CO 739 - Topics in Combinatorics: Multivariate Stable Polynomials: Theory and Applications

2016 01 0

Theory
- Borcea-Brändén Theory
  complex analysis ~ 5 weeks

Applications
- van der Waerden conjecture (1928)
  $n \times n$ matrix M with non-negative entries which is doubly stochastic (every row sums to 1 and every column sums to 1) then
  $$\operatorname{per}(M) \geq \frac{n!}{n^n}$$
  with equality if and only if
  $$M = \frac{1}{n}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$
  proved in 1981 Egorychev Falikman
  Gurvitz 2008 simple proof
- Probability and Statistical Mechanics
  "Symmetric Exclusion Process"
- Marcus-Spielman-Srivastava
  - Expander Graphs
  - Kadison-Singer
- Miscellaneous Applications

* Homework Exercises from "Multivariate Stable Polynomials ... "
* Directed Reading
* Term Paper, 8-10 pages, on some topic from literature

Multivariate Stable Polynomials

$[m] = \{1, 2, \ldots, m\}$

$\underline{x} = (x_1, x_2, \ldots, x_m)$ indeterminates

$\mathbb{C}[\underline{x}]$ polynomials in $\underline{x}$

$\mathcal{H} = \{z \in \mathbb{C} ; \operatorname{Im}(z) > 0\}$

$\overline{\mathcal{H}} = \{z \in \mathbb{C} ; \operatorname{Im}(z) \geq 0\}$

$f \in \mathbb{C}[\underline{x}]$ is <u>stable</u> if either $f \equiv 0$ identically, or for any $z_1, z_2, .., z_m \in \mathcal{H}$, $f(z_1, z_2, ..., z_m) \neq 0$.
$f \in \mathbb{C}[\underline{x}]$ is <u>real stable</u> if $f \in \mathbb{R}[\underline{x}]$ and f is stable.
$\mathfrak{S}[\underline{x}] \subseteq \mathbb{C}[\underline{x}]$ is the set of stable polynomials.

eg if $m=1$, $f(x)$ is real stable and $f \neq 0$
$f(z) \neq 0$ for all $z \in \mathcal{H}$
$f(z) \neq 0$ for all z with $\mathrm{Im}(z) < 0$
$f(x)$ is real stable if and only if f has only real roots

Hurwitz's Theorem: Let $\Omega \subseteq \mathbb{C}^m$ be non-empty, connected, and open. Let $(f_n ; n \in \mathbb{N})$ be a sequence of analytic and nonvanishing on Ω. Assume that $f_n \to f$ <u>uniformly on compact subsets of Ω</u>. Then either $f \equiv 0$ or f is nonvanishing on Ω.

Application: $\Omega = \mathcal{H}^m$.

Proposition: Let $A_1, ..., A_m, B$ be $n \times n$ matrices. If each A_j is positive semi-definite and B is Hermitian then

$$f(\underline{x}) = \det(x_1 A_1 + x_2 A_2 + \cdots + x_m A_m + B)$$

is stable.

Proof: We can assume that each A_j is positive <u>definite</u> by Hurwitz's theorem. Consider $z \in \mathcal{H}^m$, say $z_j = a_j + i b_j$ for $j \in [m]$, $b_j > 0$. Then

$$H = a_1 A_1 + \cdots + a_m A_m + B$$

is Hermitian, and

$$Q = b_1 A_1 + \cdots + b_m A_m$$

is positive definite. So Q has a positive definite square root $Q^{1/2}$.

$$\begin{aligned} f(z) &= \det(H + iQ) \\ &= \det(Q^{1/2}) \det(Q^{-1/2} H Q^{-1/2} + i) \det(Q^{1/2}) \\ &= \det(Q) \det(Q^{-1/2} H Q^{-1/2} + i) \end{aligned}$$

Note $\det(Q) \neq 0$ as Q is positive definite. So $f(z) = 0$ if and only if $-i$ is an eigenvalue of $Q^{-1/2} H Q^{-1/2}$. But $Q^{-1/2} H Q^{-1/2}$ is Hermitian, so $f(z) \neq 0$. So $f(\underline{x})$ is stable. ∎

"It seems like magic. Because it is."

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Operations preserving stability for $f \in \mathbb{C}[\underline{x}]$

(a) Permutation: for a permutation $\sigma: [m] \to [m]$, $x_i \mapsto x_{\sigma(i)}$

(b) Scaling: for $c \in \mathbb{C}$ and $\underline{a} > \underline{0}$, $f(\underline{x}) \mapsto c f(a_1 x_1, \ldots, a_m x_m)$

(c) Diagonalization: set $x_1 = x_2$

(d) Specialization: let $c \in \overline{\mathcal{H}}$ and let $x_1 = c$

If $\mathrm{Im}(c) > 0$ this is immediate.

If $c \in \mathbb{R}$ consider $c_n = c + \frac{i}{2^n}$ for $n \in \mathbb{N}$. Each $f(c_n, x_2, \ldots, x_m)$ is stable. By Hurwitz, $f(c, x_2, \ldots, x_m)$ is stable.

(e) Inversion: let $x_1 \mapsto -x_1^{-1}$

If $\deg_1(f) = d$ consider $g(x_1, \ldots, x_m) = x_1^d f(-x_1^{-1}, x_2, \ldots, x_m)$: f stable $\Rightarrow$ g stable. $z \mapsto -z^{-1}$ maps $\mathcal{H}$ to $\mathcal{H}$

(f) Differentiation (Contraction): $f \mapsto \frac{\partial}{\partial x_1} f$

Proof of (f):

$$f(\underline{x}) = \sum_{j=0}^{d} x_1^j p_j(x_2, \ldots, x_m)$$

So $d = \deg_1(f)$, so $p_d(x_2, \ldots, x_m) \not\equiv 0$. Consider

$$g_n = n^{-d} f(nx_1, x_2, \ldots, x_m) = n^{-d} \sum_{j=0}^{d} (nx_1)^j p_j(x_2, \ldots, x_m).$$

Each g_n is stable, by scaling. As $n \to \infty$, $g_n \to x_1^d p_d(x_2, \ldots, x_m)$. This is stable by Hurwitz. So $p_2(x_2, \ldots, x_m)$ is stable and not $\equiv 0$. So for any $(z_2, z_3, \ldots, z_m) \in \mathcal{H}^{m-1}$, $h(x) = f(x, z_2, \ldots, z_m) \in \mathbb{C}[x]$ has degree d, as a polynomial in x. $\frac{\partial}{\partial x_1} f(\underline{x})$ is stable if and only if $h'(x)$ is stable for all $(z_2, \ldots, z_m) \in \mathcal{H}^{m-1}$. $h(x)$ is stable by specialization.

$$h(x) = c \prod_{j=1}^{d} (x + \theta_j)$$

and each $\mathrm{Im}(\theta_j) \geq 0$.

$$\frac{h'(x)}{h(x)} = \sum_{j=1}^{d} \frac{1}{x + \theta_j}$$

Consider any $z \in \mathcal{H}$. Then $h(z) \neq 0$, since $\deg(h) = d$. $\mathrm{Im}(z + \theta_j) > 0$ for all j. $\mathrm{Im}\left(\frac{1}{z + \theta_j}\right) < 0$ for all j. So $\mathrm{Im}\left(\frac{h'(z)}{h(z)}\right) < 0$. So $h'(z) \neq 0$. So $h'(x)$ is stable. So $\frac{\partial}{\partial x_1} f$ is stable. ∎

Puzzle:

Gauss-Lucas Theorem. If $p(x) \in \mathbb{C}[x]$ then the zeros of $p'(x)$ are in the convex hull of the zeros of $p(x)$.

Univariate Stable Polynomials

"Lagrange Interpolation:" (case of simple roots)
Let $f(x), g(x)$ in $\mathbb{C}[x]$. Assume that $g(x)$ has only simple roots,

$$g(x) = c \prod_{j=1}^{d} (x - \theta_j).$$

For $1 \le j \le d$ let

$$\hat{g}_j(x) = \frac{g(x)}{x - \theta_j}.$$

Let

$$f(x) = r_d x^d + r_{d-1} x^{d-1} + \cdots + r_1 x + r_0$$

be a polynomial of degree at most d. Then there are unique $a, b_1, \ldots, b_d \in \mathbb{C}$ such that

$$f = ag + b_1 \hat{g}_1 + b_2 \hat{g}_2 + \cdots + b_j \hat{g}_j.$$

Each $\hat{g}_j$ has degree $d-1$. Coefficient of x^d in both sides: $r_d = ac$. So $a = r_d / c$. Evaluate both sides at θ_j: $f(\theta_j) = b_j \hat{g}_j(\theta_j)$. $\hat{g}_j(\theta_j) \neq 0$ since the roots are distinct. So $b_j = f(\theta_j)/\hat{g}_j(\theta_j)$.

This is equivalent to Partial Fractions on $f(x)/g(x)$.

Puzzle: What do you do if

$$g(x) = c \prod_{j=1}^{s} (x - \theta_j)^{d_j}$$

with $d_1 + \cdots + d_s = d$? Use:

$$\frac{1}{(x - \theta_j)^e}, \quad 1 \le e \le d_j, \ 1 \le j \le s.$$

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Interlacing polynomials

$f, g \in \mathbb{R}[x]$ real stable
roots of f: $\xi_1 \le \xi_2 \le \cdots \le \xi_k$
roots of g: $\theta_1 \le \theta_2 \le \cdots \le \theta_\ell$
f and g <u>interlace</u> if either

$$\xi_1 \le \theta_1 \le \xi_2 \le \theta_2 \le \cdots$$
$$\theta_1 \le \xi_1 \le \theta_2 \le \xi_2 \le \cdots$$

Note Let $h = \gcd(f,g)$. Let $f = hp$ and $g = hq$. Then pq has only simple roots

<u>Puzzle</u>: f,g are interlaced if and only if p,q are interlaced.

Note: If f,g are interlaced then

$$|\deg(f) - \deg(g)| \le 1$$

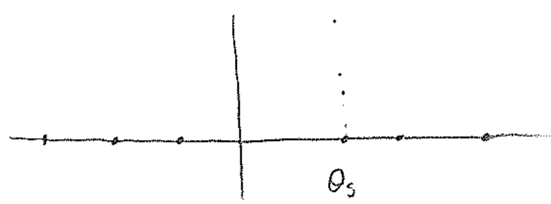
<u>Exercise</u>: Let $f,g \in \mathbb{R}[x]$ be real stable, with fg having only simple roots. Let the roots of g be $\theta_1 < \theta_2 < \cdots < \theta_\ell$. Then the following are equivalent:

1) f,g are interlaced
2) $f(\theta_1), f(\theta_2), \ldots, f(\theta_\ell)$ alternate in sign (strictly - none are zero)
3) In $f = ag + b_1\hat{g}_1 + \cdots + b_\ell \hat{g}_\ell$ all of the b_j have the same sign and are non-zero.

Note: From 3),

$$\frac{f}{g} = a + \sum_{j=1}^{\ell} b_j \frac{1}{x - \theta_j}$$

Evaluate this at $\theta_s + i/n$ and let $n \to \infty$:



$$\left(\frac{i}{n}\right)\frac{f}{g}\left(\theta_s + \frac{i}{n}\right) = \frac{ia}{n} + \frac{i}{n}\sum_{j=1}^{\ell} b_j \frac{1}{(\theta_s + \frac{i}{n})\theta_j} \longrightarrow b_s.$$

Wronskians

Wronskian of f,g is $W[f,g] = f'g - fg'$.

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{d}{dx}\left(a + \sum_{j=1}^{\ell} b_j \frac{1}{x-\theta_j}\right) = -\sum_{j=1}^{\ell} b_j \frac{1}{(x-\theta_j)^2}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2} = \frac{W[f,g]}{g^2}$$

$$W[f,g] = -g^2 \sum_{j=1}^{\ell} b_j \frac{1}{(x-\theta_j)^2} = -\sum_{j=1}^{\ell} b_j \hat{g}_j^2$$

So the following is also equivalent to 1), 2), 3):

4) $W[f,g]$ is non-zero on $\mathbb{R}$ and hence has constant sign

Note: In $f = ag + \sum_{j=1}^{\ell} b_j \hat{g}_j$ in general $b_j = 0$ if and only if $f(\theta_j) = 0$. So fg has only simple roots if and only if all $b_j \neq 0$.

Consider $f, g \in \mathbb{R}[x]$ real stable and interlacing and let $h = \gcd(f,g)$, $f = hp$, $g = hq$. So roots of pq are simple.

$$W[f,g] = W[hp,hq] = (h'p + hp')hq - hp(h'q + hq') = h^2 W[p,q]$$

It follows that stable $f, g \in \mathbb{R}[x]$ are interlacing if and only if either $W[f,g] \le 0$ on $\mathbb{R}$ or $W[f,g] \ge 0$ on $\mathbb{R}$.

Def | $f, g \in \mathbb{R}[x]$ real stable polynomials are in <u>proper position</u> if and only if $W[f,g] \le 0$ on $\mathbb{R}$. We write $f \ll g$.

If both f, g have positive leading coefficient then f, g are interlaced and the leftmost root is of f.

<u>Exercise</u>: $f, g \in \mathbb{R}[x]$ real stable. Then $cf = dg$ for some $c, d \in \mathbb{R}$ not both zero if and only if $f \ll g$ and $g \ll f$.

<u>Hermite-Kakaya-Obreschkoff</u>: Let $f, g \in \mathbb{R}[x]$. Then the following are equivalent:

1) $af + bg$ is real stable for all $a, b \in \mathbb{R}$

2) both f, g are stable, and either $f \ll g$ or $g \ll f$

Proof:

(2)$\Rightarrow$(1): Let $f, g \in \mathbb{R}[x]$ be stable and assume $\deg(f) \le \deg(g)$. Replacing f by $-f$ if necessary, we can assume $W[f,g] \le 0$ on $\mathbb{R}$. So $f \ll g$. Let $h = \gcd(f,g)$ and $f = hp$ and $g = hq$. Replace f by p and g by

9. We can assume that fg has only simple roots. So $W[f,g]<0$ on $\mathbb{R}$.

$$W[f, af+bg] = aW[f,f] + bW[f,g] = bW[f,g]$$

If $b=0$ this is $\equiv 0$. If $b \neq 0$ this has constant sign on $\mathbb{R}$.
When $b \neq 0$, f and $af+bg$ interlace. So $af+bg$ is real stable.

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Proof (restarted):

(2)$\Rightarrow$(1): It suffices to consider $f, g \not\equiv 0$ with only simple roots. Assume that $\deg f \le \deg g$. Replace f by $-f$ if necessary, and assume $f \ll g$.
f, g have only simple zeros:

$$g: \theta_1 < \theta_2 < \cdots < \theta_\ell .$$

Lagrange interpolation:

$$f = ag + \sum_{j=1}^{\ell} b_j \hat{g}_j(x), \qquad \hat{g}_j(x) = \frac{g(x)}{x-\theta_j}.$$

$$(*) \qquad W[f,g] = -g(x)^2 \sum_{j=1}^{\ell} b_j \left(\hat{g}_j(x)\right)^2$$

Assume 2): $W[f,g]<0$ on $\mathbb{R}$ since f,g are real stable with no common roots. In $(*)$, all coefficients $b_j > 0$. Consider any $\alpha, \beta \in \mathbb{R}$

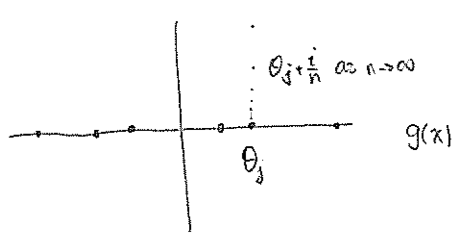
$$\alpha f + \beta g = \alpha a g + \beta g + \alpha \sum_{j=1}^{\ell} b_j \hat{g}_j$$
$$= (\alpha a + \beta) g + \sum_{j=1}^{\ell} (\alpha b_j) \hat{g}_j$$

All αb_j have the same sign. So g and $\alpha f + \beta g$ are interlaced. So $\alpha f + \beta g$ is real stable.

technically we didn't show this in the exercise, as that needed $\alpha f + \beta g$ to be stable, but it still works

(1)$\Rightarrow$(2): Assume that $\alpha f + \beta g$ is real stable for all $\alpha, \beta \in \mathbb{R}$. Prove that $f \ll g$ or $g \ll f$. We can assume that $\deg f \le \deg g$, both real stable, fg has only simple roots. $f \not\equiv 0$ and $g \not\equiv 0$ and $cf + dg \not\equiv 0$ for all $c, d \in \mathbb{R}$ not both zero.

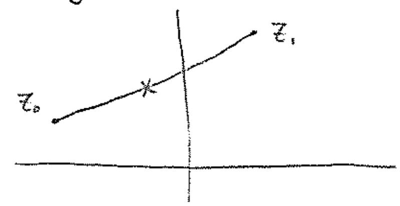
$$f = ag + \sum_{j=1}^{\ell} b_j \hat{g}_j, \quad \frac{i}{n} \frac{f(\theta_j + i/n)}{g(\theta_j + i/n)} \longrightarrow b_j \quad \text{as } n \to \infty$$



Claim: For all $z \in \mathcal{H}$, $\operatorname{Im}\left(\frac{f(z)}{g(z)}\right)$ has the same sign.

Suppose not. Then there are $z_0, z_1 \in \mathcal{H}$ with

$$\operatorname{Im}\left(\frac{f(z_0)}{g(z_0)}\right) < 0 \quad \text{and} \quad \operatorname{Im}\left(\frac{f(z_1)}{g(z_1)}\right) < 0.$$



There exists $0 \le \lambda \le 1$ such that for $z_\lambda = (1-\lambda) z_0 + \lambda z_1$,

$$\operatorname{Im}\left(\frac{f(z_\lambda)}{g(z_\lambda)}\right) = 0.$$

So

$$f(z_\lambda) + \gamma g(z_\lambda) = 0$$

for some real $\gamma \in \mathbb{R}$. Since $z_\lambda \in \mathcal{H}$ and $\alpha f + \beta g$ is stable for all $\alpha, \beta \in \mathbb{R}$, it follows that $f + \gamma g \equiv 0$. But we assumed that f, g are not scalar multiples of one another. So the claim holds.

Hence all b_j have the same sign. So either $f \ll g$ or $g \ll f$. ▪

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<u>Hermite-Biehler</u>: $f, g \in \mathbb{R}[x]$. Then f, g are stable and $f \ll g$ if and only if $g + if$ is stable.

Proof: Without loss of generality, $f \neq 0$ and $g \neq 0$. Let $p = g + if$.
Considering $ip = -f + ig$ if necessary, we may assume $\deg f < \deg g$.
Factoring out $\gcd(f,g)$ we can assume f, g have <u>no</u> common roots.

