2013 01 25

Def Let $(G,\star)$ and $(H,\diamond)$ be groups. A homomorphism from $(G,\star)$ to $(H,\diamond)$ is a function $\varphi: G \to H$ st $\forall x,y \in G$, $\varphi(a \star b) = \varphi(a) \diamond \varphi(b)$.
An isomorphism from $(G,\star)$ to $(H,\diamond)$ is a bijective homomorphism.

ex Consider $(\mathbb{Z}_4, +)$ and $(\mathbb{Z}_5^\times, \cdot)$
Define $\varphi: \mathbb{Z}_4 \to \mathbb{Z}_5^\times$ by $\varphi(i) = i+1$, clear a bijection
But not an isomorphism: $\varphi(3+3) = \varphi(2) = 3 \neq 4 \cdot 4 = \varphi(3) \cdot \varphi(3)$
Define $\psi: \mathbb{Z}_4 \to \mathbb{Z}_5^\times$ by $\psi(i) = 2^i$, a bijection
$\psi(i+j) = 2^{i+j} = 2^i \cdot 2^j = \psi(i)\psi(j)$ (~~not well defined~~, as $\mathbb{Z}_4 = \{[0], \ldots\}$ not $\{0, \ldots\}$)
Turns out to be a homomorphism $\Rightarrow$ isomorphism

ex Consider $(\mathbb{R}, +)$, $(\mathbb{R}^+, \cdot)$
$\exp: \mathbb{R} \to \mathbb{R}^+$, exp is a bijection, and $\exp(x+y) = \exp(x) \cdot \exp(y)$
so exp is a homomorphism $\Rightarrow$ isomorphism

Def Let $(G,\star), (H,\diamond)$ be groups. We say they are isomorphic if $\exists$ an isomorphism from $(G,\star)$ to $(H,\diamond)$, and write $(G,\star) \cong (H,\diamond)$.

ex $(\mathbb{Z}_4, +) \cong$ ~~is isomorphic to~~ $(\mathbb{Z}_5^\times, \cdot)$ (by ψ)
$(\mathbb{R}, +) \cong (\mathbb{R}^+, \cdot)$
$(G,\star)$ $(H,\diamond)$

Remark: Suppose $G \overset{\varphi}{\cong} H$, G countable, $G = \{a_1, \ldots\}$
Use φ to enumerate H: $H = \{\varphi(a_1), \ldots\}$

$$\begin{array}{c|cc} \star & a_1 & \\ \hline a_1 & & \\ & & a_i a_j \end{array} \qquad \begin{array}{c|cc} \diamond & \varphi(a_1) & \\ \hline \varphi(a_1) & & \\ & & \varphi(a_i a_j) \end{array}$$

ex if $|\Omega| = |\Delta|$ then $S_\Omega \cong S_\Delta$.
As $|\Omega| = |\Delta|$, $\exists$ a bijection $\theta: \Omega \to \Delta$. Define $\varphi: S_\Omega \to S_\Delta$ by $\varphi(\sigma) = (x \mapsto \theta(\sigma(\theta^{-1}(x))))$
ie $\varphi(\sigma) = \theta \circ \sigma \circ \theta^{-1}$ (show ~~hom~~ iso: exercise)

2013 01 28

Proposition: Let $\varphi: G \to H$ be a homomorphism. Then:
1) $\varphi(e_G) = e_H$;
2) $\forall g \in G,\ \varphi(g)^{-1} = \varphi(g^{-1})$. $\quad$ induction $(\Rightarrow \varphi(g^n) = \varphi(g)^n \ \forall n \in \mathbb{Z})$

Pf: 1) $\varphi(e_G)\varphi(e_G) = \varphi(e_G \cdot e_G) = \varphi(e_G) = e_H \varphi(e_G) \Rightarrow \varphi(e_G) = e_H$
2) $\varphi(g)\varphi(g^{-1}) = \varphi(gg^{-1}) = \varphi(e_G) = e_H$, sim. $\varphi(g^{-1})\varphi(g) = e_H \Rightarrow \varphi(g)^{-1} = \varphi(g^{-1})$. ■

Def Let $\varphi: G \to H$ be a homomorphism. The image of φ is $\mathrm{im}(\varphi) := \{\varphi(g);\ g \in G\}$.
The kernel of φ is $\ker(\varphi) := \{g \in G;\ \varphi(g) = e_H\}$.

