} M_\alpha\right)\otimes_R C \cong \bigoplus_{\alpha\in I}(M_\alpha\otimes C)$$

Let us consider $I = \{1,2\}$, $M_1 = M$, $M_2 = N$. We'll make a ~~b~~linear map

$$(M\oplus N)\otimes C \xrightarrow{\pi_1\otimes id} M\otimes C$$
$$(M\oplus N)\otimes C \xrightarrow{\pi_2\otimes id} N\otimes C$$

$$(M\oplus N)\otimes C \xrightarrow{(\pi_1\otimes id,\ \pi_2\otimes id)} (M\otimes C)\oplus(N\otimes C)$$

$$\begin{tikzcd} & M\oplus N & \\ M \arrow[ur, "\iota_1"] & & N \arrow[ul, "\iota_2"'] \end{tikzcd}$$

$\iota_1(m) = (m,1)$

$\iota_2(n) = (1,n)$

The inclusions give maps $\iota_1\otimes id: M\otimes C \longrightarrow (M\oplus N)\otimes C$

$\iota_2\otimes id: N\otimes C \longrightarrow$ "

This extends to a map $(M\otimes C)\oplus(N\otimes C) \longrightarrow (M\oplus N)\otimes C$? ? ? ? ?

So we get a map $h: (M\otimes C)\oplus(N\otimes C)$

where $:= \iota_1\otimes id + \iota_2\otimes id$

Notice

$$(\pi_1\otimes id, \pi_2\otimes id)\circ h(\underbrace{m\otimes c, n\otimes c'}_{\text{these generate}}) = \cdots = (m\otimes c, n\otimes c')$$

Check the other composition on pure tensors too. ☐

Corollary: If $R^X \cong R^Y \Rightarrow |X| = |Y|$

Proof: Let P be a maximal ideal of R. Let $F = R/P$. Then

$R^X\otimes_R F \cong R^Y\otimes_R F \cong F^Y$

sketch

[illegible]

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Proof: Let $P \trianglelefteq R$ be maximal, $F = R/P$ field.

$$R^X \otimes_R R/P \cong (R/P)^X = F^X$$

$R^X \cong R^Y \Rightarrow F^X \cong F^Y$ as R-modules.

Fact: If M & N are isomorphic R-modules then they are iso. as R/I-modules, where $I = \mathrm{Ann}(M) = \mathrm{Ann}(N) \trianglelefteq R$.

Proof: If $\phi: M \to N$ is an R-module isomorphism. Create R/I-module isomorphism $\Phi(m) = \phi(m)$. Check this works.

Question: What is annihilator of F^X?

Answer: $F^X = (R/P)^X$, $\mathrm{Ann}(F^X) = P = \mathrm{Ann}(F^Y)$

And so $F^X \cong F^Y$ as R-modules implies $F^X \cong F^Y$ as R/P-modules implies (A1) $|X| = |Y|$. $\square$

Algebras

Let R be a ring. ~~An R-algebra is~~

An <u>R-algebra</u> S is just a ring equipped with a ring homomorphism

$$\alpha: R \to S$$
$$\alpha(1_R) = 1_S$$

<u>Ex</u> Every ring is a $\mathbb{Z}$-algebra

$$\alpha: \mathbb{Z} \to R$$
$$\alpha(n) = n \cdot 1_R$$

<u>Ex</u> $\mathbb{C}[x,y]$ is a $\mathbb{C}$-algebra

$$\alpha: \mathbb{C} \to \mathbb{C}[x,y]$$
$$\alpha(c) = c$$

<u>Ex</u> $\mathbb{C}$ is an $\mathbb{R}$-algebra

$$\alpha: \mathbb{R} \to \mathbb{C}$$
$$\alpha(c) = c$$