1) Assume f, g stable & $f \ll g$. So fg has only simple roots. Write

$$f = ag + \sum_{j=1}^{d} b_j \hat{g}_j \qquad \text{with} \qquad g = c \prod_{j=1}^{d} (x - \theta_j), \quad \hat{g}_j = \frac{g}{x - \theta_j}.$$

Then all $b_j > 0$. Then for all $z \in \mathcal{H}$ we have

$$\operatorname{Im}\left(\frac{f(z)}{g(z)}\right) < 0 \quad \text{since} \quad \operatorname{Im}\left(\frac{b_j}{z - \theta_j}\right) < 0.$$

Note $g(z) + if(z) \neq 0$ for all $z \in \mathcal{H}$. If not, $g(z) = -if(z)$ for some $z \in \mathcal{H}$:

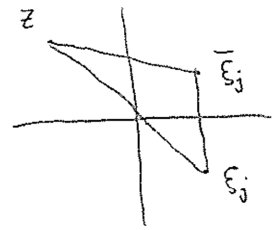
$$\frac{g(z)}{f(z)} = -i \quad \text{but} \quad \operatorname{Im}\left(\frac{g(z)}{f(z)}\right) \ge 0,$$

contradiction.

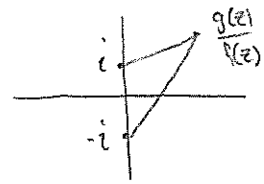
2) Conversely, assume $g, f \in \mathbb{R}[x]$ and $g + if$ is stable. Assume that

$$p(x) = \prod_{j=1}^{d} (x - \xi_j).$$

All ξ_j have $\mathrm{Im}(\xi_j) \le 0$. For any $z \in \mathcal{H}$:
$$|z - \xi_j| \ge |z - \bar{\xi}_j|.$$



So $|p(z)| \ge |p(\bar{z})|$ for all $z \in \mathcal{H}$. $f \in \mathbb{R}[x]$ so $|f(z)| = |f(\bar{z})|$ for all $z \in \mathcal{H}$. Consider $z \in \mathcal{H}$ with $f(z) \neq 0$.
$$\left| \frac{g(z)}{f(z)} + i \right| \ge \left| \frac{g(\bar{z})}{f(\bar{z})} + i \right| = \left| \frac{g(z)}{f(z)} - i \right|$$



since $f, g \in \mathbb{R}[x]$. So

(✱) $$\mathrm{Im}\left(\frac{g(z)}{f(z)} \right) \ge 0$$

for all $z \in \mathcal{H}$. Now $g(x) + yf(x)$ is real stable in $\mathbb{R}[x,y]$. If $z, w \in \mathcal{H}$ — (*)
and $g(z) + wf(z) = 0$ then
$$\frac{g(z)}{f(z)} = -w \quad \text{has} \quad \mathrm{Im}\left(\frac{g(z)}{f(z)} \right) < 0,$$
a contradiction. So $g(x) + yf(x)$ is stable. Specializing $y = 0$, we get $g(x)$ is stable. Contracting $\frac{\partial}{\partial y}$, we get $f(x)$ is stable. Scale and specialize to see $\alpha g + \beta f$ is real stable for all $\alpha, \beta \in \mathbb{R}$. By HKO, either $f \ll g$ or $g \ll f$. By (*) and Lagrange Interpolation, $f \ll g$. ∎

Multivariate HB and HKO:

Lemma 2.3: Let $f \in \mathbb{C}[\underline{x}]$. Then f is stable if and only if for all $\underline{a}, \underline{b} \in \mathbb{R}^m$ with $\underline{b} > \underline{0}$ $f(\underline{a} + t\underline{b})$ is stable in $\mathbb{R}[t]$.

Proof: Note
$$\mathcal{H}^m = \{ \underline{a} + \underline{b}t ;\ \underline{a}, \underline{b} \in \mathbb{R}^m,\ \underline{b} > \underline{0},\ \text{and } t \in \mathcal{H} \}.$$ ∎

Def] Consider $f, g \in \mathbb{R}[\underline{x}]$. Then $f \ll g$ if and only if $g + if \in \mathbb{C}[\underline{x}]$ is stable.

Proposition 2.7:
(a) Let $f, g \in \mathbb{R}[\underline{x}]$. Then $f \ll g$ if and only if $g + yf \in \mathbb{R}[\underline{x}, y]$ is stable.
(b) Let $f, g \in \mathbb{C}[\underline{x}]$ with $f \neq 0$. Then $g + yf$ is stable if and only if
f stable?
$$\mathrm{Im}\left(\frac{g(z)}{f(z)} \right) \ge 0 \quad \text{for all } z \in \mathcal{H}^m.$$

"I m a morning person. My part of the morning is between midnight and two"

Proof:

(a) If $g+yf$ is stable then

$$\begin{cases} y=0 \longrightarrow & g \text{ is stable} \\ \frac{\partial}{\partial y} \longrightarrow & f \text{ is stable} \\ y=i \longrightarrow & g+if \text{ is stable,} \end{cases}$$

so $f \ll g$.

Conversely, assume $f \ll g$. So $h = g+if$ is stable. Consider $w = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$ and $\beta > 0$. $h(x_1, \dots, x_m)$ is stable, so for all $\underline{a}, \underline{b} \in \mathbb{R}^m$ with $b > \underline{0}$, $h(\underline{a}+t\underline{b})$ is stable. $g(\underline{a}+\underline{b}t) + if(\underline{a}+\underline{b}t)$ is stable for all $\underline{a}, \underline{b} \in \mathbb{R}^m$ with $\underline{b} > \underline{0}$. By univariate HB, $f(\underline{a}+\underline{b}t) \ll g(\underline{a}+\underline{b}t)$. By HKO, $cf(\underline{a}+\underline{b}t) + dg(\underline{a}+\underline{b}t)$ is stable for all $c, d \in \mathbb{R}$. By HKO again, the roots of $\lambda f(\underline{a}+\underline{b}t)$ and $g(\underline{a}+\underline{b}t) + \mu f(\underline{a}+\underline{b}t)$. So either $\lambda f(\underline{a}+\underline{b}t) \ll g(\underline{a}+\underline{b}t) + \mu f(\underline{a}+\underline{b}t)$ or conversely.

$$W[\lambda f(\underline{a}+\underline{b}t), g+\mu f] = \lambda W[f,g] \le 0 \text{ on } \mathbb{R}$$

if $\lambda > 0$. Then $\lambda f \ll g + \mu f$. Take this with $\lambda = \beta$ and $\mu = \alpha$. By HB $(g+\alpha f) + i\beta f$ is stable in $\mathbb{C}[t]$. For all $\underline{a}, \underline{b} \in \mathbb{R}^m$ $\underline{b} > \underline{0}$ and $\alpha, \beta \in \mathbb{R}$, $\beta > 0$, $g(\underline{a}+\underline{b}t) + (\alpha + i\beta) f(\underline{a}+\underline{b}t)$ is stable. For all $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$, $g(\underline{x}) + (\alpha + i\beta) f(\underline{x})$ is stable. So $g(\underline{x}) + y f(\underline{x})$ is stable.

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(Remark about HKO:

Assume $af + bg$ stable $\forall a, b \in \mathbb{R}$, $\gcd(f,g) = 1$. Suppose $g(x) = (x-\theta)^r h(x)$ with $h(\theta) \neq 0$, $f(\theta) \neq 0$. $p = g + \varepsilon f$ as $\varepsilon \to 0$. If $\varepsilon \neq 0$ then $p(\theta) = \varepsilon f(\theta) \neq 0$. For small $\varepsilon \neq 0$, there are r roots $z_k(\varepsilon)$ of $p(x)$ with $z_k(\theta) \to \theta$ as $\varepsilon \to 0$ $(0 \le k \le r)$.

Puzzle:

$$z_k(\varepsilon) \sim \theta + \varepsilon^{1/r} e^{i(\omega + 2k\pi)} + o(\varepsilon) \quad \text{as } \varepsilon \to 0$$

So $r \le 1$. So fg has simple roots.)

(b) Assume $g+yf$ is stable. Specialize $y=0$: so g is stable. If $g \equiv 0$ it's trivial so $g \not\equiv 0$. Take any $z \in \mathcal{H}^m$. Then $g(z) \neq 0$ and $f(\underline{z}) \neq 0$. Let

$$w = -\frac{g(\underline{z})}{f(\underline{z})}.$$

Then $g(z) + wf(\underline{z}) = 0$. Since $g+yf$ is stable, and $\neq 0$, $w \notin \mathcal{H}$. So $\mathrm{Im}(w) \le 0$. So $\mathrm{Im}\left(\frac{g(\underline{z})}{f(\underline{z})}\right) \ge 0$. The argument is reversible. $\boxtimes$

Theorem 2.9 [Multivariate HKO]:
Let $f, g \in \mathbb{R}[\underline{x}]$. Then $af(\underline{x}) + bg(\underline{x})$ is stable for all $a,b \in \mathbb{R}$ if and only if f,g are stable and either $f \ll g$ or $g \ll f$.

Proof: First assume that $f \ll g$. By proposition 2.7(a), $g + yf$ is stable. Let $a,b \in \mathbb{R}$ with $b > 0$. Scale and specialize: $bg + (a+i)f$ is stable. So $(af+bg) + if$ is stable. So $f \ll af+bg$ by definition. So $af+bg$ is stable. Similarly $-af-bg$ is stable (still $b>0$). So $af+bg$ is stable for all $a,b \in \mathbb{R}$. Similarly if $g \ll f$. by Lemma 2.3

Conversely, assume $af+bg$ is stable $\forall a,b \in \mathbb{R}$. Fix any $\underline{a}, \underline{b} \in \mathbb{R}^m$ with $\underline{b} > \underline{0}$. Let $\tilde f(t) = f(\underline{a} + \underline{b}t)$ and $\tilde g(t) = g(\underline{a} + \underline{b}t)$. These are stable. $a\tilde f + b\tilde g$ is stable $\forall a,b \in \mathbb{R}$. By univariate HKO, either $\tilde f \ll \tilde g$ or $\tilde g \ll \tilde f$. First, assume that $\tilde f \ll \tilde g$ for all $\underline{a}, \underline{b} \in \mathbb{R}^m$ with $\underline{b} > \underline{0}$. $\tilde g + i\tilde f$ is stable for all $\underline{a}, \underline{b} \in \mathbb{R}^m$ with $\underline{b} > \underline{0}$ by univariate HB. So $g(\underline{x}) + if(\underline{x})$ is stable by lemma 2.3. So $f \ll g$ by definition. Similarly, if $\tilde g \ll \tilde f$ for all $\underline{a}, \underline{b} \in \mathbb{R}^m$ with $\underline{b} > \underline{0}$ then $g \ll f$.

Remaining case: $\tilde f_0 \ll \tilde g_0$ for some $\underline{a}_0, \underline{b}_0 \in \mathbb{R}^m$ with $\underline{b}_0 > \underline{0}$, $\tilde g_1 \ll \tilde f_1$ for some $\underline{a}_1, \underline{b}_1 \in \mathbb{R}^m$ with $\underline{b}_1 > \underline{0}$. For $0 \le \lambda \le 1$ let

$$\underline{a}_\lambda = (1-\lambda)\underline{a}_0 + \lambda \underline{a}_1, \quad \underline{b}_\lambda = (1-\lambda)\underline{b}_0 + \lambda \underline{b}_1.$$

For all λ, either $\tilde f_\lambda \ll \tilde g_\lambda$ or $\tilde g_\lambda \ll \tilde f_\lambda$. As λ varies from 0 to 1, the roots vary continuously. There exists $0 \le \lambda \le 1$ such that $c\hat f_\lambda = d\hat g_\lambda$ for some $c,d \in \mathbb{R}$ not both zero. By hypothesis $h = cf - dg$ is stable. $h(\underline{a}_\lambda + \underline{b}_\lambda t) = c\tilde f_\lambda - b\tilde g_\lambda \equiv 0$ in $\mathbb{R}[t]$, so $h(\underline{a}_\lambda + i\underline{b}_\lambda) = 0$. But $\underline{a}_\lambda + i\underline{b}_\lambda \in \mathcal{H}^m$. Thus $h \equiv 0$. So $cf = dg$, in $\mathbb{C}[\underline{x}]$. So $f \ll g$ and $g \ll f$. ∎

Preview of chapter 3: multiaffine polynomial in $\mathbb{C}[\underline{x}]$ every variable occurs at most to the 1st power

$$f(\underline{x}) = \sum_{S \subseteq [m]} c(S)\, \underline{x}^S \qquad \underline{x}^S = \prod_{i \in S} x_i$$

Shorthand notation:

$$f^i = f|_{x_i = 0} \text{ "deletion"} \qquad f_i = \frac{\partial}{\partial x_i} f \text{ "contraction"}$$

$$f = f^i + x_i f_i$$

For $i \ne j$, $f = f^{ij} + x_i f_i^j + x_j f_j^i + x_i x_j f_{ij}$

$$\Delta_{ij} f = f_i^j f_j^i - f_{ij} f^{ij} \text{ "Raleigh difference"}$$

Theorem: Let $f \in \mathbb{R}[\underline{x}]$ be multiaffine. Then the following are equivalent: 2016 01 20

(a) f is stable

(b) $\Delta_{ij} f(\underline{a}) \geq 0$ for all $\underline{a} \in \mathbb{R}^m$ and all $\{i,j\} \subseteq [m]$ (f is strongly Rayleigh)

(c) Either $m \leq 1$ or $\exists \{i,j\} \subseteq [m]$ such that f_i, f^i, f_j, f^j are strongly Rayleigh and $\Delta_{ij} f(\underline{a}) \geq 0$ for all $\underline{a} \in \mathbb{R}^m$

Strong Rayleigh property:

$$\forall \{i,j\} : \forall \underline{a} \in \mathbb{R}^m : \Delta_{ij} f(\underline{a}) \geq 0$$

Proof

(b) $\Rightarrow$ (c)

If f is strongly Rayleigh then f^i and f_i are strongly Raleigh

$$\Delta_{jk} f_i(\underline{a}) = \lim_{a_i \to \infty} \frac{1}{a_i} \Delta_{jk} f(\underline{a}) \geq 0$$

$$\Delta_{jk} f^i(\underline{a}) = \Delta_{jk} f(\underline{a})|_{a_i = 0} \geq 0$$

(a) $\Rightarrow$ (b) $m=1$ trivial, $m=2$ indeterminates x, y. $f = A + Bx + Cy + Dxy$ with $A, B, C, D \in \mathbb{R}$. When is this stable? Suppose $z, w \in \mathbb{C}$ with $z \in \mathcal{H}$ and $f(z,w) = 0$. Take $f = 0$, solve for y in terms of x.

$$0 = (A + Bx) + (C + Dx)y$$

$$y = -\frac{A+Bx}{C+Dx} = -\frac{(A+Bx)(C+D\bar{x})}{|C+Dx|^2}$$

$$\mathrm{Im}((A+Bx)(C+D\bar{x})) = \mathrm{Im}(AC + BCx + AD\bar{x} + BDx\bar{x})$$
$$= (BC - AD)\,\mathrm{Im}(x)$$

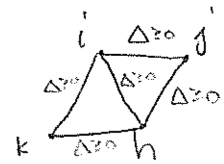
$$\mathrm{Im}(y) = \frac{-1}{|C+Dx|^2}(BC - AD)\,\mathrm{Im}(x)$$

f is stable if and only if $BC - AD \geq 0$.

General case $m \geq 2$: Assume that $f \in \mathbb{R}[\underline{x}]$ is stable. Let $\{i,j\} \subseteq [m]$. Let $\underline{a} \in \mathbb{R}^m$. Specialize $x_k = a_k$ for each $k \in [m] \setminus \{i,j\}$. The result is 2-variable real stable. Apply the $m=2$ argument: $\Delta_{ij} f(\underline{a}) \geq 0$.

(c) $\Rightarrow$ (b)

Let $\{i,j\} \subseteq [m]$ be such that f^i, f_i, f^j, f_j are strong Rayleigh, and $\Delta_{ij} f(\underline{a}) \geq 0$ for all $\underline{a} \in \mathbb{R}^m$



Consider $\Delta_{ij} f$ as a polynomial in x_h with $x_k = a_k \in \mathbb{R}$ specialized for all $k \notin \{h, i, j\}$.

$$\Delta_{ij} = f^i_j f^j_i - f_{ij} f^{ij}$$

$$f^i_j = f^{hi}_j + x_h f^i_{hj} \quad \text{etc.} \quad f_{ij} = f^h_{ij} + x_h f_{hij}$$

I know you're all graduate students but I'm going to pretend you're responsible adults?

$\Delta_{ij} = Ax_h^2 + Bx_h + C$, $A_{hij} = f_{ih}^{j} f_{jh}^{i} - f_{ijh} f_h^{ij} = \Delta_{ij} f_h$

$B_{hij} =$ mess

$C_{hij} = f_i^{jh} f_j^{ih} - f_{ij}^h f^{ijh} = \Delta_{ij} f^h$

By hypothesis, $\Delta_{ij} f = Ax_h^2 + Bx_h + C$ does not change sign for $x_h \in \mathbb{R}$.
So the descriminant

$$D_{hij} = B_{hij}^2 - 4A_{hij}C_{hij} \le 0$$

for all $a \in \mathbb{R}^m$.

Fact: D_{hij} is invariant under all permutations of the subscripts.

Consider $\Delta_{hi} f$ as a polynomial in x_j

$$\Delta_{hi} f = A_{jhi} x_j^2 + B_{jhi} x_j + C_{jhi}$$

Discriminant $D_{jhi} = D_{hij} \le 0$ for all $a \in \mathbb{R}^m$. $\Delta_{hi} f$ does not change sign for $x_j \in \mathbb{R}$ Since f_i and f^i are strongly Rayleigh $\Delta_{hi} f \ge 0$ for all $a \in \mathbb{R}$.

$$A_{jhi} = \Delta_{hi} f_j \ge 0 \qquad B^2 - 4AC \le 0$$
$$C_{jhi} = \Delta_{hi} f^j \ge 0$$

Check f_h, f^h are strong Rayleigh.

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Corollary 2.10
Theorem 3.1((b)⇒(a))
$\Big\}$ up to you

<u>Applications</u>

- spanning trees in graphs
- Gurvits's proof of van der Waerden

Let $G = (V, E)$ be a finited connected loopless graph (multiple edges are okay). $\underline{x} = \{x_e \,;\, e \in E\}$ indeterminates.
For a spanning tree (T, V) let

$$\underline{x}^T = \prod_{e \in T} x_e$$

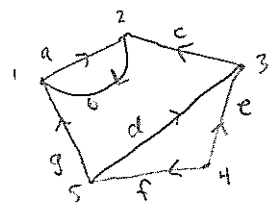
Let $\mathcal{T}(G)$ be the set of all spanning trees of G. Let

$$T(G; \underline{x}) = \sum_{T \in \mathcal{T}(G)} \underline{x}^T$$

be the <u>spanning tree enumerator</u>. This $T(G; \underline{x})$ is stable. Why?