Proposition: Let $\varphi: G \to H$ be a homomorphism. Then $\mathrm{im}(\varphi) \leq H$, $\ker(\varphi) \leq G$.
Proof: Exercise, use subgroup criterion

Group Actions

Def A group action is a triple $(G, A, \cdot)$ where G is a group, A is a non-empty set, and $\cdot : G \times A \to A$ st $g \cdot (h \cdot a) = (g * h) \cdot a \ \forall a \in A, g, h \in G$, and $e_G \cdot a = a \ \forall a \in A$. $\quad (G = (G, *))$

ex Let G be a group, take $A = G$, $g \cdot a = ga \ \forall a, g \in G$
First property is commutativity, second is def. of identity

ex Let G be a group, $H \leq G$, take $(H, G, \cdot)$, $h \cdot g = hg$

ex Let $\Omega \neq \emptyset$. Recall $(S_\Omega, \circ)$. Take $(S_\Omega, \Omega, \cdot)$, $\sigma \cdot a = \sigma(a)$
$\sigma_1(\sigma_2(a)) = (\sigma_1 \circ \sigma_2)(a)$ ✓ $\quad id(a) = a$ ✓

2013 01 30

Def) Let $(G, A, \cdot)$ be a group action, $g \in G$. Define $\sigma_g : A \to A$ by $\sigma_g(a) = g \cdot a$.

Claim: " . Then σ_g is a bijection.

Pf: Suppose $\sigma_g(a) = \sigma_g(b)$. Then $g \cdot a = g \cdot b \Rightarrow g^{-1} \cdot (g \cdot a) = g^{-1} \cdot (g \cdot b) \Rightarrow (g^{-1}g) \cdot a = (g^{-1}g) \cdot b \Rightarrow e \cdot a = e \cdot b \Rightarrow a = b$.

Now let $b \in A$. Consider $g^{-1} \cdot b \in A$. Then $g \cdot (g^{-1} \cdot b) = \cdots = b$. ∎

Remark: $\sigma_g \in S_A$. Is $\sigma : G \to S_A$, $\sigma(g) = \sigma_g$, a homomorphism.

Yes, it is, as we see here:

Def) Given a group action $(G, A, \cdot)$, the corresponding homomorphism $\sigma : G \to S_A$ is the permutation representation of $(G, A, \cdot)$.

Lemma: Let $(G, A, \_)$ be a group action. Then:

1) $\sigma : G \to S_A$ is a group homomorphism;
2) σ determines $(G, A, \_)$;
3) Every homo. $\psi : G \to S_A$ is the perm. rep. of some group action $(G, A, \_)$.

Pf: (2): $g \cdot a = (\sigma(g))(a)$, so $\cdot$ is recovered from σ

3) Define $\cdot : G \times A \to A$ by $g \cdot a = (\psi(g))(a)$

Show $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$:

Show $1 \cdot a = a$:

Def) Given a group action $(G, A, \cdot)$, the kernel of the action is the kernel of its perm rep, $\sigma : G \to S_A$.

2013 02 01

Recall: Given any group action $(G, A, \cdot)$, we get an induced homomorphism $\sigma: G \to S_A$

Conversely, any homomorphism $\psi: G \to S_A$ comes from some action $(G, A, \cdot)$.

ex Let $G = S_A$. Consider the group action $(S_A, A, \cdot)$ where $\tau \cdot a = \tau(a)$

What is the corresponding homomorphism?

Well, $\sigma: S_A \to S_A$ given by $\sigma(\tau) = a \mapsto \tau \cdot a = \tau(a)$, ie $\sigma(\tau) = \tau \Rightarrow \sigma = id$

ex Consider $(S_A, A, \cdot)$ where $\tau \cdot a = a$. Note this is an action (check).

