Semi-Direct Products

2013 04 01

Recall:

Theorem: Suppose $\exists H, K \trianglelefteq G$ st $H \cap K = \{1\}$, $HK = G$. Then $G \cong H \times K$

New:

Theorem: Suppose $\exists H, K < G$ st $H \trianglelefteq G$, $H \cap K = \{1\}$, $HK = G$.
Then $\exists$ a group homomorphism $\varphi : K \to \mathrm{Aut}(H)$ st $G \cong H \rtimes_\varphi K$.

Proof: As in the prev thm pf, $\forall g \in G$ $\exists! h \in H, k \in K$ st $g = hk$. $(\star)$
Define $\varphi_k : H \to H$ by $\varphi_k(h) = khk^{-1} \in kHk^{-1} = H$.
By A3Q3b, this is an isomorphism, ie $\varphi_k \in \mathrm{Aut}(H)$.
Define $\varphi(k) = \varphi_k$, ie $\varphi : K \to \mathrm{Aut}(H)$.
~~Remains to~~ show φ is a group homomorphism.

$\varphi_{k_1 k_2}(h) = (k_1 k_2) h (k_1 k_2)^{-1} = k_1 k_2 h k_2^{-1} k_1^{-1} = k_1 \varphi_{k_2}^{(h)} k_1^{-1} = \varphi_{k_1}(\varphi_{k_2}(h))$

$\Rightarrow \varphi_{k_1 k_2} = \varphi_{k_1} \circ \varphi_{k_2}$ as req. $\Rightarrow$ by A8, $H \rtimes_\varphi K$ is a thing

Remains to show $G \cong H \rtimes_\varphi K$.
Define $\alpha : H \rtimes_\varphi K \to G$ by $\alpha((h,k)) = hk$.
By $(\star)$, α is a bijection. Thus we conclude by showing homomorphism.

$\alpha((h_1,k_1)(h_2,k_2)) = \alpha((h_1 \varphi_{k_1}(h_2), k_1 k_2)) = \alpha((h_1 k_1 h_2 k_1^{-1}, k_1 k_2))$
$= h_1 k_1 h_2 k_1^{-1} k_1 k_2 = h_1 k_1 h_2 k_2 = \alpha((h_1,k_1))\alpha((h_2,k_2))$. ∎

Applications

Let $|G| = pq$, $p < q$ primes. We saw if $p \nmid q-1$ then G is cyclic.
Now don't assume $p \nmid q-1$. Let H = Sylow p-subg. K = S q-sub.
Then $[G:H] = p$ is smallest prime divisor $\Rightarrow H \trianglelefteq G$.
Clearly $H \cap K = \{e\}$, $|HK| = |H||K| / |H \cap K| = |G| \Rightarrow HK = G$.
Thus by thm, $\exists \varphi : K \to \mathrm{Aut}(H)$ st $G \cong H \rtimes_\varphi K$.

As $|H|, |K|$ prime $\Rightarrow$ $H \cong \mathbb{Z}_q$, $K \cong \mathbb{Z}_p$. By A9, $\exists \psi : \mathbb{Z}_p \to \mathrm{Aut}(\mathbb{Z}_q)$
st $G \cong H \rtimes_\varphi K \cong \mathbb{Z}_q \rtimes_\psi \mathbb{Z}_p$.

So our task amounts to classifying homs $\psi : \mathbb{Z}_p \to \mathrm{Aut}(\mathbb{Z}_q)$ st $p < q$
Recall $\mathrm{Aut}(\mathbb{Z}_q) \cong \mathbb{Z}_q^\times$ which is cyclic of order $q-1$ by last class. (as q prime $\Rightarrow$ field)
So $\mathrm{Aut}(\mathbb{Z}_q)$ is a cyclic group of order $q-1$.

$\ker(\psi) \leq \mathbb{Z}_p$, so

$$\begin{array}{ccc} & \ker(\psi) & \\ \| & & \diagdown\!\!\diagdown \\ \{0\} & \text{or} & \mathbb{Z}_p \\ \downarrow & & \downarrow \\ \psi \text{ inj.} & & \psi \text{ constant} \to \psi(k) = id \end{array}$$

If $p \nmid q-1$, then $|\mathrm{im}(\psi)| \mid |\mathrm{Aut}(\mathbb{Z}_q)| = q-1$
so $\mathrm{im}(\psi) \neq p \Rightarrow \psi$ constant.

By A8, $\mathbb{Z}_q \rtimes_\psi \mathbb{Z}_p \cong \mathbb{Z}_q \times \mathbb{Z}_p \cong \mathbb{Z}_{pq}$.

If $p \mid q-1$, can still have $\mathbb{Z}_q \rtimes_\psi \mathbb{Z}_p \cong \mathbb{Z}_{pq}$ as above.
So assume ψ injective. How many such ψ's are there?
Note ψ is an iso from $\mathbb{Z}_p$ to a subgroup of $\mathrm{Aut}(\mathbb{Z}_q)$.
Fact: Since $\mathrm{Aut}(\mathbb{Z}_q)$ is cyclic and p divides its order, $\mathrm{Aut}(\mathbb{Z}_q)$ has exactly one subgroup, S, of order p.

$$\mathbb{Z}_p \xrightarrow[\psi]{\cong} S \leq \mathrm{Aut}(\mathbb{Z}_q).$$

By A9Q5, As any such ψ's have same image, the semi-direct products they define are isomorphic.
So in case 2, we only have one (up to $\cong$) semi-direct product.