Signed Incidence Matrix

Fix an orientation $v \to w$ or $w \to v$ for each edge $\{v,w\} \in E$

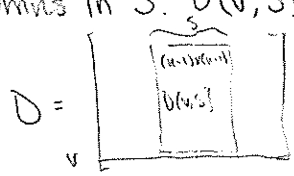


D is indexed $V \times E$

$$D_{v,e} = \begin{cases} 1 & \text{if } e \text{ points in to } v \\ -1 & \text{if } e \text{ points out of } v \\ 0 & \text{otherwise} \end{cases}$$

$$D = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & & & 1 \\ 1 & -1 & 1 & 0 & & & 0 \\ 0 & 0 & -1 & 1 & \cdots & & 0 \\ 0 & 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & -1 & & & -1 \end{bmatrix} \end{array}$$

Every column of D sums to 0. In fact $\operatorname{rank}(D) = |V| - 1$.
More precisely, delete any row $v \in V$ of D. Consider a set $S \subseteq E$ of size $|S| = |V| - 1$. Let $D(v,S]$ be the submatrix of D obtained by deleting row v keeping only the columns in S. $D(v,S]$ is $(n-1)\times(n-1)$



Fact: $\det D(v,S] \neq 0$ if and only if (V,S) is a spanning tree. Moreover, $\det D(v,S] \in \{-1,0,1\}$

Proof: If (V,S) is not a tree then since $|S| = |V| - 1$ then (V,S) is not connected and it contains a cycle. Let C be the edges of a cycle in (V,S). Then the columns of D indexed by C are linearly dependent. So, since $C \subseteq S$, the columns of $D(v|S]$ are linearly dependent. So $\det(v|S] = 0$.
Conversely, assume that (V,T) is a tree. If $|V| = 1$ it's trivial (0×0 matrix). Otherwise $|V| \geq 2$ and (V,T) has at least two leaves. Let $v \in V$ and delete row v from D. Let w be any leaf not equal to v.

" that's just by MATH 239 .. or oxygen" ..."if you're talking to a physicist ... which I don't recommend"

Let $e = \{u, w\}$ be the edge incident with w: $u \xrightarrow{e} w$.

$$D(v,T] = \begin{array}{c} \\ v \\ \\ \\ \\ u \\ w \end{array} \begin{array}{c} \begin{array}{cccc} & & & e \end{array} \\ \begin{bmatrix} & & & 0 \\ & & & \vdots \\ & & & 0 \\ & & & -1 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \end{array}$$

$$\det D(v|T] = \det(vw|T \setminus e]$$

$T \setminus e$ is the set of edges of a spanning tree of $G \setminus w$. By induction, $\det D(v|T] \in \{-1, 1\}$. $\blacksquare$

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Let $R \subseteq V$ and $S \subseteq E$ with $|R| + |S| = |V|$ and $R \neq \emptyset$. Then $D(R|S]$ is a square submatrix of D, and $\det D(R|S] \neq 0$ if and only if (V, S) is a spanning forest such that each component contains exactly one vertex in R. If $\det D(R|S] \neq 0$ then it is ± 1.

Laplacian matrix

$L = D_\circ X D_\circ^\dagger$, $D_\circ$ is signed incidence matrix with row $v \in V$ deleted

$X = \operatorname{diag}(x_e : e \in E)$, diagonal matrix of inderterminates

$M^\dagger$ is the conjugate transpose of M

L does not depend on the orientation chosen to define D, or deletion for $D_\circ$

Binet-Cauchy Formula

Let M be $r \times n$, and P be $n \times r$, matrices. Then

$$\det(MP) = \sum_{\substack{S \subseteq [n] \\ |S| = r}} \left(\det M(|S]\right) \cdot \left(\det P[S|)\right).$$

So

$$\det(L) = \det(D_\circ X D_\circ^\dagger) = \sum_{\substack{S \subseteq E \\ |S| = r}} \det D_\circ(|S] \cdot \det(X D_\circ^\dagger [S|))$$

$$= \sum_{\substack{S \subseteq E \\ |S| = |V|-1}} \det D_\circ(|S] \cdot \underline{x}^S \cdot \det D_\circ^\dagger [S|)$$

$$= \sum_{\substack{S \subseteq E \\ |S| = |V|-1}} \underline{x}^S \left| \det D_\circ(|S] \right|^2$$

$$= \sum_{T \in \mathcal{T}(G)} \underline{x}^T$$

$$= T(G; x)$$

Theorem: For any connected (loopless) graph G, $T(G;\underline{x})$ is stable.

Proof: $T(G;\underline{x}) = \det(L) = \det(D_0 X D_0^\dagger)$. Let $\underline{z} \in \mathcal{H}^E$. We must show that $T(G;\underline{z}) \neq 0$. We will show that substituting $x_e = z_e \in \mathcal{H}$ for each $e \in E$, the Laplacian matrix is invertible. Consider any $\underline{w} \in \mathbb{C}^{V \setminus \{v\}}$. Assume $\underline{w} \neq \underline{0}$. Show that $L\underline{w} \neq \underline{0}$. Then it follows that L is invertible. We will show that $\underline{w}^\dagger L \underline{w} \neq 0$.

$$\underline{w}^\dagger L \underline{w} = \underline{w}^\dagger D_0 L D_0^\dagger \underline{w}$$
$$= \sum_{e \in E} |(\underline{w}^\dagger D_0)_e|^2 x_e \in \mathcal{H}$$

see notes for alternate proof

So $\underline{w}^\dagger L \underline{w} \neq 0$. So L is invertible. So $T(G;\underline{x})$ is stable. ■

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So $T(G;\underline{x})$ has the strong Rayleigh property: $\Delta_{ef} T(G;\underline{a}) \geq 0$ for all $e \neq f \in E$ and $\underline{a} \in \mathbb{R}^E$.

Consider the case that all $a_e > 0$. We can choose $T \in \mathcal{T}(G)$ with probability

$$\frac{\underline{a}^T}{T(G;\underline{a})}$$

$$T = T(G,\underline{x}) = T^e + x_e T_e$$

evaluate at $\underline{x} = \underline{a}$ and divide by T.

$$1 = \frac{T^e}{T} + \frac{x_e T_e}{T}$$

T^e/T = Prob[tree doesn't use e]

$x_e T_e/T$ = Prob[tree does use e]

$$\Delta_{ef} = T_e^f T_f^e - T_{ef} T^{ef} = T_e T_f - T_{ef} T$$

$$\frac{x_e x_f \Delta_{ef}}{T^2} = \left(\frac{x_e T_e}{T}\right)\left(\frac{x_f T_f}{T}\right) - \left(\frac{x_e x_f T_{ef}}{T}\right) \geq 0 \quad \text{at } \underline{x} = \underline{a} > \underline{0}$$

$$\left(\frac{x_e T_e}{T}\right)\left(\frac{x_f T_f}{T}\right) \geq \left(\frac{x_e x_f T_{ef}}{T}\right)$$

So

Prob[tree uses e] · Prob[tree uses f] ≥ Prob[tree uses both e & f]

These events are "<u>negatively correlated</u>".

Let

$$Z(\underline{x}) = \sum_{S \subseteq [m]} c(S)\,\underline{x}^S$$

be multiaffine with non-negative coefficients.

$$\mathcal{S} = \text{supp}(Z) = \{S \subseteq [m];\ c(S) \neq 0\}.$$

Fix $\underline{a} \in \mathbb{R}^m$ with $\underline{a} > \underline{0}$. Probability measure $\mu_{\underline{a}}$ on $\mathcal{S}$,

$$\mu_{\underline{a}}(S) = \frac{c(S)\,\underline{a}^S}{Z(\underline{a})}.$$

If Z is stable then for all $i \neq j$ in $[m]$ and all $\underline{a} \in \mathbb{R}^m$ with $\underline{a} > \underline{0}$, then the events $i \in S$ and $j \in S$ are negatively correlated.

* Are there examples other than graphs?

$T(G;\underline{x}) = \det(D_0 X D_0^T)$ is stable

For any $r \times m$ matrix M of rank r over $\mathbb{C}$, $\det(MXM^*)$ is stable.

By Binet-Cauchy

$$\det(MXM^*) = \sum_{\substack{S \subseteq [m] \\ |S| = r}} |\det M(|S])|^2\, \underline{x}^S$$

$$= \sum_{B \in \mathcal{M}} |\det M(|B])|^2\, \underline{x}^B$$

where $\mathcal{M}$ is the set of bases of the matroid represented by M.

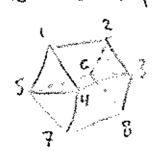
If $|\det(|B])|^2 = 1$ for all $B \in \mathcal{M}$ then

$$\det(MXM^*) = \sum_{B \in \mathcal{M}} \underline{x}^B.$$

These are known as "complex totally unimodular matroids" (CTU).

* graphs are CTU
* regular matroids are CTU (real & CTU)
  representable over $GF(2)$ and $GF(3)$
* CTU representable over $GF(3)$ & $GF(4)$
* A few more examples are known

eg



Vámos V_8

Coplanar 4-sets: 1234, 1256, 3478, 5678, 3456

If V_8 were representable over any field then 1278 would also be coplanar. It's not.

But $B(V_8, \underline{x})$ is stable.

For the negative correlation property we only need $B = B(\mathcal{M}; \underline{x})$ has $\Delta_{ij} B(\underline{a}) \geq 0$ for all $i \neq j$ in $[m]$ and $\underline{a} \in \mathbb{R}^m$, $\underline{a} > \underline{0}$. This is the Rayleigh condition.

*A binary matroid is Rayleigh if and only if it has no S_8 minor.

$$S_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \text{ over } GF(2)$$

* A binary matroid is strong Rayleigh if and only if it's regular.

* every matroid of rank (or corank) 3 is Raleigh.

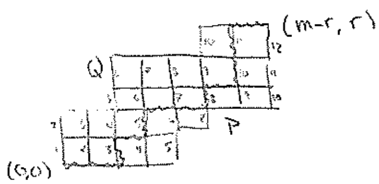
Theorem: If

$$Z = \sum_{S \subseteq [m]} c(S) \underline{x}^S$$

has the Rayleigh property, has non-negative coefficients, and is homogeneous then $\mathrm{supp}(Z)$ is the set of bases of a matroid.

2016 01 29

Lattice path matroids



Puzzle: The collection of sets recording lattice paths between P and Q is the set of bases of a matroid $M[P,Q]$ of rank r on $\{1,2,\ldots,m\}$

Yan: Lattice path matroids have the Rayleigh property. In fact for any $\{e,f\} \subseteq [m]$: $\Delta_{ef} B(\underline{x})$ has only positive coefficients

Matroid Inequalities

Let $\mathcal{M} = (E, \mathcal{B})$ be a matroid, $|E| = m$, $\mathcal{B}$ set of bases

$$B = B(\mathcal{M}) = \sum_{B \in \mathcal{B}} \underline{x}^B \text{ basis enumerator}, \quad \underline{x}^B = \prod_{e \in B} x_e$$

Assume $\mathcal{M}$ is regular.

Stanley (1981) Let $S \subseteq E$, $|S| = d$. For each $0 \le k \le d$, let N_k be the number of bases B such that $|B \cap S| = k$. Then, for all $1 \le k \le d-1$,

$$(*) \qquad \frac{N_k^2}{\binom{d}{k}^2} \ge \frac{N_{k-1}}{\binom{d}{k-1}} \cdot \frac{N_{k+1}}{\binom{d}{k+1}}$$

logarithmic concavity

Hence

$$\left(\frac{N_k}{\binom{d}{k}} \; ; \; 0 \le k \le d \right)$$

is unimodal. Hence $(N_k; 0 \le k \le d)$ is unimodal.

Godsil (1982): If $\mathcal{M}$ is regular then

$$(**) \qquad \sum_{k=0}^{d} N_k t^k$$

has only real roots.

Newton's inequalities: If $(**)$ has only real roots then $(*)$ holds.

Choe-W. (2006): If $B(\mathcal{M})$ is stable then $(**)$ has only real roots.

Proof: Let, in $B(\mathcal{M})$,

$$x_e = \begin{cases} t & \text{if } e \in S, \\ 1 & \text{if } e \notin S. \end{cases}$$

By diagonalization and specialization, the result is stable. This polynomial is $(**)$. ▫

Let $S \subseteq E$ and $R \subseteq E$ with $R \cap S = \emptyset$, $|S| = d$, $|R| = c$. Let N_{jk} be the number of bases B such that $|B \cap R| = j$, $|B \cap S| = k$. Then

$$\sum_{j=0}^{c} \sum_{k=0}^{d} N_{jk} r^j s^k$$

is stable if $B(\mathcal{M})$ is stable. Let

$$x_e = \begin{cases} r & e \in R \\ s & e \in S \\ 1 & \text{else} \end{cases}$$

"I take my administrative duties very, seriously" *wink*

Grace-Walsh-Szegő Theorem

A <u>circular region</u> $\mathcal{A}$ is a proper connected subset of $\mathbb{C}$ that is either open or closed and is bounded by either a circle or a straight line. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. Lines are circles on $\hat{\mathbb{C}}$ that pass through ∞.

statement may be flawed, see before proof later

Let $f(x_1, x_2, \ldots, x_m)$ be a polynomial that is invariant under all permutations of the variables:

$$\sigma f(x_1, \ldots, x_m) = f(x_{\sigma 1}, \ldots, x_{\sigma m}) = f$$

If $\mathcal{A}$ is a circular region and f is a symmetric polynomial and either $\deg f = m$ or $\mathcal{A}$ is convex, then

$$\forall z_1, \ldots, z_m \in \mathcal{A} : \exists z \in \mathcal{A} : f(z_1, z_2, \ldots, z_m) = f(z, z, \ldots, z)$$

2016 02 01

Question: Given a linear transformation $T : \mathbb{C}[\underline{x}] \to \mathbb{C}[\underline{x}]$, which T have the property that if f is stable then $T(f)$ is stable?

Let $\mathbb{C}[\underline{x}]^{MA}$ be the set of multiaffine polynomials.
1st step: $T : \mathbb{C}[\underline{x}]^{MA} \to \mathbb{C}[\underline{x}]$ linear transformations that are <u>stability preservers</u>

<u>Ingredients</u>

Recall:

Proposition 2.7(b): $0 \neq f \in \mathbb{C}[\underline{x}]$ stable and $g \in \mathbb{C}[\underline{x}]$. Then $g + yf$ is stable if and only if $\mathrm{Im}\left(\frac{g(z)}{f(z)}\right) \geq 0$ for all $\underline{z} \in \mathcal{H}^m$.

Exercise 3.3 and Proposition 2.12(b) will be used in the degenerate case of the theorem.

Lemma [Lieb-Sokal, ~1980s]: If $g + yf \in \mathbb{C}[\underline{x}, y]$ is stable and $\deg_{x_i}(f) \leq 1$ then $g - \frac{\partial}{\partial x_i} f$ is stable.

Proof: By permutation, $i = 1$. Specializing $y = 0$, g is stable. If $\partial_1 f \equiv 0$ we're done, so $\partial_1 f \not\equiv 0$, so $f \not\equiv 0$. By contraction, $f = \frac{\partial}{\partial y}(g + yf)$ is stable.

(*) — If $z_1, z_2 \in \mathcal{H}$ then $z_1 - z_2^{-1} \in \mathcal{H}$. Hence $yf(x_1 - y^{-1}, x_2, \dots, x_m)$ is stable as it is $-\partial_1 f + yf$. So for any $z \in \mathcal{H}^m$,

$$\mathrm{Im}\left(\frac{g - \partial_1 f}{f}\right) = \mathrm{Im}\left(\frac{g}{f}\right) + \mathrm{Im}\left(\frac{-\partial_1 f}{f}\right) \geq 0$$

since each term is nonnegative by proposition 2.7(b). By 2.7(b) in the other direction, $g - \partial_1 f + yf$ is stable. Specializing $y=0$ gives $g - \partial_1 f$ is stable. ∎

Theorem 3.4 [1st instance of Borcea-Brändén "Master Theorem"]:
Let $T: \mathbb{C}[\underline{x}]^{MA} \to \mathbb{C}[\underline{x}]$ be a linear transformation. Then T preserves stability if and only if either:
(a) $T(f) = \eta(f) \cdot p$ for some linear $\eta: \mathbb{C}[\underline{x}]^{MA} \to \mathbb{C}$ and stable p; or
(b) $T(\underline{x} + \underline{y})^{[m]}$ is stable in $\mathbb{C}[\underline{x}, \underline{y}]$.
Here $y = (y_1, \dots, y_m)$ and

$$(\underline{x} + \underline{y})^{[m]} = \prod_{i=1}^{\infty} (x_i + y_i)$$

and

$$T(\underline{x} + \underline{y})^{[m]} = \sum_{S \subseteq [m]} T(\underline{x}^S)\, \underline{y}^{[m] \setminus S}.$$

Proof: If (a) holds then T preserves stability since $T(f)$ is stable for all $f \in \mathbb{C}[\underline{x}]^{MA}$. Assume (b). Invert all the y variables. $y_i \mapsto -y_i^{-1}$. So $\underline{y}^{[m]} T(\underline{x} - \underline{y}^{-1})^{[m]}$ is stable. Let $f(t_1, t_2, \dots t_m)$ be stable and multiaffine. So $\underline{y}^{[m]} T(\underline{x} - \underline{y}^{-1})^{[m]} f(\underline{t})$ is stable in $\mathbb{C}[\underline{x}, \underline{y}, \underline{t}]$. This equals

$$\sum_{S \subseteq [m]} T(\underline{x}^S)(-\underline{y}^S) f(\underline{t}).$$

Apply Lieb-Sokal: replace each y_i by $-\frac{\partial}{\partial t_i}$. The result is stable. Specialize to $\underline{t} = 0$. The result is stable.

$$\sum_{S \subseteq [m]} T(\underline{x}^S) \underbrace{\frac{\partial}{\partial \underline{t}^S} f(\underline{t})\Big|_{\underline{t}=0}}_{\text{coefficient of } t^S \text{ in } f(\underline{t}).} = T(f(\underline{x})).$$

Conversely, assume that T preserves stability. For any $\underline{w} \in \mathcal{H}^m$. $(\underline{x} + \underline{w})^{[m]}$ is stable, so $T(\underline{x} + \underline{w})^{[m]}$ is stable. If $\exists\, \underline{w} \in \mathcal{H}^m$ such that $T(\underline{x} + \underline{w})^{[m]} \equiv 0$ then by Exercise 3.3 & Proposition 2.12(b), we end up in case (a).