Well, $\sigma(\tau) = id$.

Def Let $(G, A, \cdot)$ be a group action. Let $a, b \in A$. We say "G moves a to b" if $\exists g \in G$ st $g \cdot a = b$. We write $a \sim b$.

Proposition: For any action $(G, A, \cdot)$, $\sim$ is an equivalence relation.

Proof: Well $e \cdot a = a$, so $a \sim a$. If $a \sim b$, then $g \cdot a = b$, so $g^{-1} \cdot b = a \Rightarrow b \sim a$. If $a \sim b, b \sim c$, then $g \cdot a = b$, $h \cdot b = c$, so $(hg) \cdot a = c \Rightarrow a \sim c$. ∎

ex Consider $(G, G, \cdot)$ where $g \cdot a = ga$. Note $(ba^{-1}) \cdot a = b$, so $a \sim b$ $\forall a, b \in G$.

So we only have one equivalence class.

ex Let G be a group, $H \leq G$. Consider $(H, G, \cdot)$, $\cdot$ being *left-multiplication*.

Note $a \sim b \Leftrightarrow \exists h \in H$ st $ha = b \Leftrightarrow \exists h = ba^{-1} \Leftrightarrow ba^{-1} \in H \Leftrightarrow Ha = Hb$.

$\therefore$ The equivalence classes are the right cosets of H.

ex Consider $(S_A, A, \cdot)$, $\tau \cdot a = \tau(a)$. Note $a \sim b$ $\forall a, b \in G$, as $(a\ b) = \tau$ works. Only 1 eq. class.

ex $A = \{1, \ldots, 8\}$, $\tau = (1\ 2\ 3)(4\ 5)(6\ 7) \in S_8$. Let $G = \langle \tau \rangle \leq S_8$. Note $|G| = 6$

Well $1 \sim 2 \sim 3$, $4 \sim 5$, $6 \sim 7$, $8 \sim 8$. We have 4 eq. cl., corresponding to the cycles of τ.

Def Given an action $(G, A, \cdot)$, the equivalence classes of $\sim$ are called the orbits of the action. $\mathcal{O}$ notation

An action is said to be transitive when it has only one orbit.

Remark: For $a \in A$, the orbit containing a is $\mathcal{O} = \{g \cdot a ; g \in G\} = G \cdot a$

2013 02 04

Def] Let $(G,A,\cdot)$ be a group action, $a\in A$. The stabilizer of a is $G_a := \{g\in G;\ g\cdot a = a\}$

Proposition: Let $(G,A,\cdot)$ be a group action, $a\in A$. Then $G_a \le G$ and $|G\cdot a| = [G:G_a]$.

Proof: First exercise.

Note $g_1\cdot a = g_2\cdot a \Leftrightarrow a = (g_1^{-1}g_2)\cdot a \Leftrightarrow g_1^{-1}g_2 \in G_a \Leftrightarrow g_1G_a = g_2G_a$.

~~So the size of G~~ So $|G\cdot a| = \#$ of left cosets of $G_a = [G:G_a]$. ∎

Corollary: Let $(G,A,\cdot)$ be a group action, G finite. Then $|G\cdot a| \mid |G|$.

Pf: $|G\cdot a| = [G:G_a] = |G|/|G_a|$ by Lagrange's Thm. ∎

Recall: The kernel of $(G,A,\cdot)$ is the kernel of the cor. hom $\sigma: G\to S_A$.

So $\ker(\sigma) = \{g\in G;\ \sigma(g) = id\} = \{g\in G;\ \sigma_g(a) = a\ \forall a\in A\}$
$= \{g\in G;\ g\cdot a = a\ \forall a\in A\} = \{g\in G;\ g\in G_a\ \forall a\in A\} = \bigcap_{a\in A} G_a$

why is this here?