This proves:
Proposition: If p, q are primes, $p < q$.
(1) if $p \nmid q-1$ then $\exists!$ group of order pq, $\mathbb{Z}_{pq}$ (up to $\cong$)
(2) if $p \mid q-1$ then $\exists$ exactly two groups of order pq: $\mathbb{Z}_{pq}$, $\mathbb{Z}_q \rtimes_\psi \mathbb{Z}_p$ (up to $\cong$)

Special case: $p = 2$
There are 2 groups of order $2q$: $\mathbb{Z}_{2q}$ and $\mathbb{Z}_q \rtimes_\psi \mathbb{Z}_2$
D_{2q} is not abelian $\Rightarrow D_{2q} \cong \mathbb{Z}_q \rtimes_\psi \mathbb{Z}_2$

show $G \cong H \Rightarrow Aut(G) \cong Aut(H)$

Groups of order 30 — 2013 04 03

$\mathbb{Z}_{30} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ — abelian

D_{30}, $D_{10} \times \mathbb{Z}_3$, $D_6 \times \mathbb{Z}_5$ — non abelian

First we show pair-wise non-isomorphic.

D_{30}: possible orders: we have a cyclic subgroup of order 15 $\rightarrow$ 1, 3, 5, 15 and 15 of order 2

$D_{10} \times \mathbb{Z}_3$ has elm of order 6

$D_6 \times \mathbb{Z}_5$ " 10

Claim: There are no other groups of order 30

Proof: Let $|G|=30$. Let P, Q be Sylow 3,5-subgr. resp.

So $n_3 \in \{1,10\}$, $n_5 \in \{1,6\}$. If neither $=1$ then get $10(3-1)+6(5-1) > 30$ ※

So either $P \triangleleft G$ or $Q \triangleleft G \Rightarrow PQ \leq G$ (see Feb 13)

Also $P \cap Q = \{e\}$ $|PQ| = \cdots = 15$. Let $H = PQ$

Let K be a Sylow 2-subgroup. ~~Note G is the idp of H,K.~~

Note $H \cap K = \{e\}$, $|HK| = \cdots = 30 \Rightarrow HK = G$, $H \triangleleft G$ as $[G:H]=2$.

So $G \cong H \rtimes_\varphi K$ for some hom $\varphi: K \to Aut(H)$

Well, $K \cong \mathbb{Z}_2$ and $H \cong \mathbb{Z}_{15}$ as H is cyclic (Mar 13, Monday Thm)

$\Longrightarrow G \cong \mathbb{Z}_{15} \rtimes_\psi \mathbb{Z}_2$. Recall $Aut(\mathbb{Z}_5) \cong \mathbb{Z}_{15}$

We need only classify such ψ. Must have $\psi(0) = id \in Aut(\mathbb{Z}_5)$

depends only on $\psi(1)$.

$\psi(1)$ is either id, or is an elm of 2

(1) When $\psi(1) = id$, we saw $G \cong \mathbb{Z}_{15} \times \mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$. (A8)

(2) When $|\psi(1)| = 2$, note $\mathbb{Z}_{15} \cong \mathbb{Z}_3 \times \mathbb{Z}_5 \Rightarrow Aut(\mathbb{Z}_{15}) \cong Aut(\mathbb{Z}_3 \times \mathbb{Z}_5)$

Exercise: $Aut(\mathbb{Z}_3 \times \mathbb{Z}_5) \cong Aut(\mathbb{Z}_3) \times Aut(\mathbb{Z}_5)$ ← (requires cyclic and coprime)

$\Rightarrow \cong \mathbb{Z}_3^\times \times \mathbb{Z}_5^\times \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ (as 3,5 prime)

The elms of order 2 in $\mathbb{Z}_2 \times \mathbb{Z}_4$ are $(1,0)$, $(0,2)$, $(1,2)$

$\therefore$ there are at most 3 $\mathbb{Z}_{15} \rtimes_\varphi \mathbb{Z}_2$ which are not $\cong \mathbb{Z}_{30}$, as required.

Digging ~~Deeper~~: In ②, the possible ψ ~~are~~ $: \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_3 \times \mathbb{Z}_5)$

i $\quad \psi_1((a,b)) = (a,-b)$

ii $\quad \psi_1((a,b)) = (-a,b)$

iii $\quad (-a,-b)$

i $\to$ $(\mathbb{Z}_3 \times \mathbb{Z}_5) \rtimes_{\psi_1} \mathbb{Z}_2 \cong$?!

consider $((0,0),1) = x$, $y = ((1,0),0)$

note $xy = yx$

key: ψ_1 "doesn't disturb" $\mathbb{Z}_3 \Rightarrow$

$(\mathbb{Z}_3 \times \mathbb{Z}_5) \rtimes_{\psi_1} \mathbb{Z}_2 \cong \mathbb{Z}_3 \times (\mathbb{Z}_5 \rtimes_\theta \mathbb{Z}_2)$