Otherwise, $\forall \underline{w} \in \mathcal{H}$: $T(\underline{x}+\underline{w})^{[m]} \not\equiv 0$. So $\forall \underline{z} \in \mathcal{H}^m$: $T(\underline{z}+\underline{w})^{[m]} \neq 0$.
So $T(\underline{x}+\underline{y})^{[m]}$ is stable in $\mathbb{C}[\underline{x},\underline{y}]$. ▨

2016 02 03

Polarization

Let $f(x) \in \mathbb{C}[x]$ be a univariate polynomial of degree at most m. There is a unique polynomial in m variables $F(x_1, \ldots x_m)$ such that
* F is multiaffine
* F is a symmetric polynomial
* F diagonalizes to f, that is $F(x,x,..,x) = f(x)$

This F is denoted $\mathrm{Pol}_m f(\underline{x})$; the m^{th} polarization of f.

eg// $f = x^d$ $\quad \mathrm{Pol}_m(x^d) = \binom{m}{d}^{-1} e_d(x_1, \ldots, x_m)$
In general,

$$\mathrm{Pol}_m\left(\sum_{i=0}^{d} c_i x^i\right) = \sum_{i=0}^{d} c_i \binom{m}{i}^{-1} e_i(x_1, \ldots, x_m)$$

Theorem [Grace-Walsh-Szegö]: Let $f(x) \in \mathbb{C}[x]$ have degree at most m. Let A be a circular region. Assume that either $\deg f = m$ or A is convex. Then, for any $z_1, z_2, \ldots z_m \in A$ there exists $z \in A$ such that

$$\mathrm{Pol}_m f(z_1, \ldots, z_m) = f(z).$$

Möbius Transformations

$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ Riemann sphere
Transformations $z \mapsto \varphi(z)$ on $\hat{\mathbb{C}}$

$$\varphi(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$

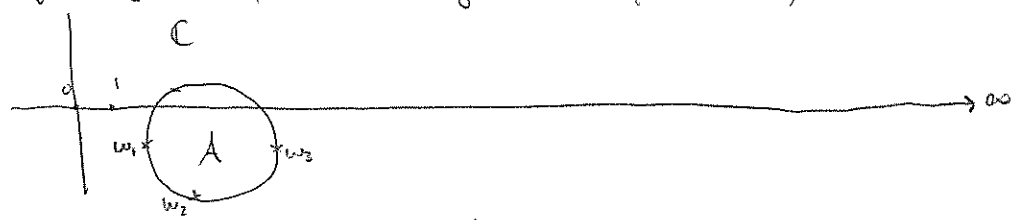
Functorial composition makes this a group $\mathcal{M}$, Möbius group.

$$GL(2,\mathbb{C}) \to \mathcal{M}$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \left(z \mapsto \frac{az+b}{cz+d}\right)$$

is surjective with kernel $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $a \neq 0$, is a group homomorphism
$\mathcal{M} = PGL(2,\mathbb{C})$ is the (complex) 2×2 projective general linear group

Key Fact $\forall$ distinct $z_1, z_2, z_3 \in \hat{\mathbb{C}}$, $\forall$ distinct $w_1, w_2, w_3 \in \hat{\mathbb{C}}$, $\exists$ a unique $\varphi \in \mathcal{M}$ such that $\varphi(z_1) = w_1$, $\varphi(z_2) = w_2$, $\varphi(z_3) = w_3$.
$\mathcal{M}$ is sharply transitive on triples of distinct points.
<u>Puzzle</u>. Do this for $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$.

Corollary: Any two open circular regions are equivalent by $\mathcal{M}$



Let w_1, w_2, w_3 be on the boundary of A. Let $\varphi \in \mathcal{M}$ such that $\varphi(0) = w_1$, $\varphi(1) = w_2$, $\varphi(\infty) = w_3$, or $\psi \in \mathcal{M}$ such that $\psi(0) = w_2$, $\psi(1) = w_1$, $\psi(\infty) = w_3$. One of these maps $\mathcal{H}$ to A.

§4.1 Consider $f(x) \in \mathbb{C}[x]$ of degree at most m and an open circular region A and $z_1, \dots z_m \in A$. Let $F(\underline{x}) = \mathrm{Pol}_m f(\underline{x})$. Let $v = F(z_1, \dots z_m)$. Consider $\tilde{f} = f - v$, and $\tilde{F}(\underline{x}) = \mathrm{Pol}_m \tilde{f}(\underline{x})$. Note $\tilde{F}(z_1, \dots, z_m) = 0$. We want to find $z \in A$ such that $\tilde{f}(z) = 0$.
Map A to $\mathcal{H}$ by some $\varphi \in \mathcal{M}$. If A is convex or $\deg f = m$ then the transformed polynomial $G(x_1, \dots x_m)$ is stable if and only if $F(x_1, \dots, x_m)$ is non-vanishing on A.

Lemma: For $f(x) \in \mathbb{C}[x]$ of degree at most m, $\mathrm{Pol}_m f(\underline{x})$ is stable if and only if $f(x)$ is stable.
This is equivalent to G-W-S.

One direction is easy
If $F(x_1, \dots x_m) = \mathrm{Pol}_m f(\underline{x})$ is stable (and $\not\equiv 0$) then $F(z_1, \dots, z_m) \neq 0$ for all $z \in \mathcal{H}^m$. So $f(z) = F(z, \dots, z) \neq 0$ for all $z \in \mathcal{H}$. So f is stable and $\not\equiv 0$.

Strategy: Given $f(x) = \prod_{i=1}^{d} (x - \theta_i)$ stable, $d \leq m$,
$$F_0(\underline{x}) = \prod_{i=1}^{d} (x_i - \theta_i)$$
is multiaffine, diagonalizes to $f(x)$, and is stable.

But it is not invariant under all permutations of $(x_1,\dots,x_m)$. Note

$$\left(\frac{1}{m!}\sum_{\sigma\in S_m}\sigma\right)F_0(\underline{x}) = \mathrm{Pol}_m f(\underline{x})$$

This operator

$$\frac{1}{m!}\sum_{\sigma\in S_m}\sigma$$

preserves stability.

2016 02 05

Proof (of lemma): If $\mathrm{Pol}_m f(\underline{x})$ is stable then $f(x)$ is stable, by diagonalization. Conversely, let $f(x)=c\prod_{i=1}^{d}(x-\theta_i)$ with $c\neq 0$, $d\leq m$, and $\mathrm{Im}(\theta_i)\leq 0$, for each $1\leq i\leq d$. Define a sequence of polynomials $F_0(\underline{x}), F_1(\underline{x}),\dots$ in $\mathbb{C}[\underline{x}]$, each of which is stable, with $\lim_{k\to\infty}F_k(\underline{x})=\mathrm{Pol}_m f(\underline{x})$. Then convergence is uniform on compact sets, so Hurwitz implies $\mathrm{Pol}_m f(\underline{x})$ is stable.
Starting point:

$$F_0(\underline{x}) = c\prod_{i=1}^{d}(x_i-\theta_i)$$

This is stable, multiaffine, and diagonalizes to $f(x)$. However, it is not symmetric under all permutations. Pick two variables x_i and x_j, say x_1 and x_2. Symmetrize by x_1 and x_2.

$$F(\underline{x}) \longmapsto \hat{F}(\underline{x}) = \frac{1}{2}\left(F(x_1,x_2,x_3,\dots,x_m)+F(x_2,x_1,x_3,\dots,x_m)\right).$$

This operation $F\longmapsto\hat{F}$ preserves stability. Linear transformation $T:\mathbb{C}[\underline{x}]^{MA}\to\mathbb{C}[\underline{x}]$ preserves stability if and only if $T(\underline{x}+\underline{y})^{[m]}$ is stable in $\mathbb{C}[\underline{x},\underline{y}]$. Let $\tau=(1\,2)$ be the transposition exchanging x_1 and x_2. Let T_{12}^{λ}: $\lambda\in[0,1]$ $\mathbb{C}[\underline{x}]^{MA}\to\mathbb{C}[\underline{x}]$ be given by $T_{12}^{\lambda}(F)=(1-\lambda)F+\lambda\tau F$. This preserves stability. We only need to show that $T_{12}^{\lambda}(\underline{x}+\underline{y})^{[m]}$ is stable.

$$=(1-\lambda)(x_1+y_1)(x_2+y_2)\prod_{i=3}^{m}(x_i+y_i)+\lambda(x_2+y_1)(x_1+y_2)\prod_{i=3}^{m}(x_i+y_i)$$

This is stable if and only if

$$(1-\lambda)(x_1+y_1)(x_2+y_2)+\lambda(x_2+y_1)(x_1+y_2)$$

is stable. Exercise 4.4: For all $0\leq\lambda\leq 1$ it is stable.
Start with $f(x)=c\prod_{i=1}^{d}(x-\theta_i)$. Let $F_0(\underline{x})=c\prod_{i=1}^{d}(x_i-\theta_i)$. For each $n\in\mathbb{N}$, let $\tau_n=(i_n,j_n)$ be a transposition, and let $F_{n+1}(\underline{x})=T_{i_nj_n}^{1/2}F_n(\underline{x})$.

Then each $F_n(\underline{x})$ is multiaffine, diagonalizes to $f(x)$, and is stable, by exercise 4.4. The issue remaining is convergence to $\mathrm{Pol}_m f(\underline{x})$.
Measuring imbalance: For $F(\underline{x}) \in \mathbb{C}[\underline{x}]^{MA}$:

$$F(x) = \sum_{S \subseteq [m]} c(S) \underline{x}^S,$$

the ij-imbalance is

$$\omega_{ij}(F) = \frac{1}{2} \sum_{\substack{S \subseteq [m] \\ |S \cap \{i,j\}| = 1}} \left| c(S) - c(S \triangle \{i,j\}) \right| \geq 0.$$

The total imbalance is

$$\|F\| = \sum_{\{i,j\} \subseteq [m]} \omega_{ij}(F) \geq 0.$$

We have $\|F\|=0$ if and only if $\omega_{ij}(F)=0$ for all ij, if and only if F is symmetric under all permutations.
Choose $\{i_n, j_n\}$ to maximize the summand $\omega_{i_n j_n}(F_n)$,

$$\omega_{i_n j_n}(F_n) \geq \binom{m}{2}^{-1} \|F_n\|$$

Then

$$\|F_{n+1}\| \leq \left(1 - \binom{m}{2}\right)^{-1} \|F_n\|$$

by exercise 4.6 (which is more general).
eg. let

$$P = \sum_{S \subseteq [m]} c(S) \underline{x}^S, \quad Q = T_{12}^{1/2} P = \frac{1}{2} \sum_{S \subseteq [m]} \left(c(S) + c(\tau S)\right) \underline{x}^S$$

$\omega_{12}(Q) = 0, \quad \omega_{13}(Q) + \omega_{23}(Q) \leq \omega_{13}(P) + \omega_{23}(P), \quad \omega_{34}(Q) = \omega_{34}(P)$
So $\|Q\| \leq \|P\| - \omega_{12}(P)$. So as $n \to \infty$, $\|F_n\| \to 0$. So $F_n \to \mathrm{Pol}_m f$. ∎

2016 02 08

If Ω is a set with the action of a group G, then Ω^G is the set of G-invariant elements.
eg. $\mathbb{C}[x]^{S_m}$ is the space of symmetric polynomials.

G-W-S: isomorphism of $\mathbb{C}$-vector spaces

$$\mathbb{C}[X]^{S_m} \underset{\mathrm{diag}}{\overset{\mathrm{Pol}_m}{\rightleftarrows}} \left(\mathbb{C}[\underline{x}]^{MA}\right)^{S_m}$$

$$X^d \longleftrightarrow \binom{m}{d}^{-1} e_d(\underline{x})$$

$f(x) \in \mathbb{C}[x]$ is stable if and only if $\mathrm{Pol}_m f(\underline{x})$ is stable.

Extend this to have more than one variable on LHS. Fix $\kappa : [m] \to \mathbb{N}$.

Look at $\mathbb{C}[\underline{x}]^{\le\kappa}$. This has basis $\{\underline{x}^\alpha;\ \alpha \in \mathbb{N}^m,\ \alpha \le \kappa\}$. Let

$$I(\kappa) = \{(i,j);\ i \in [m] \text{ and } 1 \le j \le \kappa(i)\}$$

index set for $\underline{u} = \{u_{ij};\ (i,j) \in I(\kappa)\}$. For each $i \in [m]$ let $K(i) = \{(i,j);\ 1 \le j \le \kappa(i)\}$.

Polarize x_i to the $\kappa(i)$-th order, $\mathrm{Pol}^{(i)}_{\kappa(i)}$. For $i \ne j$ these commute.

$$\mathrm{Pol}_\kappa : \mathbb{C}[\underline{x}]^{\le\kappa} \to (\mathbb{C}[\underline{u}]^{MA})^\Gamma \hookrightarrow \mathbb{C}[\underline{u}]^{MA}$$

where

$$\Gamma = \mathfrak{S}(K(1)) \times \cdots \times \mathfrak{S}(K(m)),$$

a Young subgroup of $\mathfrak{S}(I(\kappa))$, by

$$\mathrm{Pol}_\kappa = \mathrm{Pol}^{(m)}_{\kappa(m)} \circ \cdots \circ \mathrm{Pol}^{(1)}_{\kappa(1)}.$$

Also

$$\Delta : \mathbb{C}[\underline{u}]^{MA} \to \mathbb{C}[\underline{x}]$$
$$\Delta(u_{ij}) = x_i$$

extended algebraically. Then we have mutually inverse isomorphisms

$$\mathrm{Pol}_\kappa : \mathbb{C}[\underline{x}]^{\le\kappa} \underset{\Delta}{\overset{\mathrm{Pol}_\kappa}{\rightleftarrows}} (\mathbb{C}[\underline{u}]^{MA})^\Gamma.$$

Note $\Delta(\underline{u}^S) = \underline{x}^{\alpha_S}$ where $\alpha_S(i) = |S \cap K(i)|$ for all $i \in [m]$.

<u>G-W-S</u>: $f(\underline{x})$ is stable if and only if $\mathrm{Pol}_\kappa f(\underline{u})$ is stable.

Theorem 5.2: Fix $\kappa : [m] \to \mathbb{N}$. Let $T : \mathbb{C}[\underline{x}]^{\le\kappa} \to \mathbb{C}[\underline{x}]$ be linear. Then T preserves stability if and only if either

(a) $T(f) = \eta(f) \cdot p$ for some $\eta : \mathbb{C}[\underline{x}]^{\le\kappa} \to \mathbb{C}$ and stable $p \in \mathbb{C}[\underline{x}]$, or

(b) $T(\underline{x}+\underline{y})^\kappa = T\prod_{i=1}^m (x_i+y_i)^{\kappa(i)}$ is stable.

Proof: Define $\tilde{T} : \mathbb{C}[\underline{u}]^{MA} \to \mathbb{C}[\underline{x}]$ as follows. For $S \subseteq I(\kappa)$:

$$\tilde{T}(\underline{u}^S) = T(\underline{x}^{\alpha_S}) = T \circ \Delta(\underline{u}^S).$$

So $\tilde{T} = T \circ \Delta$.

$$\begin{tikzcd}
(\mathbb{C}[\underline{u}]^{MA})^\Gamma \arrow[r, hook] & \mathbb{C}[\underline{u}]^{MA} \arrow[dl, "\Delta"'] \arrow[d, "\tilde{T}"] \\
\mathbb{C}[\underline{x}]^{\le\kappa} \arrow[u, "\mathrm{Pol}_\kappa"] \arrow[r, "T"'] & \mathbb{C}[\underline{x}]
\end{tikzcd}$$

Since the diagram commutes, $\tilde{T}$ (restricted to $(\mathbb{C}[\underline{u}]^{MA})^{\Gamma}$) is $T\circ \mathrm{Pol}_K$.
⊛ T preserves stability if and only if $\tilde{T}$ preserves stability.
Since $T = \tilde{T}\circ \mathrm{Pol}_K$ and Pol_K preserves stability, if $\tilde{T}$ preserves stability then so does T. Conversely, $\tilde{T} = T\circ\Delta$ so same thing.
By theorem 3.4, $\tilde{T}$ preserves stability if case (a) or

$$\tilde{T}(\underline{u}+\underline{v})^{I(K)} = \tilde{T}\prod_{(i,j)\in I(K)}(u_{ij}+v_{ij}).$$

Case (a): $\tilde{T}(f) = \tilde{\eta}(f)\cdot p(\underline{x})$. Define $\eta : \mathbb{C}[\underline{x}]^{\leq K} \to \mathbb{C}$ by $\eta(f) = \tilde{\eta}(\mathrm{Pol}_K f)$. So T is in case (a). Conversely, if $T(f) = \eta(f)\cdot p$ then let $\tilde{\eta} = \eta\circ\Delta$. $\tilde{T}$ is in case (a) if and only if T is in case (b).

$$\begin{aligned}
\tilde{T}(\underline{u}+\underline{v})^{I(K)} &= T\circ\Delta(\underline{u}+\underline{v})^{I(K)} \\
&= T\prod_{i=1}^{m}\prod_{j=1}^{K(i)}(x_i+v_{ij}) \\
&= T\circ \mathrm{Pol}_K^{(y)}\prod_{i=1}^{m}(x_i+y_i)^{K(i)} \\
&= \mathrm{Pol}_K^{(y)}\, T(\underline{x}+\underline{y})^{K}
\end{aligned}$$

$\tilde{T}(\underline{u}+\underline{v})^{I(K)}$ is stable if and only if $T(\underline{x}+\underline{y})^{K}$ is stable. ▨

2016 02 10

Note on exercises:
- must do: 2.5, 2.11, 4.3, 4.5, 5.2(b), 5.5, 7.5
- may do: 2.8, 3.5, 4.4
- don't do: 2.6, 3.6 (5.2(d))

Theorem 5.3: Let $T : \mathbb{C}[\underline{x}] \to \mathbb{C}[\underline{x}]$ be linear. Then T preserves stability if and only if either
(a) $T(f) = \eta(f)p$ for some $\eta : \mathbb{C}[\underline{x}] \to \mathbb{C}$ and stable p, or
(b) $T(e^{-\underline{x}\underline{y}})$ is in $\mathfrak{S}[\underline{x},\underline{y}]$.

Here

$$T(e^{-\underline{x}\underline{y}}) = T\sum_{\alpha:[m]\to\mathbb{N}}\frac{(-\underline{x}\underline{y})^{\alpha}}{\alpha!} = \sum_{\alpha:[m]\to\mathbb{N}}(-1)^{|\alpha|}\,T(\underline{x}^{\alpha})\,\frac{\underline{y}^{\alpha}}{\alpha!}$$

where

$$\alpha! = \prod_{i=1}^{m} \alpha(i)!, \quad (-\underline{x}\underline{y})^{\alpha} = (-1)^{|\alpha|} \prod_{i=1}^{m} (x_i y_i)^{\alpha(i)}, \quad |\alpha| = \sum_{i=1}^{m} \alpha(i).$$

Also $\mathfrak{S}[\underline{x},\underline{y}]$ stable polynomials in $\mathbb{C}[\underline{x},\underline{y}]$, and $\overline{\mathfrak{S}[\underline{x},\underline{y}]}$ is its closure in $\mathbb{C}[[\underline{x},\underline{y}]]$ with respect to convergence of sequences which converge uniformly on compact subsets of $\mathbb{C}^{2m}$.