Prop: Let $\varphi: G\to H$ be a group hom. Then φ is inj iff $\ker(\varphi)=\{e\}$; $\varphi(G)\le H$; φ inj $\Rightarrow$ φ iso from $G\to\varphi(G)$ ∎

Def) Let $(G,A,\cdot)$ be a group action. It is said to be faithful if its kernel is $\{e\}$.

Rmk: Suppose $(G,A,\cdot)$ is faithful. Then $\sigma: G\to S_A$ has $\ker(\sigma)=\{e\} \Rightarrow$
$G \cong \sigma(G) \le S_A$.

ex $(G,G,\cdot)$, $\cdot$ left mult. What is ker? Pick $a\in A$. Then $G_a = \{\cdots\} = \{e\}$
So $\ker(\sigma) = \bigcap_{a\in A} G_a = \bigcap\{e\} = \{e\}$ so this action is faithful

Theorem (Cayley's Theorem): Every group is isomorphic to some subgroup of a symmetric group.

2013 02 06

relabel

1 2 - - 7 8

ex $G = D_8$; $\{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ st $r^4 = 1$, $s^2 = 1$, $rs = sr^3$

$\sigma_1 = id$, $\sigma_r = (1\ r\ r^2\ r^3)(s\ sr^3\ sr^2\ sr)$

$\sigma_{r^2} = (1\ r^2)(r\ r^3)(s\ sr^2)(sr^3\ sr) = \sigma_r^2$

$\sigma_{r^3} = (r^3\ r^2\ r\ 1)(sr\ sr^2\ sr^3\ s) = \sigma_r^{-1}$

We get an induced action of D_8 on [8]

$\sigma_r = (1\,2\,3\,4)(5\,8\,7\,6)$, $\sigma_{r^2} = (1\,3)(2\,4)(5\,7)(6\,8)$, $\sigma_{r^3} = (1\,4\,3\,2)(5\,6\,7\,8)$

$\sigma_s = (1\,5)(2\,6)(3\,7)(4\,8)$, $\sigma_{sr} = (1\,6)(2\,7)(3\,8)(4\,5)$, $\sigma_{sr^2} = (1\,7)(2\,8)(3\,5)(4\,6)$

$\sigma_{sr^3} = (1\,8)(2\,5)(3\,6)(4\,7)$, $\sigma_1 = id$ Must be a subgroup, say $H_1 \leq S_8$

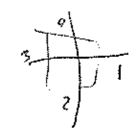
If G is finite, G is isomorphic to a subgroup of $S_{|G|}$.

ex A better representation of D_8 :  D_8 acts naturally on [4]

$r \cdot i = i+1 \mod 4$, $s \cdot 1 = 1$, $s \cdot 2 = 4$, $s \cdot 3 = 3$, $s \cdot 4 = 2$

$H_1 \cong D_8 \cong H_2$

This is an action (no need to check it's too natural) and is faithful.
So this action induces an isomorphism from D_8 to a subgroup of S_4.
Now: $\sigma_1 = id$, $\sigma_r = (1\ 2\ 3\ 4)$, $\sigma_{r^2} = (1\ 3)(2\ 4)$, $\sigma_{r^3} = (1\ 4\ 3\ 2)$
$\sigma_s = (2\ 4)$, $\sigma_{sr} = (1\ 4)(2\ 3)$, $\sigma_{sr^2} = (1\ 3)$, $\sigma_{sr^3} = (1\ 2)(3\ 4)$ $\Big\} \in H_2 \leq S_4$

ex D_8 has other actions: D_8 acts naturally on the edges of the square. 

$S \leq G_3$ ← rigid motions of $\mathbb{R}^3$

Def) S is the group of all symmetries of "the cube", under composition.

Remark: S acts on: - the vertices, - the edges, - the faces 

all are faithful, by intuition

so we get subgroups of

S_8, S_{12}, S_6 respectively

Consider faces, fix one, say f. Then $|S_f| = 4$

And $|O| = 6$

know $|O| = [G : G_a]$

$\Rightarrow 6 = |F| = [S : S_f] = |S|/|S_f| = |S|/4$

$\Rightarrow |S| = 24$