Note: $T(e^{-\underline{x}\underline{y}})$ is in $\mathbb{C}[\underline{x}][[\underline{y}]]$.

Theorem 5.4: Let

$$F(\underline{x},\underline{y}) = \sum_{\alpha:[m]\to\mathbb{N}} P_\alpha(\underline{x})\underline{y}^{\alpha} \in \mathbb{C}[\underline{x}][[\underline{y}]].$$

Then $F(\underline{x},\underline{y})$ is in $\overline{\mathfrak{S}[\underline{x},\underline{y}]}$ if and only if $\forall \beta:[m]\to\mathbb{N}$ the polynomial

$$\sum_{\alpha\le\beta} (\beta)_\alpha P_\alpha(\underline{x})\underline{y}^{\alpha}$$

is stable in $\mathbb{C}[\underline{x},\underline{y}]$. Here

$$(\beta)_\alpha = \begin{cases} \dfrac{\beta!}{(\beta-\alpha)!} & \text{if } \alpha\le\beta \\ 0 & \text{if } \alpha \not\le \beta \end{cases}$$

Exercise 5.5: Theorems 5.1 and 5.4 together imply Theorem 5.3. (Need exercise 5.2 also.)

Exercise 5.2(a): $T:\mathbb{C}[\underline{x}]\to\mathbb{C}[\underline{x}]$ preserves stability if and only if $\forall \kappa:[m]\to\mathbb{N}$, $T:\mathbb{C}[\underline{x}]^{\le\kappa}\to\mathbb{C}[\underline{x}]$ preserves stability.

Lemma 5.6: Fix $\beta:[m]\to\mathbb{N}$. Define $T:\mathbb{C}[\underline{y}]\to\mathbb{C}[\underline{y}]$ by $T(\underline{y}^{\alpha}) = (\beta)_\alpha \underline{y}^{\alpha}$ and extending linearly. This preserves stability.

Proof: It suffices to show that for all $\kappa:[m]\to\mathbb{N}$, $T:\mathbb{C}[\underline{y}]^{\le\kappa}\to\mathbb{C}[\underline{y}]$ preserves stability. By theorem 5.2(b) it suffices to show that $T(\underline{y}+\underline{u})^{\kappa}$ is stable. This is

$$T\prod_{i=1}^{m}(y_i+u_i)^{\kappa(i)} = T\prod_{i=1}^{m}\left(\sum_{j=0}^{\kappa(i)}\binom{\kappa(i)}{j} y_i^{j} u_i^{\kappa(i)-j}\right)$$

$$= \prod_{i=1}^{m} \sum_{j=0}^{\kappa(i)} j! \binom{\kappa(i)}{j}\binom{\beta(i)}{j} y_i^j u_i^{\kappa(i)-j} \qquad (*)$$

It suffices to show that each factor is stable. Each is a homogeneous polynomial with positive real coefficients. Let $t = y/u$. Consider

$$\sum_{j=0}^{k} j! \binom{k}{j}\binom{b}{j} t^j$$

Each factor in $(*)$ is stable if and only if $\forall k, b \in \mathbb{N}$ this polynomial has only real roots (i.e. univariate real stable). Let

$$g(t) = \left(1 + \frac{d}{dt}\right)^k t^b.$$

Then $f(t) = t^b g(1/t)$, by induction on k. It suffices to show that $g(t)$ is stable. It suffices to show that $1 + \frac{d}{dt}$ preserves stability on $\mathbb{C}[t]^{\leq a}$ for all $a \in \mathbb{N}$.

$$\left(1 + \frac{d}{dt}\right)(t+u)^a = (t+u)^a + a(t+u)^{a-1} = (t+u)^{a-1}(t+u+a).$$

This is stable. Done. $\blacksquare$

2016 02 12

Note for previous proof: if $y, u \in \mathcal{H}$ then y/u can be any complex number not in $(-\infty, 0]$. With $p(y,u) = u^d f(y/u)$, if f is stable it can only vanish on $(-\infty, 0]$.

Theorem: Let $f(\underline{x}, \underline{y}) \in \mathbb{C}[\underline{x}][[\underline{y}]]$,

$$F(\underline{x}, \underline{y}) = \sum_{\alpha \in \mathbb{N}^m} P_\alpha(\underline{x}) \underline{y}^\alpha.$$

Then $F(\underline{x}, \underline{y}) \in \overline{\mathfrak{S}[\underline{x}, \underline{y}]}$ if and only if for every $\beta \in \mathbb{N}^m$

$$\sum_{\alpha \leq \beta} (\beta)_\alpha P_\alpha(\underline{x}) \underline{y}^\alpha$$

is stable in $\mathbb{C}[\underline{x}, \underline{y}]$.

Lemma: $T_\beta : \mathbb{C}[\underline{y}] \to \mathbb{C}[\underline{y}]$ by $T_\beta(\underline{y}^\alpha) = (\beta)_\alpha \underline{y}^\alpha$ preserves stability for any $\beta \in \mathbb{N}^m$.

Proof (of theorem): First, assume that $F(\underline{x}, \underline{y}) \in \overline{\mathfrak{S}[\underline{x}, \underline{y}]}$. Let $F_n(\underline{x}, \underline{y}) \to F(\underline{x}, \underline{y})$ be stable polynomials converging to $F(\underline{x}, \underline{y})$ (uniformly on compact sets). Consider any $\beta \in \mathbb{N}^m$. So by the lemma, each of $T_\beta(F_n(\underline{x},\underline{y}))$ is stable. The limit of $T_\beta(F_n(\underline{x},\underline{y}))$ is

$$T_\beta(F(\underline{x},\underline{y})) = \sum_{\alpha \le \beta} (\beta)_\alpha P_\alpha(\underline{x}) \underline{y}^\alpha,$$

a polynomial which is a limit of stable polynomials. So this is also stable by Hurwitz.

Conversely, assume that for all $\beta \in \mathbb{N}^m$

$$\sum_{\alpha \le \beta} (\beta)_\alpha P_\alpha(\underline{x}) \underline{y}^\alpha$$

is stable. Consider this for $\beta = n\mathbb{1} = (n,n,\dots,n)$ as $n \to \infty$. Apply this to $\underline{y}/n$.

$$F_n(\underline{x},\underline{y}) = \sum_{\alpha \le n\mathbb{1}} (n\mathbb{1})_\alpha P_\alpha(\underline{x}) \frac{\underline{y}^\alpha}{n^{|\alpha|}} \longrightarrow F(\underline{x},\underline{y})$$

as $n \to \infty$, as

$$\frac{(n\mathbb{1})_\alpha}{n^{|\alpha|}} = \prod_{i=1}^{m} \frac{\alpha(i)!\binom{n}{\alpha(i)}}{n^{\alpha(i)}} \longrightarrow 1$$

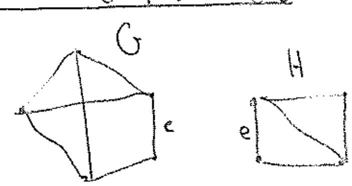
as $n \to \infty$. Convergence is uniform on compact sets. See paper for details. ▪

eg. x^r is stable. Its polarization is stable by GWS:

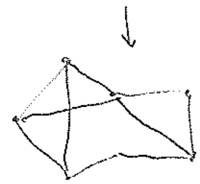
$$\mathrm{Pol}_m(x^r) = \binom{m}{r}^{-1} e_r(x_1,\dots,x_m) = \binom{m}{r}^{-1} \sum_{\substack{S \subseteq [m] \\ |S| = r}} \underline{x}^S.$$

This is $\binom{m}{r}^{-1}$ times the generating function for bases of a uniform matroid $\mathcal{U}_{r,m}$.

2-sums of matroids



$$G = \sum_{T \in \mathcal{T}(G)} \underline{x}^T \text{ is stable, same for } H$$



$G \oplus_e H$

Let f,g be stable with variable sets that intersect only on $\{x_i\}$. Assume f,g are both multiaffine.

$$f = f^i + x_i f_i, \quad g = g^i + x_i g_i$$

Two-sum of f,g: $f \oplus_i g = f^i g_i + f_i g^i$.

If f,g are stable, then $f \oplus_i g$ is stable.

2016 02 22

Bröndén
Haglund
Visontai
Wagner

Monotone Column Permanent Conjecture (MCPC)

Let $A=(a_{ij})$ be an $n\times n$ real matrix with descending down columns (Monotone column): $\forall\, 1\le j\le n$ $a_{ij}\ge a_{i+1,j}$ for $1\le i\le n-1$. Let J_n be the $n\times n$ all-ones matrix. Let z be an indeterminate.

For $H=(h_{ij})$ an $n\times n$ matrix, the <u>permanent</u> of H is

$$\mathrm{per}(H)=\sum_{\sigma\in S_n}\prod_{i=1}^{n} h_{i\sigma(i)}.$$

MCPC (1995): Let A be an $n\times n$ monotone column matrix. Then $\mathrm{per}(zJ_n+A)\in\mathbb{R}[z]$ has only real roots.

Haglund-Ono-Wagner (1997): True if all $a_{ij}\in\{0,1\}$.

e.g. $$A=\begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&1&1\\0&0&0&1\end{bmatrix}\qquad zJ_4+A=\begin{bmatrix}z&z+1&z+1&z+1\\z&z&z+1&z+1\\z&z&z+1&z+1\\z&z&z&z+1\end{bmatrix}$$

Let $Z_n=\mathrm{diag}(z_1,\dots,z_n)$

$$Z_4=\begin{bmatrix}z_1&0&0&0\\0&z_2&0&0\\0&0&z_3&0\\0&0&0&z_4\end{bmatrix}\qquad Z_4J_4+A=\begin{bmatrix}z_1&z_2+1&z_3+1&z_4+1\\z_1&z_2&z_3+1&z_4+1\\z_1&z_2&z_3+1&z_4+1\\z_1&z_2&z_3&z_4+1\end{bmatrix}$$

Note $\mathrm{per}(Z_nJ_n+A)\big|_{z_i=z}=\mathrm{per}(zJ_n+A)$.

Haglund-Visontai: Multivariate MCPC: $\mathrm{per}(J_nZ_n+A)$ is stable.

First step: Suffices to prove this for $\{0,1\}$ monotone column matrix (Wednesday)

A <u>Ferrers matrix</u> is a $\{0,1\}$ monotone column matrix that is weakly increasing left to right along rows.

It suffices to prove MMCPC for Ferrers matrices.

Let $y_j=(z_j+1)/z_j$ for $1\le j\le n$ and let $A=(a_{ij})$ be a Ferrers matrix.

$Y_n=\mathrm{diag}(y_1,\dots,y_n)$. $AY_n+J_n-A_n=(a_{ij}y_j+1-a_{ij})$. So

$$\mathrm{per}(z_j+a_{ij})=z_1\cdots z_n\,\mathrm{per}(a_{ij}y_j+1-a_{ij})$$

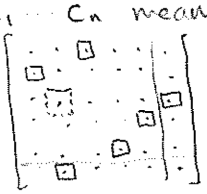
Fact: $\mathrm{per}(z_j + a_{ij})$ is stable in $\mathbb{R}[\underline{z}]$ if and only if $\mathrm{per}(a_{ij}y_j + (1-a_{ij}))$ is stable in $\mathbb{R}[\underline{y}]$.

eg. (cont.) $AY_n + J_n - A = \begin{bmatrix} 1 & y_2 & y_3 & y_4 \\ 1 & 1 & y_3 & y_4 \\ 1 & 1 & y_3 & y_4 \\ 1 & 1 & 1 & y_4 \end{bmatrix}$ (does not have the following symmetry)

Symmetry of Ferrers matrices: $A = (a_{ij}) \mapsto A^\vee = (1 - a_{ji})$. Note $A^{\vee\vee} = A$.
Let $\underline{x} = (x_1, \dots, x_n)$ be indeterminates associated with rows. Let $B(A) = (a_{ij}y_j + (1-a_{ij})x_i)$

eg. (cont.) $B(A) = \begin{bmatrix} x_1 & y_2 & y_3 & y_4 \\ x_2 & x_2 & y_3 & y_4 \\ x_3 & x_3 & y_3 & y_4 \\ x_4 & x_4 & x_4 & y_4 \end{bmatrix}$

Note $B(A^\vee; \underline{x}, \underline{y}) = B(A; \underline{y}, \underline{x})^T$. So $\mathrm{per}(B(A^\vee; \underline{x}, \underline{y})) = \mathrm{per}\, B(A; \underline{y}, \underline{x})$.
Claim: For a Ferrers matrix A, $\mathrm{per}(B(A); \underline{x}, \underline{y})$ is stable in $\mathbb{R}[\underline{x}, \underline{y}]$.
It suffices to prove that $\mathrm{per}(B(A))$ is stable when $a_{nn} = 0$ (or else 2016 02 24
apply $A \mapsto A^\vee$).
one-line notation: $\sigma: c_1 \cdots c_n$ means $\sigma(i) = c_i$ for $1 \le i \le n$.
eg. $\sigma: 316542$



Induction on n:
$n=1$: $A = [0]$, $B(A) = x_1$, $\mathrm{per}(B(A)) = x_1$ is stable.
map $\pi: S_n \to S_{n-1}$ $\quad \sigma: c_1 \cdots c_n$
If $c_n = n$ then $\pi(\sigma): c_1 \cdots c_{n-1}$ Otherwise, replace n by c_n
eg. (cont.) $\pi(\sigma): 31254$
Each permutation $\gamma \in S_{n-1}$ has $|\pi^{-1}(\gamma)| = n$; once in the first case and $n-1$ times in the second case.
Let A° be A with row n and column n removed.

$$\mathrm{per}(B(A)) = \sum_{\gamma \in S_n} \prod_{i=1}^n b_{i\sigma(i)}$$
$$= \sum_{\gamma \in S_{n-1}} \sum_{\sigma \in \pi^{-1}(\gamma)} \prod_{i=1}^n b_{i\sigma(i)} = \begin{pmatrix}\text{case (1)} \\ \text{terms}\end{pmatrix} + \begin{pmatrix}\text{case (2)} \\ \text{terms}\end{pmatrix}$$

$\begin{pmatrix}\text{case (1)}\\ \text{terms}\end{pmatrix} = x_n \operatorname{per}(B(A^\circ))$

$\begin{pmatrix}\text{case (2)}\\ \text{terms}\end{pmatrix}$: let k be the number of 0's in column n of A

The last column of $B(A)$ is
$$\begin{bmatrix} y_n \\ \vdots \\ y_n \\ x_{n-k+1} \\ \vdots \\ x_n \end{bmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} y_n \\ \vdots \\ y_n \end{matrix}}\right\} n-k \\ \left.\vphantom{\begin{matrix} x \\ \vdots \\ x \end{matrix}}\right\} k \end{array}$$

$\frac{\partial}{\partial x_3} \operatorname{per}(B(A^\circ))$

* sums over occurences of x_3 in $B(A^\circ)$
* sums over $\gamma \in S_{n-1}$ using that box
* deletes the factor of x_3 from $\prod_{i=1}^{n-1} b_{i\gamma(i)}$ in $\operatorname{per}(B(A^\circ))$

Replace 3 by h. Multiply this by
$$\begin{cases} x_n \cdot y_n & \text{if } 1 \le h \le n-k \\ x_n \cdot x_h & \text{if } n-k+1 \le h \le n. \end{cases}$$

$$\begin{pmatrix}\text{case (2)}\\ \text{terms}\end{pmatrix} = x_n \left(y_n \sum_{h=1}^{n-k} \frac{\partial}{\partial x_h} + \sum_{h=n-k+1}^{n-1} x_n \frac{\partial}{\partial x_h} + \sum_{j=1}^{n-1} y_n \frac{\partial}{\partial y_j} \right) \operatorname{per}(B(A^\circ)).$$
$$= x_n \underline{\partial} \operatorname{per}(B(A^\circ))$$

So
$$\operatorname{per}(B(A)) = x_n (1 + \underline{\delta}) \operatorname{per}(B(A^\circ))$$

By induction, $\operatorname{per}(B(A^\circ))$ is stable. Suffices to show that $1+\underline{\delta}$ preserves stability.

Since each of the last k rows is constant, and $\operatorname{per}(B(A^\circ))$ is multi-affine,
$$x_n(1+\underline{\delta}) \operatorname{per}(B(A^\circ)) = k x_n \operatorname{per}(B(A^\circ)) + x_n y_n \left(\sum_{h=1}^{n-k} \frac{\partial}{\partial x_h} + \sum_{j=1}^{n-1} \frac{\partial}{\partial y_j} \right) \operatorname{per}(B(A^\circ)).$$

It suffices to show that
$$T = k + z_m \sum_{j=1}^{m-1} \frac{\partial}{\partial z_j}$$
preserves stability in $\mathbb{C}[\underline{z}]$. By Borcea-Brändén it suffices to show that
$$T(\underline{z} + \underline{w})^{[m-1]}$$
is stable. This is
$$\left(k + z_m \sum_{j=1}^{m-1} \frac{1}{z_j + w_j} \right) \prod_{j=1}^{m-1} (z_j + w_j) \qquad (*)$$

Assume that all $z_j, w_j \in H$. So $\prod_{j=1}^{m-1} (z_j + w_j) \neq 0$. Then

$$\frac{k}{z_m} + \sum_{j=1}^{m-1} \frac{1}{z_j + w_j}$$

has strictly negative real part since $k \geq 1$ and $z_m \in \mathcal{H}$. So it is not zero. So $(*)$ is stable. ∎

2016 02 26

Puzzle: Let $f \in \mathbb{R}[\underline{x}]$ be stable. Then $\{g \in \mathbb{R}[\underline{x}];\ f \ll g\}$ and $\{g \in \mathbb{R}[\underline{x}];\ g \ll f\}$ are convex cones containing f.
Hint: Section 2.4, HB, HKO, Wronskian condition

Proposition: Let V be a finite dimensional vector space, $\varphi: V^m \to \mathbb{R}$ multilinear, and $e_1, \dots, e_m, v_1, \dots, v_m \in V$. Then the set K of all v such that $\varphi(v_1 + xe_1, \dots, v_m + xe_m)$ is stable is either empty or a convex cone containing e_1 and $-e_1$.

Proof: Assume $K \neq \emptyset$. For $v \in V$ let $F_v(\underline{x}) = \varphi(v_1, v_2 + x_2 e_2, \dots, v_m + x_m e_m)$. Then $\varphi(v_1 + x_1 e_1, \dots, v_m + x_m e_m) = F_{v_1} + x_1 F_{e_1}$ by multilinearity, and this is stable if and only if $F_{e_1} \ll F_{v_1}$. So $K = \{v \in V;\ F_{e_1} \ll F_{v_1}\}$. Note $F_{\alpha v + \beta w} = \alpha F_v + \beta F_w$, so K is a convex cone, and $F_{v_1} + x_1 F_{e_1}$ is stable and non-zero for some v_1. So F_{e_1} is stable if and only if $(x_1 + 1) F_{e_1}$ and $(x_1 - 1) F_{e_1}$ are stable. Thus $e_1, -e_1$ are in K. ∎

Proposition: Assume that MMCPC holds for Ferrers matrices. Then MMCPC holds for all MC matrices.

Proof: We have $\mathrm{per}(z_j + a_{ij})$ is stable for all $\{0,1\}$ MC matrices. Let $V \cong \mathbb{R}^n$ be the vector space of column matrices of length n. Let $\varphi: V^n \to \mathbb{R}$ be given by $\varphi(u_1, \dots, u_m) = \mathrm{per}[u_1 | \cdots | u_m]$. Let $e_j = \mathbb{1}$ for $j = 1, \dots, m$. Note that $\varphi(v_1 + z_1 e_1, \dots, v_m + z_m e_m) = \mathrm{per}(J_n Z_n + A) = \mathrm{per}(z_j + a_{ij})$ where $A = [v_1 | \cdots | v_n]$. By the proposition, if each v_j is MC vector, then $\varphi(v_1 + z_1 e_1, \dots, v_n + z_n e_n)$ is stable and nonzero.
Now we proceed by induction, Base-case: Let A be any MC matrix, and $v_1, \dots, v_{k-1}$ be the first $k-1$ columns of A. Let $v_k, \dots, v_n$ be any $\{0,1\}$ MC vectors. Then $\varphi(v_1 + x_1 e_1, \dots, v_n + x_n e_n)$ is (non-zero and) stable ($k=1$ trivial)
Inductive-case: Assume true for k. Apply the proposition to column k. The set of vectors v_k st $\varphi(\dots, v_i + x_i e_i, \dots)$ is stable, is a convex cone containing $\pm e_k$, and all $\{0,1\}$ MC vectors. Note: any MC vector is a non-neg. comb. of $-e_k = -\mathbb{1}$ and some $\{0,1\}$ MC vectors. The k^{th} column of A has the above form, so $\varphi(\dots, v_i + x_i e_i, \dots)$ is stable. ∎

2016 02 29

Van der Waerden Conjecture (1928)

á la Gurvitz [G-2008]

$A = (a_{ij})$ an $n \times n$ matrix of nonnegative reals

stochastic: every column sums to 1.

Let $\underline{v} \in \mathbb{R}^n$ be a column vector that sums to 1.

Then $A\underline{v}$ also sums to 1. So $\underline{v} \longmapsto A\underline{v}$, discrete time evolution on probability distributions in $\mathbb{R}^n$.

doubly stochastic: all rows and columns sum to 1.

eg. · $\frac{1}{n} J_n$

· for any $\sigma \in S_n$, "permutation matrix"

$$(A\sigma)_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i) \\ 0 & \text{otherwise} \end{cases}$$

eg. · $\mathrm{per}(\frac{1}{n} J_n) = \frac{n!}{n^n}$

· $\mathrm{per}(A_\sigma) = 1$

Van der Waerden Conjecture: For any $n \times n$ doubly stochastic matrix A, $\mathrm{per}(A) \geq \frac{n!}{n^n}$, with equality if and only if $A = \frac{1}{n} J_n$.

Inequality proved by Falikman (1981)

Characterization of equality by Egorychev (1981)

Gurvitz (2008) - huge generalization using stable polynomials

(1946) (1958)

Birkoff-von Neumann Theorem Set of $n \times n$ doubly stochastic matrices is the convex hull of $n \times n$ permutation matrices

(König 1936 actually though)

Gurvitz's strategy:

* a weird inequality for univariate stable polynomials
* induct on the number of variables
* contract and specialize to get $\mathrm{per}(A)$

Jensen's Inequality [see HLP, Thm 90]: $I \subseteq \mathbb{R}$ an interval, $\rho: I \to \mathbb{R}$ is concave if $\forall a,b \in I$:

$$\rho\left(\frac{a+b}{2}\right) \geq \frac{\rho(a)+\rho(b)}{2}.$$

It is strictly concave if equality holds if and only if $a=b$.

eg. $I=(0,\infty)$, Let $a,b>0$, $(\sqrt{a}-\sqrt{b})^2 \geq 0$, equality only if $a=b$.

$$\frac{a+b}{2} = \lambda\sqrt{ab} \geq \sqrt{ab}$$

where $\lambda \geq 1$.

So log is strictly concave.

Jensen's Inequality. Let $I \subseteq \mathbb{R}$ be an interval, $\rho: I \to \mathbb{R}$ concave. Let $a_1,\dots,a_n \in I$ and $b_1,b_2,\dots,b_n > 0$ with $\sum_{i=1}^n b_i = 1$. Then

$$\rho\left(\sum_{i=1}^n b_i a_i\right) \geq \sum_{i=1}^n b_i \rho(a_i). \qquad (*)$$

If ρ is strictly concave then equality holds in $(*)$ if and only if $a_1 = \cdots = a_n$.

Capacity of a polynomial $f \in \mathbb{R}[\underline{x}]$ (Gurwitz) with nonnegative coefficients.

$$\operatorname{cap}(f) = \inf_{\underline{c}>0} \frac{f(c_1,\dots,c_m)}{c_1\cdots c_m}$$

G-function: $G(0)=1$ and for integer $d \geq 1$,

$$G(d) = \left(1-\frac{1}{d}\right)^{d-1}.$$

So $G(1) = 0^0 = 1$, $G(2) = \frac{1}{2}, \dots$. G is a decreasing function of d.

Proposition: Let

$$f = \sum_{i=0}^d b_i x^i$$

be a polynomial of degree d with all $b_i \geq 0$. If f is real stable then

$$b_1 = f'(0) \geq G(d)\operatorname{cap}(f).$$

Equality holds if and only if either $d \leq 1$ or $f(x) = b_d(x+\xi)^d$ for some $\xi > 0$.

Proof: If $\mathrm{cap}(f)=0$ then there is nothing to prove. If $d=0$ then $f'(0)=0=G(0)\,\mathrm{cap}(f)$ since $\mathrm{cap}(f)=\inf_{c>0}\frac{b_0}{c}=0$. If $d=1$ then

$$f'(0)=b_1=\inf_{c>0}\frac{b_0+b_1c}{c}=G(1)\,\mathrm{cap}(f).$$

So assume $d\geq 2$. If $b_0=f(0)=0$ then

$$b_1=f'(0)=\lim_{c\to 0}\frac{f(c)}{c}\geq \mathrm{cap}(f)\geq G(d)\,\mathrm{cap}(f)$$

since G is a decreasing function of d. So assume $b_0>0$. Rescale so that $b_0=1$.

$$f(x)=\prod_{i=1}^{d}(1+a_ix)$$

and $a_1+\cdots+a_d=b_1$. For any $c>0$ we have $\frac{f(c)}{c}\geq \mathrm{cap}(f)$. So

$$\begin{aligned}\frac{\log(\mathrm{cap}(f)c)}{d} &\leq \frac{\log(f(c))}{d}\\ &=\sum_{i=1}^{d}\frac{1}{d}\log(1+a_ic)\\ &\leq \log\left(\sum_{i=1}^{d}\frac{1}{d}(1+a_ic)\right)\\ &=\log\left(1+\frac{b_1c}{d}\right).\end{aligned}$$

So for all $c>0$

$$\mathrm{cap}(f)\,c\leq\left(1+\frac{b_1c}{d}\right)^d.$$

Let

$$g(x)=\left(1+\frac{b_1}{d}x\right)^d.$$

So for any $c>0$, $\mathrm{cap}(f)\leq\frac{g(c)}{c}$. So $\mathrm{cap}(f)\leq\mathrm{cap}(g)=\frac{b_1}{G(d)}$, by elementary calculus, attained at

$$c_*=\frac{d}{b_1(d-1)}.\qquad\blacksquare$$

2016 03 04

Proposition: Let $f\in\mathbb{R}[x]$ be non-zero with non-negative coefficients, which is stable of degree d. Then $f'(0)\geq G(d)\,\mathrm{cap}(f)$; equality holds if and only if $f(x)=b_d(x+\varepsilon)^d$ for some $b_d,\varepsilon>0$.

Proposition: Let $f(\underline{x}) \in \mathbb{R}[\underline{x}]$ be non-zero, non-negative, stable, and homogeneous of degree m. Let $g = \frac{\partial}{\partial x_m} f(\underline{x})|_{x_m=0}$ (which is stable, homogeneous of degree $m-1$). Let d be the degree x_m in f. Then $\text{cap}(g) \geq G(d)\,\text{cap}(f)$.

Proof: $d=0$ is trivial. Fix $c_1, \dots, c_{m-1} > 0$ and let $p_c(x) = f(c_1, \dots, c_{m-1}, x)$. By the previous proposition, $p_c'(x)|_{x=0} \geq G(d)\,\text{cap}(p_c) \geq G(d)\,\text{cap}(f)$. If $m=1$, $g = \text{cap}(g)$ is constant. If $m \geq 2$ then for any $\underline{c} > \underline{0}$, $g(c_1, \dots, c_{m-1}) \geq G(d)\,\text{cap}(f)$. Let $b = (c_1 \cdots c_{m-1})^{-\frac{1}{m-1}}$. Note

$$\frac{g(c_1, \dots, c_{m-1})}{c_1 \cdots c_{m-1}} = g(bc_1, \dots, bc_{m-1}) \geq G(d)\,\text{cap}(f).$$

Thus $\text{cap}(g) \geq G(d)\,\text{cap}(f)$. ■

Theorem [Gurvitz]: Let $f \in \mathbb{R}[\underline{x}]$ be stable with non-negative coefficients, m variables, homogeneous of degree m. Let d_i be the degree of x_i in f, and $e_i = \min\{i, d_i\}$. Then

$$\underline{\partial}^{[m]} f(\underline{0}) \geq \text{cap}(f) \prod_{i=2}^{m} G(e_i)$$

where

$$\underline{\partial}^{[m]} = \prod_{i=1}^{m} \frac{\partial}{\partial x_i}.$$

Corollary: $\underline{\partial}^{[m]} f(\underline{0}) \geq \frac{m!}{m^m} \cdot \text{cap}(f)$ with equality if and only if $f(\underline{x}) = c(a_1x_1 + \dots + a_m x_m)^m$, $c, a_i > 0$.

Proof: Let $g_m = f$ and for $1 \leq i \leq m$ let $g_{i-1} = \frac{\partial}{\partial x_i} g_i(\underline{x})|_{x_i=0}$. Note that $g_0 = \text{cap}(g_0)$. By the previous proposition,

$$\text{cap}(g_{i-1}) \geq G(\deg_i(g_i)) \cdot \text{cap}(g_i) \qquad (*)$$

and $\deg_i(g_i) \leq \deg_i(f) = d_i$ and $\deg_i(g_i) \leq$ total degree of $g_i = i$ (by indexing from m to 0).

So $\deg_i g_i \leq \min\{i, d_i\} = e_i$. Therefore $G(\deg_i g_i) \geq G(e_i)$. Multiply $(*)$ for $1 \leq i \leq m$:

$$\underline{\partial}^{[m]} f(\underline{0}) \geq \text{cap}(f) \prod_{i=1}^{m} G(e_i).$$ ■

Note: $e_i = \min\{i, d_i\} \leq i$ so $G(e_i) \geq G(i) = (1 - 1/i)^{i-1} = \left(\frac{i-1}{i}\right)^{i-1}$, so

$$\prod_{i=1}^{m} G(e_i) \geq \prod_{i=1}^{m} G(i) = \frac{m!}{m^m} \quad \text{(Puzzle: prove this.)}$$

2016 03 07

Gurvitz

Let $f \in \mathbb{R}[x_1, \dots, x_m]$ have non-negative coefficients, homogeneous of degree m, stable. Then

$$\underline{\partial}^{[m]} f(\underline{0}) \geq \frac{m!}{m^m} \operatorname{cap}(f).$$

Equality holds if and only if $f(\underline{x}) = (a_1 x_1 + \cdots + a_m x_m)^m$ with all $a_i > 0$.

Van der Waerden

$A = (a_{ij})$ doubly stochastic, $m \times m$

$$f_A(\underline{x}) = \prod_{j=1}^{m} (a_{1j} x_1 + \cdots + a_{mj} x_m)$$

Check: $\operatorname{per}(A) = \underline{\partial}^{[m]} f(\underline{0}) = [x_1 \cdots x_m] f_A(\underline{x})$.

Gurvitz: $\operatorname{per}(A) \geq \frac{m!}{m^m} \operatorname{cap}(f_A)$, equality holds if and only if $f_A(\underline{x}) = (a_1 x_1 + \cdots + a_m x_m)^m$. In this case $a_1 + \cdots + a_m = 1$ since A is stochastic. The i^{th} row of A sums to 1, so $m a_i = 1$. So each $a_i = \frac{1}{m}$. So $A = \frac{1}{m} J_m$.

Lemma: Let $f \in \mathbb{R}[x_1 \dots x_m]$ be homogeneous of degree m with non-negative coefficients. Assume that for all $1 \leq i \leq m$,

$$\frac{\partial}{\partial x_i} f(1) = 1.$$

Then $\operatorname{cap}(f) = 1$.

Proof: Let $f(\underline{x}) = \sum_\alpha b(\alpha) \underline{x}^\alpha$, sum over exponent vectors $\alpha : [m] \to \mathbb{N}$ with $|\alpha| = m$, each $b(\alpha) \geq 0$. Note if $b(\alpha) \neq 0$ then $\sum_{i=1}^m \alpha(i) = m$.

$$1 = \frac{\partial}{\partial x_i} f(1) = \sum_\alpha b(\alpha) \alpha(i)$$

Averaging this over all $1 \leq i \leq m$,

$$1 = \frac{1}{m}(m) = \frac{1}{m} \sum_\alpha b(\alpha) \sum_{i=1}^m \alpha(i) = \sum_\alpha b(\alpha) = \frac{f(1)}{1} \geq \operatorname{cap}(f).$$

Conversely, consider any $c_1, \dots, c_m > 0$

$$\log f(\underline{c}) = \log \sum_\alpha b(\alpha) \underline{c}^\alpha \underset{\text{Jensen}}{\geq} \sum_\alpha b(\alpha) \log(\underline{c}^\alpha) = \sum_{i=1}^m \log(c_i) \underbrace{\sum_\alpha b(\alpha)\alpha(i)}_{=1}$$

$$= \sum_{i=1}^m \log(c_i)$$

so

$$\frac{f(\underline{c})}{c_1 \cdots c_m} \geq 1 \qquad \text{so} \qquad \operatorname{cap}(f) \geq 1. \qquad \blacksquare$$

Note: $\operatorname{cap}(f_A) = 1$:

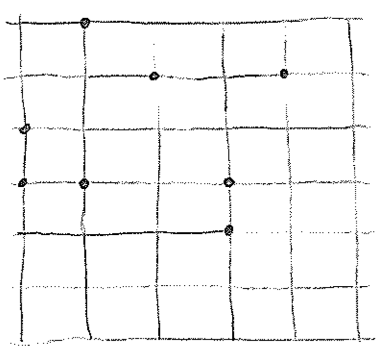
$$\frac{\partial}{\partial x_i} f_A(\underline{x}) = \sum_{j=1}^{m} a_{ij} \prod_{h \neq j} (a_{1h} x_1 + \cdots + a_{mh} x_m) \Big|_{\underline{x}=\underline{1}}$$

$$= \sum_{j=1}^{m} a_{ij} = 1$$

since A is stochastic and since A is doubly stochastic.

Symmetric Exclusion Process

$S^{-1}(1)$ occupied sites



Λ a set of "sites" (lattice)
E a set of 2-element subsets of Λ
(Λ, E) is a simple graph

state: $S : \Lambda \to \{0,1\}$

$$S(v) = \begin{cases} 0 & \text{means } v \text{ is vacant} \\ 1 & \text{means } v \text{ is occupied} \end{cases}$$

dynamic. For every edge $e = \{i,j\}$ there is a rate $\lambda_{ij} > 0$.
Each edge has an exponential "clock" of rate λ_{ij}.

$$\text{p.d.f.}(T) = \lambda_{ij} e^{-\lambda_{ij} t}$$

all independent of one another. Let $\tau_{ij} = (i\ j)$ acting on Λ. When $e = \{i,j\}$ rings, replace S by $S \circ \tau_{ij} : \Lambda \to \{0,1\}$

$$\begin{cases} i \mapsto j \mapsto S(j) \\ j \mapsto i \mapsto S(i) \\ h \mapsto h \mapsto S(h) & h \neq i,j \end{cases}$$

Negative Correlation

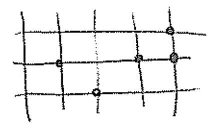
Let i,j be any two sites. Let E_i be the event that site i is occupied, $S(i) = 1$. $\Pr[E_i]\Pr[E_j] \geq \Pr[E_i E_j]$.
Fixed time, deterministic state: each $\Pr[E]$ is either 0 or 1. This holds trivially.

Conjecture [Liggett, Pemantle 199?]: If at time $t=0$ the state is deterministic then for all $t>0$ the state is negatively correlated.

2016 03 09

Symmetric Exclusion Process (SEP)

Λ a finite set of sites, say $\Lambda = [m] = \{1, \dots, m\}$
E a set of 2-element subsets of Λ
$\lambda_{ij} > 0$ for each $\{i,j\} \in E$.
state: $S : \Lambda \to \{0,1\}$
set of all states $\Omega = \{0,1\}^\Lambda$
probability distribution $\varphi : \Omega \to [0,1]$ with $\sum_{S \in \Omega} \varphi(S) = 1$
φ is deterministic if there is an S_0 such that



$$\varphi(S) = \begin{cases} 1 & \text{if } S = S_0 \\ 0 & \text{if } S \neq S_0 \end{cases}$$

$\underline{x} = (x_1, \dots, x_m)$ variables indexed by Λ
for a state $S : \Lambda \to \{0,1\}$, $\underline{x}^S = \prod_{i \in S} x_i^{S(i)}$
partition function of a distribution φ:

$$Z(\varphi) = Z(\varphi; \underline{x}) = \sum_{S \in \Omega} \varphi(S)\, \underline{x}^S$$

For positive reals $\underline{a} > \underline{0}$ define $\varphi^{\underline{a}} : \Omega \to [0,1]$ by putting

$$\varphi^{\underline{a}}(S) = \frac{\varphi(S)\, \underline{a}^S}{Z(\varphi; \underline{a})}$$

$\underline{a}$ is an "external field".
We saw negative correlation last time. Now negative association.

Negative Association

Event any subset of Ω
Increasing event $\mathcal{E}$: $S \in \mathcal{E}$ and $S \le S' \Rightarrow S' \in \mathcal{E}$
eg// Let $K \subseteq \Lambda$, let $\mathcal{E}_K$ be the event that $S(i) = 1$ for all $i \in K$, is increasing.
Events $\mathcal{F}, \mathcal{G}$ are disjointly supported if there is a partition $\Lambda = A \cup B$ and events $\mathcal{F}' \subseteq \{0,1\}^A$, $\mathcal{G}' \subseteq \{0,1\}^B$ such that

$$\mathcal{F} \rightleftharpoons \mathcal{F}' \times \{0,1\}^B, \quad \mathcal{G} \rightleftharpoons \{0,1\}^A \times \mathcal{G}'$$

negative association for $\varphi : \Omega \to [0,1]$: For any two disjointly supported increasing events $\mathcal{F}, \mathcal{G}$, $\Pr[\mathcal{F}]\Pr[\mathcal{G}] \ge \Pr[\mathcal{F} \cap \mathcal{G}]$

Theorem [Feder-Mihail]:
Let S be a class of functions φ which are non-negative valued.
(i) Each φ has domain $\{0,1\}^\Lambda$ for some finite set $\Lambda = \Lambda(\varphi)$
(ii) Each $Z(\varphi) = \sum_{S \in \Lambda} \varphi(S) \underline{x}^S$ is homogeneous (partition function)
(iii) For each $i \in \Lambda(\varphi)$. $Z(\varphi)|_{x_i=0}$ and $\frac{\partial}{\partial x_i} Z(\varphi)$ are partition functions for elements of S
(iv) For any $\varphi \in S$ and $a > 0$, φ^a is negatively correlated
Then: For all $\varphi \in S$ and $a > 0$, φ^a is negatively associated

<u>Time Evolution of SEP</u>
φ_0 initial distribution at time 0.
$\varphi_t : \Omega \to [0,1]$ distribution of states at time t.
There is a semigroup of operators $T(t): \mathbb{R}[\underline{x}] \to \mathbb{R}[\underline{x}]$ such that
$$Z(\varphi_{t_1+t}) = T(t) Z(\varphi_{t_1}) \text{ for any } t_1 \text{ and } t$$
$T(t_1 + t_2) = T(t_1) T(t_2)$
Look at one edge $e \in \{1,2\}$

$1 \overset{\lambda_{12}>0}{\longrightarrow} 2$ $\qquad e^{\lambda_{12}(\tau_{12}-1)t}$ this is the semigroup for one edge

For all edges $\exp(t\mathcal{L}) = T(t)$
$$\mathcal{L} = \sum_{ij \in E} \lambda_{ij} (\tau_{ij} - 1)$$

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Conjecture [Liggett-Pemantle]: If initial state of SEP is deterministic then state at time t is negatively associated $\forall t \geq 0$.

Proof [Borcea-Brandén-Liggett, 2008]: Find a class S of of probability measures on $\Omega = \{0,1\}^\Lambda$:
① * if φ is deterministic then φ is in S
② * if φ is in S then φ is negatively associated (NA)
③ * time evolution of SEP preserves membership in S
φ is in S if and only if $Z(\varphi)$ is homogeneous and real stable (with non-negative coefficients)
For ①: there is a state S_0 st $\varphi(S) = \begin{cases} 1 & S = S_0 \\ 0 & \text{otherwise} \end{cases}$
Then $Z(\varphi) = \underline{x}^{S_0}$ is homogeneous & real stable

For ②: Apply Feder-Mihail

(i) Each φ has domain $\{0,1\}^\Lambda$ for some finite set $\Lambda(\varphi)$ ✓

(ii) Each $Z(\varphi)$ is homogeneous ✓

(iii) If φ is in S then $Z(\varphi)|_{x_i=0}$ and $\frac{\partial}{\partial x_i} Z(\varphi)$ are partition function of members of S ✓

(iv) For φ in S and $\underline{a} > \underline{0}$ indexed by $\Lambda(\varphi)$: $\varphi^{\underline{a}}$ is NC.

Then by F.M, every φ in S is such that every $\varphi^{\underline{a}}$ is NA.

For $s \in \Omega = \{0,1\}^\Lambda$ and $\underline{a} > \underline{0}$,

$$\varphi^{\underline{a}}(s) = \frac{\varphi(s)\underline{a}^s}{Z(\varphi;\underline{a})}.$$

NC: $\Pr[\mathcal{E}_i] \cdot \Pr[\mathcal{E}_j] \geq \Pr[\mathcal{E}_{\{i,j\}}]$ — ⊛

$$Z(\varphi;\underline{a}) = \sum_{s\in\Omega} \varphi(s)\underline{a}^s$$

$$\Pr[\mathcal{E}_i] = \sum_{\substack{s\in\Omega \\ s(i)=1}} \varphi(s)\underline{a}^s = a_i \frac{\partial}{\partial x_i} Z(\varphi;\underline{a}), \quad \Pr[\mathcal{E}_j] = a_j \frac{\partial}{\partial x_j} Z(\varphi;\underline{a})$$

$$\Pr[\mathcal{E}_{\{i,j\}}] = a_i a_j \frac{\partial^2}{\partial x_i \partial x_j} Z(\varphi;\underline{a})$$

⊛ follows from $a_i a_j \Delta_{ij} Z(\varphi;\underline{a}) \geq 0$ for all $\underline{a} > \underline{0}$.

Note

$$\Delta_{ij} Z = Z_i^j Z_j^i - Z_{ij} Z^{ij} = Z_i Z_j - Z_{ij} Z$$

as polynomials, using

$$Z_i = Z_i^j + x_j Z_{ij}, \quad Z = Z^{ij} + x_i Z_i^j + x_j Z_j^i + x_i x_j Z_{ij}$$

For ③

Time evolution of SEP. $Z(\varphi_t) = T(t) Z(\varphi_0)$ for all $t \geq 0$. $T(t): \mathbb{R}[x] \to \mathbb{R}[\underline{x}]$ one-parameter semigroup of operators, $T(t_1+t_2) = T(t_1) + T(t_2)$

Infinitesimal generator

$$\mathcal{L} = \sum_{\{i,j\}\in E} \lambda_{ij}(\tau_{ij} - 1) \quad \text{so} \quad T(t) = \exp(t\mathcal{L}).$$

<u>Trotters Product Formula</u>

For two semigroups with infinitesimal operators $\mathcal{L}_1, \mathcal{L}_2$, $T_1(t) = \exp(t\mathcal{L}_1)$, $T_2(t) = \exp(t\mathcal{L}_2)$,

$$T(t) = \exp(t(\mathcal{L}_1 + \mathcal{L}_2)) = \lim_{n\to\infty} \left(T_1(t/n) T_2(t/n)\right)^n$$

The point is: If $T_1(t)$ and $T_2(t)$ preserve stability for all $t \geq 0$ then

$T(t)$ preserves stability for all $t \geq 0$, by Hurwitz's Theorem.
It suffices to show that one term in $\mathcal{L}$ preserves stability; then induct on the number of terms in $\mathcal{L}$. ($\tau = (1\ 2)$, $\lambda > 0$ rate)

$$\exp(t\lambda(\tau - 1)) = \sum_{k=0}^{\infty} \frac{t^k \lambda^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \tau^j (-1)^{k-j}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{t^k \lambda^k (-1)^k}{k!} \left(2^{k-1} - 2^{k-1}\tau\right)$$

$$= 1 + \frac{1}{2} \sum_{k=1}^{\infty} \frac{t^k (2\lambda)^k}{k!} (1-\tau)(-1)^k$$

$$= \frac{1}{2} + \frac{1}{2}\exp(-2\lambda t) - \frac{1}{2}(\exp(-2\lambda t) - 1)\tau$$

$$= \frac{1}{2}(1 + \exp(-2\lambda t)) + \frac{1}{2}(1 - \exp(-2\lambda t))$$

since

$$\tau^k = \begin{cases} 1 & k \text{ even} \\ \tau & k \text{ odd} \end{cases} \quad \text{and} \quad \sum_{\substack{j=0 \\ \text{even}}}^{k} \binom{k}{j} = 2^{k-1} = \sum_{\substack{j=0 \\ \text{odd}}}^{k} \binom{k}{j}.$$

This preserves stability by results in §4. ∎

2016 03 14

Symmetric homogenization

Let

$$Z(\underline{x}) = \sum_{S \in \Lambda} \varphi(S)\, \underline{x}^S$$

be a multiaffine polynomial of degree d. Let $y_1, \ldots, y_d$ be new variables.

$$e_j(\underline{y}) = \sum_{J \in \binom{[d]}{j}} \underline{y}^J,$$

j^{th} elementary symmetric function in $\underline{y}$ for $0 \leq j \leq d$.

$$Z_{sh}(\underline{x}, \underline{y}) = \sum_{S \in \Lambda} \varphi(S)\, \underline{x}^S \binom{d}{d-|S|}^{-1} e_{d-|S|}(\underline{y}).$$

Note $Z_{sh}(\underline{x}, \underline{y})$ is homogeneous of degree d, $Z_{sh}(\underline{x}, \mathbb{1}) = Z(\underline{x})$.

Factoid: $Z_{sh}(\underline{x}, \underline{y})$ is stable if and only if $Z(\underline{x})$ is stable

Proof: missing, see Wednesday

For SEP, we can start at φ_0 with $Z(\varphi_0)$ any stable polynomial. $Z_{sh}(\varphi_0)$ is stable, homogeneous multiaffine. So $\exp(t\mathcal{L})Z_{sh}(\varphi_0)$ is stable for all $t \geq 0$. So it is NA for all $t > 0$ by Feder-Mihail. Set all $y = \mathbb{1}$. Result is still NA.

Particle Creation/Annihilation

$i \in \Lambda$ a site of the lattice

multiaffine a

$\square_i^*, \square_i : \Omega \to \Omega$

$$(\square_i(s))(j) = \begin{cases} s(j) & \text{if } j \neq i \\ 0 & \text{if } j = i \end{cases} \qquad (\square_i^*(s))(j) = \begin{cases} s(j) & \text{if } j \neq i \\ 1 & \text{if } j = i \end{cases}$$

time evolution semigroup for $\square$ with rate $\theta > 0$, $\exp(t\theta(\square - 1))$

Note $\square^2 = \square$, $(\square^*)^2 = \square^*$

$$\begin{aligned} \exp(t\theta(\square - 1)) &= \sum_{k=0}^{\infty} \frac{\theta^k t^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \square^j (-1)^{k-j} \\ &= 1 + \sum_{k=1}^{\infty} \frac{\theta^k t^k}{k!} \left((-1)^k + \square \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} \right) \\ &= e^{-\theta t} + (1 - e^{-\theta t}) \square \end{aligned}$$

By Borcea-Brändén, this preserves stability. Similarly for $\square^*$.

Asymmetric Exclusion Process

$$i \bullet \underset{\lambda_{ij} > 0}{\overset{\lambda_{ji} > 0}{\rightleftarrows}} \bullet j$$

If for any pair of sites $\lambda_{ij} \neq \lambda_{ji}$ then $\exp(t\mathcal{L})$ does not preserve stability. (for that 2-point process)

from 2016 03 16:

Proof (of Factoid): $(\Rightarrow)$ holds by specialization

$(\Leftarrow)$ False: $x^2 - 1$, $n = d = 2$: $x^2 - y_1 y_2$, $x = y_1 = y_2 =$ anything in H. We need f to have non-negative coefficients. One can reduce to the case that f is multiaffine (Polarization) Suffices to show that $y^d f(\underline{x}/y)$ is stable in $\mathbb{R}[\underline{x}, y]$ (then GWS implies $f_{sh}(\underline{x}, y)$ is stable). Induct on the # of $\underline{x}$ variables. $m = 1$:

include proof in term paper

$f = a + bx$, $g = (ay + bx)y^{d-1}$, $a, b > 0$ ✓ induction step: check $\Delta_{ij} f(\underline{a}) \geq 0$ for all $\underline{a} \in \mathbb{R}^{m+1}$ for some i, j indexing x_i, x_j

<u>Stability and Matroids</u>

Let

$$Z(\underline{x}) = \sum_{S \subseteq E} c(S)\, \underline{x}^S$$

be multiaffine, homogeneous, and real stable, with nonnegative coefficients.

$$\operatorname{supp}(Z) = \{S \subseteq E ; c(S) > 0\}.$$

Theorem: Then $\operatorname{supp}(Z)$ is the set of bases of a matroid. [COSW, 2004]

For any matroid $\mathcal{M}$,

$$\sum_{B \in \mathcal{B}(\mathcal{M})} \underline{x}^B$$

is stable? False. Counter-example: 

2016 03 16

Proof (sketch × 2/3) (of theorem): Induct on $|E|$. $|E|=1$: $Z(\underline{x}) = c x_1$ with $c>0$, $\operatorname{supp}(Z) = \{\{1\}\} = \mathcal{B}(\mathcal{U}_{1,1})$. $|E|=2$:

$$Z(\underline{x}) = a x_1 + b x_2 \rightsquigarrow \mathcal{U}_{1,2}$$

or

$$Z(\underline{x}) = c x_1 x_2 \longrightarrow \mathcal{U}_{2,2}$$

Now induct. Check basis exchange axiom. Let $A, B \in \operatorname{supp}(Z)$. Let $e \in A$. Find $f \in B$ such that $(A \setminus e) \cup f \in \operatorname{supp}(Z)$. — ⊛

Two easy reductions:

- There is an $i \in E$ with $i \notin A \cup B$. Specialize $x_i = 0$. Then $Z' = Z|_{x_i = 0}$ is stable both $A, B \in \operatorname{supp}(Z')$, which by induction is (bases of) a matroid. So ⊛ holds in $\operatorname{supp}(Z')$, hence in $\operatorname{supp}(Z)$.
- There is $j \in E$ with $j \in A \cap B$. Then $\hat{Z} = \frac{\partial}{\partial x_j} Z$ is stable. $\hat{A} = A \setminus \{j\}$ and $\hat{B} = B \setminus \{j\}$ are in $\operatorname{supp}(\hat{Z})$. Induct.

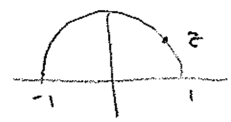
Remaining case: $A \cup B = E$ and $A \cap B = \emptyset$.

There is at least one more set in $\operatorname{supp}(Z)$. Otherwise, $Z = \underline{x}^A + \underline{x}^B$. 2016 03 18

Now specialize

$$x_i = \begin{cases} 1 & \text{if } i \in B \\ x & \text{if } i \in A \end{cases}$$

This yields $x^r + 1$ being stable. If $r=1$, it is stable. If $r \geq 2$, it is not, since (-1) has an r^{th} root in $\mathcal{H}$:



$z = e^{i\pi/r}$

In the $r=1$ case, $\mathrm{supp}(Z) = \{\{a\},\{b\}\} = \mathcal{B}(\mathcal{U}_{1,2})$.
So if $r \geq 2$ then there is another set $C = \mathrm{supp}(Z)$
Consider any $a \in A$. If $a \notin C$ then (by induction) there is a $c \in C \setminus A$ such that $(A \setminus a) \cup c$ is in $\mathrm{supp}(Z)$. Necessarily $c \in B$, so we get the exchange axiom in this case
If $a \in A \cap C$ then set $x_i = 0$ for all $i \in B \setminus C$. Some grunge shows that there exists C' with $a \in A \setminus C'$ and $A \cap C' \neq \emptyset$, $C \cap C' \neq \emptyset$. Then apply basis exchange on A, C' by induction. ∎

Question Does every matroid arise in this way?
Let $\mathcal{B}$ be the set of bases of a matroid M. Is

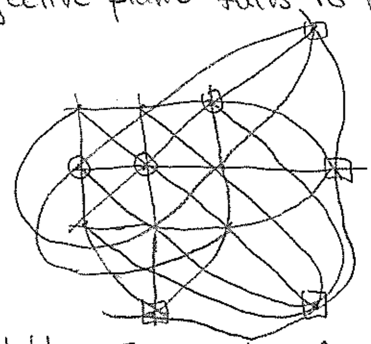
$$B(M;\underline{x}) = \sum_{B \in \mathcal{B}(M)} \underline{x}^B$$

stable? Then M is a <u>half-plane property</u> (HPP) matroid. Is there a set of positive coefficients $\{c(B) ; B \in \mathcal{B}(M)\}$ such that

$$\sum_{B \in \mathcal{B}(M)} c(B)\underline{x}^B$$

is stable? (<u>weak</u> HPP)
Every finite projective plane fails to have the HPP.

PG(2,3)
(q=3)



□ - ℓ
o - B

Bases: noncolinear triples

$B(M)$ is not stable. Fix any line ℓ, set

$$x_i = \begin{cases} 1 & \text{if } i \notin \ell \\ x & \text{if } i \in \ell \end{cases}$$

$$B(M) \mapsto Ax^2 + Bx + C$$

$$\left.\begin{aligned} A &= \binom{q+1}{2} q^2 \\ B &= (q+1)q^2(q^2-q)/2 \\ C &= q^2(q^2-1)(q^2-q)/6 \end{aligned}\right\} \text{ check } B^2 - 4AC < 0$$

So $\mathcal{B}(M)$ is not stable.

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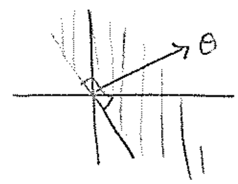
$$Z(\underline{x}) = \sum_{S \subseteq E} c(S)\, \underline{x}^S$$

is <u>H_θ-stable</u> if either $Z \equiv 0$ or $z_i \neq 0$ and $|\arg(z_i) - \theta| < \pi/2$ for all $1 \le i \le m$ implies that $Z(\underline{z}) \neq 0$.



Fact: If $Z(\underline{x})$ is homogeneous then H_θ-stability doesn't depend on θ.
$H_{\pi/2}$ is stability and H_0 is Hurwitz stability

<u>Phase Theorem</u>
Let $f(\underline{x}) = \sum_\alpha c(\alpha)\, \underline{x}^\alpha$ be Hurwitz stable with <u>definite parity</u> if $c(\alpha) \neq 0$ and $c(\beta) \neq 0$ then $|\alpha| = |\beta|$. Then there is a phase θ such that $e^{-i\theta} f(\underline{x})$ has non-negative coefficients.

So if $Z(\underline{x}) = \sum_{B \in \mathcal{B}(M)} c(B)\, \underline{x}^B$ is stable where M is some matroid, we may assume $c(B) > 0$ for each basis B of M. Given M, is there a stable polynomial $Z(\underline{x})$ with support $\mathcal{B}(M)$? If so, M has the <u>Weak Half-Plane-property</u>.
If M can be represented over $\mathbb{C}$, then it is WHPP.

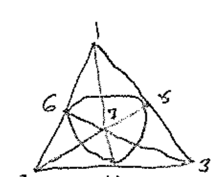
Proof: Let M represent M over $\mathbb{C}$. M is $r \times m$ of rank r. $X = \operatorname{diag}(x_1, \dots, x_m)$.
$Z(\underline{x}) = \det(MXM^\dagger)$ is stable by cor 2.2.

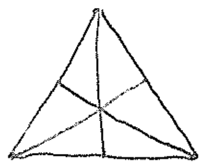
$$Z(\underline{x}) = \sum_{S \in \binom{[m]}{r}} \det(M|_S) \det((XM)|_S)^\dagger = \sum_{S \in \binom{[m]}{r}} |\det M|_S|^2\, \underline{x}^S = \sum_{B \in \mathcal{B}(M)} c(B)\, \underline{x}^B$$

with $c(B) = |\det M|_B|^2$.

ex Fano Plane

$$F_7 \quad \begin{array}{|ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array} \quad \text{over } GF(2)$$





F_7^- : non-Fano matroid

$\det(MXM^t) = B(F_7) + 4x_4x_5x_6$ is stable,

support is $B(F_7^-)$

Question. For which $c > 0$ is $B(F_7) + cx_4x_5x_6$ stable?

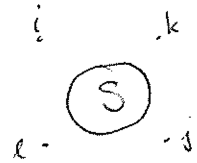
Answer. Brändén. Exactly for $c = 4$.

More generally, given a matroid $\mathcal{M}$ and $c : \mathcal{B}(\mathcal{M}) \to (0, \infty)$, when is $\sum_{B \in \mathcal{B}(\mathcal{M})} c(B)\, \underline{x}^B$ stable?

A <u>degenerate quadruple</u> is a 4-tuple of bases in $\mathcal{M}$ of the form

$$(B_1, B_2, B_3, B_4) = (S \cup \{i,k\}, S \cup \{i,\ell\}, S \cup \{j,\ell\}, S \cup \{j,k\})$$

for some set S and at most one of $S \cup \{i,j\}$ and $S \cup \{k,\ell\}$ is a basis.



Brändén. If Z is stable and $(B_1, \dots, B_4)$ form a degenerate quadruple then $c(B_1)c(B_3) = c(B_2)c(B_4)$.

$\mathcal{M}$ is WHPP $\quad$ $\mathcal{B}(\mathcal{M})$ has the Rayleigh property $\Rightarrow$ $\mathcal{B}(\mathcal{M})$ is negatively correlated

$\Uparrow \qquad\qquad \Uparrow$

$\mathcal{M}$ is HPP $\Leftrightarrow$ $\mathcal{B}(\mathcal{M})$ has the strong Rayleigh property

Every matroid of rank 3 has the Rayleigh property.

Every matroid with a 2-transitive automorphism group is negatively correlated.

Let $\{e, f\} \subseteq E$. Say $\mathcal{M}$ has m elements and rank r, and b bases.

$$\{(i, B) ;\ i \in [m],\ B \in \mathcal{B}(\mathcal{M}),\ i \in B\}$$

$$mM_e = b \cdot r \text{ for any } e \in E.$$

$$\binom{m}{2} M_{ef} = b\binom{r}{2}, \quad M = b.$$

$$M_eM_f - MM_{ef} \geq 0$$

April 16th exercises

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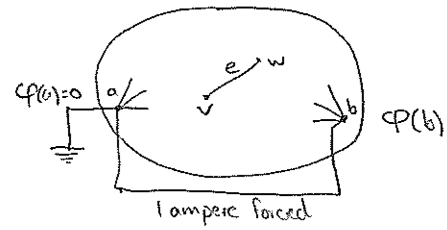
Electrical Networks

Linear Resistive Networks, Kirchhoff (1847)



I - current
φ - electrical current
V - voltage difference
R - resistance

Ohm's Law: in e from v to w the current is proportional to the potential difference, $V = RI$.

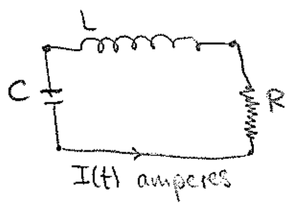
$$Y = \frac{1}{R} = \frac{I}{V} \text{ is conductance ("admittance")}$$

Kirchhoffs Current Law (KCL): At every vertex, sum of current pointing in is zero

Kirchhoffs Voltage Law (KVL): There is a potential function $\varphi : V \to \mathbb{R}$ satisfying KCL and Ohm's Law.

RLC

- resistance - inductance - capacitance



$V = IR$ resistor R
$V = L\frac{dI}{dt}$ inductor L
$I = C\frac{dV}{dt}$ capacitor C

Ohm, KCL $\Rightarrow$ coupled differential equation

Laplace transform

$$\mathcal{L}\{f\} = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt$$

$s = \sigma + i\omega$ complex frequency
$Re(s) = \sigma > 0$ exponential decay
$\omega \in \mathbb{R}$ frequency

$V(s)$, $R(s)$, $L(s)$, $C(s)$
Fact: $\mathcal{L}\{f'\} = s\mathcal{L}\{f\}$ by integration by parts
Hence

$$V(s) = I(s)\underbrace{\left(R + sL + \frac{1}{sC}\right)}_{Y = \text{ admittance in } \mathbb{C}(s)}$$

"You wouldn't threaten me because I would take you out."

Let G be any connected graph, $a,b \in V$. Ground $\varphi(a)=0$. Force 1 ampere through G from b to a. Orient G arbitrarily.

Let $\mathbb{F}$ be any field. Let D be the $V \times E$ signed incidence matrix

$$D_{v,e} = \begin{cases} 1 & \xrightarrow{e} v \\ -1 & v \xrightarrow{e} \\ 0 & \text{otherwise} \end{cases}$$

KCL:

$$\sum_{e: u \text{ to } v} j_e = 0 \quad \text{so} \quad Dj = \underbrace{\delta_a - \delta_b}_{\text{driving current}}$$

Ohm's Law: $y_e(\varphi(v) - \varphi(w)) = j_e$, $-YD^T\varphi = j$ where $Y = \mathrm{diag}(y_e; e \in E)$.

Then $DYD^T\varphi = \delta_b - \delta_a$.

$\varphi(a)=0$, a last row of D. Then $D^T\varphi$ does not depend on last row of D. D_a delete last row of D, $L_a = D_a Y D_a^T$ non-singular.

$$\det(L_a) = \sum_{\substack{\text{spanning tree} \\ T \text{ of } G}} y^T = T(G).$$

Effective admittance of G between a,b is

$$\mathcal{Y}_{ab}(y) = \frac{T(H \setminus f)}{T(H/f)}$$

where $H = G \cup \{f\}$, a new edge between a,b.

2016 03 28

Random Walks

$G = (V,E)$ a connected loopless simple graph

$y = (y_e; e \in E)$ positive edge-weights

Total weight of edges at $v \in V$ is

$$C_v = \sum_{\substack{e \in E \text{ inc} \\ \text{at } v}} y_e.$$

If at vertex v at time n, then at time $n+1$, move to vertex w with probability y_e/C_v if there is an edge $e = \{u,v\}$

Let $p: V \to [0,1)$ be a probability distribution on V. $\mathbb{1}^T p = 1$.

Time evolution $p_n \mapsto p_{n+1}$

$$p_{n+1}(w) = \sum_{v \in V} \frac{y_{wv}}{C_v} p_n(v), \quad p_{n+1} = AC^{-1}p_n$$

where $C = \mathrm{diag}(C_v; v \in V)$ and "we can decide what's true when we get to that point in the proof"

$$A_{vw} = \begin{cases} y_{vw} & \text{if } vw \in E, \\ 0 & \text{otherwise}, \end{cases}$$

weighted adjacency matrix.
If G is not bipartite then there is a unique solution to $\underline{p} = AC^{-1}\underline{p}$.
This $\underline{p}$ is the steady-state.
$(I - AC^{-1})\underline{p} = 0$ so $\underline{p}$ is in the nullspace of $I - AC^{-1}$. So $\det(I - AC^{-1}) = 0$.
Note $I - AC^{-1} = (C - A)C^{-1}$ and $C - A = DYD^T$ where $Y = \mathrm{diag}(y_e : e \in E)$ and D is any signed incidence matrix. DYD^T has nullity 1, is spanned by $\mathbb{1}$. So take $\underline{p} \propto C\mathbb{1}$:

$$(I - AC^{-1})\underline{p} = (C-A)C^{-1}C\mathbb{1} = (C-A)\mathbb{1} = \underline{0}.$$

Let $Z = \mathbb{1}^T\underline{p} = \mathbb{1}^T C \mathbb{1}$. In steady state, $\underline{p} = \frac{1}{Z}C\mathbb{1}$. The proportion of time random walk spends at vertex v is C_v / Z.
$C - A = DYD^T$ Laplacian matrix
Restrict to all $y_e = 1$ and G is k-regular. Then $C = kI$ and A is the adjacency matrix of G.
$(kI - A)\underline{u} = \underline{0}$. Rate of decay to steady state $\approx$ smallest positive eigenvalue of DYD^T.

2016 03 29

<u>Schur-Szegö Composition</u> (special case)

Let

$$P(x) = \sum_i c_i x^i \quad \text{and} \quad K(x) = \sum_{j=0}^{d} \binom{d}{j} u_j x^j$$

be polynomials (in one variable x) with $\deg P \overset{?}{\geq} d = \deg K$. Let

$$Q(x) = \sum_{j=0}^{d} c_j u_j x^j.$$

Proposition: If $P(x)$ is Hurwitz stable and all zeros of $K(x)$ satisfy $|\arg(z) - \pi| \leq \alpha$ for some fixed $\alpha > 0$, then $Q(x)$ has no zeros in the sector $|\arg(z)| < \frac{\pi}{2} - \alpha$.

Proof: The right half plane $|\arg(z)| < \pi/2$ is a circular region $\mathcal{H}_0$. $P(x)$ is non-vanishing on $\mathcal{H}_0$. Let $F(x_1, \ldots, x_d)$ be the d-th polarization of $P(x)$.

First case: $K(0) \neq 0$

$$K(x) = C\prod_{k=1}^{d}(1+\theta_k x) = \sum_{j=0}^{d}\binom{d}{j}u_j x^j.$$

The θ_i are negative inverse roots of $K(x)$, so $|\arg(\theta_k)| \leq \alpha$.

Note:

$$K(x) = C\prod_{K=1}^{d}(1+\theta_k x) = C\sum_{j=0}^{d} e_j(\theta_1, \dots, \theta_d) x_j$$

So

$$u_j = \binom{d}{j}^{-1} e_j(\theta_1, \dots, \theta_d), \quad \text{for } 0 \leq j \leq d.$$

On the other hand

$$F(x_1, \dots, x_d) = \sum_i c_i \binom{d}{i}^{-1} e_i(x_1, \dots, x_d).$$

Now $F(\theta_1 x_1, \dots, \theta_d x_d)$ is nonvanishing for $|\arg x_i| < \frac{\pi}{2} - \alpha$ (since $|\arg(\theta_i x_i)| < \pi/2$ if this holds). Now diagonalize $x_1 = \dots = x_d = x$.

$$F(\theta_1 x, \dots, \theta_d x) = \sum_i c_i \binom{d}{i}^{-1} e_i(\theta_1, \dots, \theta_d) x^i = \sum_i c_i u_i x^i = Q(x).$$

Second case: If 0 is a zero of $K(x)$ of multiplicity $r \geq 1$

$$K(x) = Cx^r \prod_{k=1}^{d-r}(1+\theta_k x)$$

each $|\arg(\theta_k)| \leq \alpha$.

$$K_N(x) = CN^{-r}(1+Nx)^r \prod_{k=1}^{d-r}(1+\theta_k x) \to K(x)$$

as $N \to \infty$ for $N > 0$. For $N > 0$ we've just done this. By Hurwitz's Theorem we are done. ∎

Subgraph Counting Polynomials

Let $G = (V, E)$ be a graph, $\underline{\lambda} = \{\lambda_e ; e \in E\}$ positive edge weights, $\underline{x} = \{x_v ; v \in V\}$ indeterminates.

Enumerator for all spanning subgraphs:

$$\Omega = \prod_{vw \in E}(1 + \lambda_e x_v x_w)$$

is Hurwitz stable (if $|\arg(x_v)| < \pi/2$ for all $v \in V$ then $\Omega \neq 0$).

For each $v \in V$ let

$$K_v(x_v) = \sum_{j=0}^{d} \binom{d}{j} u_j^{(v)} x^j$$

be its "key polynomial", where $d = \deg(G, v)$. Number the vertices $\{1, .., n\}$. Apply Schur-Szegö composition inductively to $\Omega(G, \underline{\lambda}; \underline{x})$ and each $K_v(x_v)$

$$\Omega = \sum_{H \subseteq E} \underline{\lambda}^H \underline{x}^{\deg(H)} \quad \text{where} \quad \underline{\lambda}^H = \prod_{e \in H} \lambda_e \ \& \ \underline{x}^f = \prod_{v \in V} x_v^{f(v)}.$$

The result is

$$Z(G, \underline{\lambda}, \underline{u}; \underline{x}) = \sum_{H \subseteq E} \lambda^H u_{\deg(H)} \underline{x}^{\deg(H)} \quad \text{where} \quad \underline{u}_{\deg(H)} = \prod_{v \in V} u^{(v)}_{\deg(H, v)}.$$

eg// $u_0 = u_1 = 1$ for all $v \in V$

$K_v(x_v) = 1 + d x_v$ if $\deg(G, v) = d$

$$\underline{u}_{\deg(H)} = \begin{cases} 1 & \text{if } H \text{ is a matching} \\ 0 & \text{if } H \text{ is not a matching} \end{cases}$$

$$\underline{x}_{\deg(H)} = \prod_{\substack{\text{saturated} \\ \text{vertices of } H}} x_v$$

"Multivariate Weighted matching polynomial"

$$Z(G, \underline{\lambda}, \underline{u}; \underline{x}) = \sum_{\substack{\text{matchings} \\ H}} \underline{\lambda}^H \underline{x}^{\deg(H)}$$

2016 04 01

To get from Ω to Z we use Schur-Szegö inductively.
Order vertices $V = \{1, 2, .., n\}$ arbitrarily. Let $F_0(\underline{x}) = \Omega(G, \underline{\lambda}; \underline{x})$.
For $1 \le v \le n$ let $F_v(\underline{x})$ be the Schur-Szegö composition of $F_{v-1}(\underline{x})$ and $K_v(\underline{x})$. Then $F_n(\underline{x}) = Z(G, \underline{\lambda}, \underline{u}; \underline{x})$.
Let $\xi_1, \xi_2, ..., \xi_n$ have positive real part. So $\Omega(G; \underline{\xi}) \neq 0$. Let $\eta_1, \eta_2, ..., \eta_n$ be such that $|\arg(\eta_v)| < \frac{\pi}{2} - \alpha$.
Assume that all roots of all keys $K_v(t)$ have $|\arg(z) - \pi| \le \alpha$.
We want to show $F_n(\eta_1, \eta_2, ..., \eta_n) \neq 0$.
Inductively, show that for all $1 \le v \le n$, $F_{v-1}(\eta_1, ..., \eta_{v-1}, x_v, \xi_{v+1}, ... \xi_n)$ is Hurwitz stable of degree at most $\deg(G, v)$ in x_v.

eg Each vertex has at most two consecutive permissible degrees.

$$K_v(t) = \binom{d}{j} t^j + \binom{d}{j+1} t^{j+1}$$

has only non-positive roots. Theorem applies with $\alpha = 0$.

$$Z(G, \underline{\lambda}, \underline{u}; \underline{x}) = \sum_{H \subseteq E} \underline{\lambda}^H \underline{u}_{\deg(H)} \underline{x}^{\deg(H)}$$

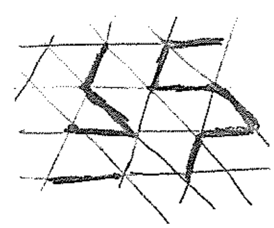
is Hurwitz stable. Diagonalize:

$$\sum_{H \subseteq E} \underline{\lambda}^H \underline{u}_{\deg(H)} y^{|H|}$$

has only non-negative roots.

eg// $u_0 = u_1 = 1$, $u_j = 0$ for all $j \geq 3$, $u_2 = u$

If v has degree d then



$$K_v(t) = 1 + dt + \binom{d}{2} u t^2$$

$$Z(y) = \sum_{H \subseteq E} u^{\#\text{vertices of degree } 2} t^{|H|}$$

roots of K_v are at

$$\frac{d \pm \sqrt{d^2 - 4\binom{d}{2} u}}{2\binom{d}{2} u}$$

If $u \leq d^2 / (4\binom{d}{2})$ then all roots are real, then all roots of $Z(y)$ are real.

2016 04 04

Spanning Graph Enumerator

$$\Omega(G, \underline{\lambda}; \underline{x}) = \prod_{uv \in E} (1 + \lambda_e x_u x_v)$$

If all $\lambda_e > 0$ then Ω is Hurwitz stable.

If all $\lambda_e \in \mathbb{C}$ have $|\lambda_e| = 1$ then $1 + \lambda_e x_v x_w \neq 0$ if $|x_v| < 1$ and $|x_w| < 1$, also $1 + \lambda_e x_v x_w \neq 0$ if $|x_v| > 1$ and $|x_w| > 1$.

So $\Omega(G, \underline{\lambda}; \underline{x})$ is non-vanishing if all $\underline{x}$ are in the circular region

$$\mathcal{D} = \{z \in \mathbb{C} ; |z| < 1\},$$

or in the circular region

$$\mathcal{E} = \{z \in \mathbb{C} ; |z| > 1\}.$$

If all edge weights λ_e have $|\lambda_e|=1$ and all vertex keys $K_v(t)$ have all roots on the unit circular, and $\deg K_v = \deg(G,v)$ then $Z(G,\underline{\lambda},\underline{u};y)$ has all roots on the unit circle.

Let $G=(V,E)$ be a graph. Let $v\in V$ have degree $d(v)$. Consider the key polynomials

$$K_v(t) = 1+t+t^2+\cdots+t^{d(v)} = \frac{1-t^{d(v)+1}}{1-t}.$$

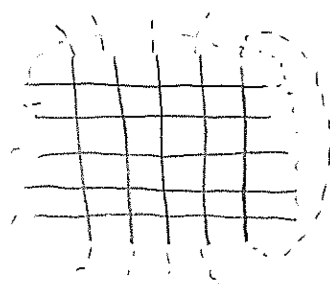
This has all roots of unit modulus

$$K_v(t) = \sum_{j=0}^{d} \binom{d}{j} u_j t^j$$

so $u_j = \binom{d}{j}^{-1}$ for $0 \le j \le d$. Set all $\lambda_e = 1$. Thus

$$Z(G;y) = \sum_{H\subseteq E} \frac{y^{|H|}}{\prod_v \binom{\deg(G,v)}{\deg(H,v)}}$$

has all its roots on the unit circle. Set $y=e^{i\theta}$ with $-\pi<\theta\le\pi$.



$C_5 \square C_5$

Question: For $C_n \square C_n$, the cartesian product of two cycles, as $n\to\infty$ this "tends to" $\mathbb{Z}\square\mathbb{Z}$ in "the limit". $Z(C_n\square C_n;y)$ has finitely many zeros all in $(-\pi,\pi]$, where $y=e^{i\theta}$. Dividing by the total number of zeros of Z, as $n\to\infty$ does the limit of the counting measure of roots of Z exist? What is the limit measure? What about $\mathbb{Z}^r$, $r\ge 3$?