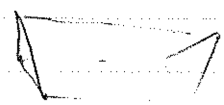
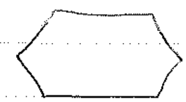
(2014 07 24)

# Lift-and-Project Methods

Recall the lifted-SDP representation of TH(G):

$$TH(G) = \{ x \in \mathbb{R}^V ;\ \underbrace{Y \in \mathbb{S}_+^{\{0\}\cup V},\ Ye_0 = \binom{1}{x} = \mathrm{diag}(Y),\ Y_{ij} = 0\ \forall \{i,j\} \in E}_{\text{this set is in a much higher dimensional space compared to STAB(G)}} \}$$

$\mathbb{R}^3$  5 facets

projection onto $\mathbb{R}^2$  6 facets

Sometimes going from an n-dimensional space to an $n^2$- or $n^3$-dim. space can result in exponential gain in terms of representations of facets of the polytope in $\mathbb{R}^d$.

Let's try to generalize this SDP rep. for TH(G) which is used for STAB(G), to arbitrary polytopes with 0,1 extreme points.

Note that in the SDP rep. of TH(G),

$$Y = \begin{bmatrix} 1 & \cdots & x_i & \\ \vdots & & \vdots & \\ x_i & & x_i & \\ & & & \end{bmatrix}$$

If $x_i = 0$, $Y \succeq 0 \Rightarrow Ye_i = 0$

If $x_i > 0$, $\frac{1}{x_i} Ye_i = \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \\ \vdots \end{bmatrix}$ $j^{th}$ component is zero $\forall \{i,j\} \in E$

If $Y = \binom{1}{x}(1\ x^T)$

$\Rightarrow \binom{1}{x} \in \{0,1\}^{\{0\}\cup V}$

$\Rightarrow$ $i^{th}$ col. of $Y$, $Ye_i = x_i \binom{1}{x}$

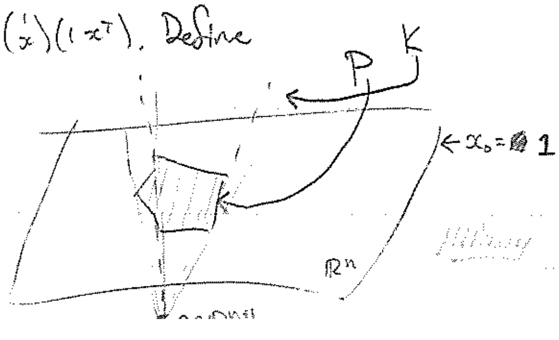
Suppose we are given polytope $P \subseteq [0,1]^n$ (like FRAC(G)). We are interested in

$$P_I := \mathrm{conv}(P \cap \{0,1\}^n). \text{ (like STAB(G))}$$

Think about $\{0,1\}$ vectors $x$ in $P_I$ and their representations $\binom{1}{x}(1\ x^T)$. Define

$$K := cl\{\lambda \binom{1}{x} ; x \in P,\ \lambda > 0\}$$

$$N_+(K) := \{ Ye_0 ; Y \in \mathbb{S}_+^{n+1},\ Ye_0 = \mathrm{diag}(Y)^*,\ Ye_i,\ Y(e_0 - e_i) \in K \}$$




*encodes $\{0,1\}$ if rank(Y)=1

If $Y = \binom{1}{x}(1 \; x^T)$, then

$$Y(e_0 - e_i) = \binom{1}{x} - x_i \binom{1}{x} = \overbrace{(1-x_i)}^{\geq 0} \binom{1}{x}.$$

$N_+(K) \subseteq K$ since $Ye_0 = \underbrace{Y(e_0 - e_i)}_{\in K} + \underbrace{Ye_i}_{\in K} \in K$ ← conve

Further note that

$$Y(e_0 - e_i) = \begin{array}{c} \\ \\ i \\ \\ \end{array} \begin{bmatrix} 1-x_i \\ \vdots \\ 0 \\ \vdots \end{bmatrix} \qquad Ye_i = x_i \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \end{bmatrix}$$

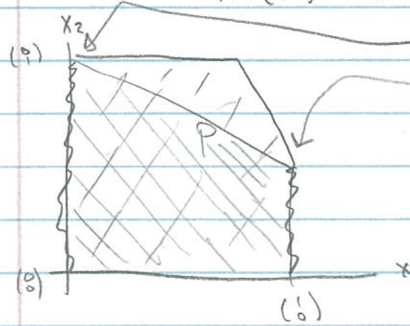
$\hookrightarrow \in K \cap \{\binom{x_0}{x} \in \mathbb{R}^{n+1};\ x_i = 0\}$ $\quad \hookrightarrow \in K \cap \{\binom{x_0}{x} \in \mathbb{R}^{n+1};\ x_i = x_0$

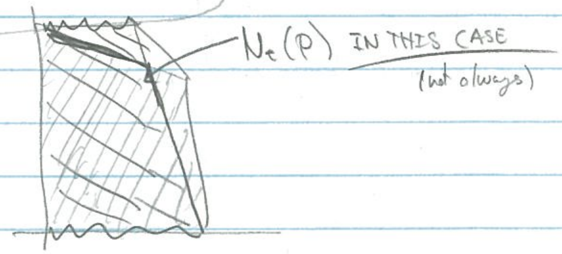
in $\mathbb{R}^n$, $P \cap \{x \in \mathbb{R}^n;\ x_i = 1\}$

$$\Rightarrow N_+(K) \subseteq \left(K \cap \{\binom{x_0}{x};\ x_i = 0\}\right) + \left(K \cap \{\binom{x_0}{x};\ x_i = x_0\}\right)$$

In the space of $P$, $P_I$,

$$N_+(P) \subseteq \operatorname{conv}\left(\underbrace{(P \cap \{x;\ x_i = 0\})}\, \cup \underbrace{(P \cap \{x;\ x_i = 1\})}\right)$$




Theorem 7.10: $P \supseteq N_+(P) \supseteq N_+^2(P) \supseteq \cdots \supseteq N_+^n(P) = P_I$.

Theorem: If $\exists$ a poly time separation oracle for $P$, then we can optimize any linear function 2014 07 29
over $N_+^k(P)$, approximately, in poly. time for all $k = O(1)$.

spelling

The above method is due to Lo? & Schijver (1991). There are many other ~~approaches~~ methods, including one due to Sherali? and Adams.

$$P := \{x \in \mathbb{R}^n;\ Ax \leq b,\ 0 \leq x \leq \bar{e}\}$$

Multiply each inequality by

$$f(J_1, J_2) := \prod_{i \in J_1} x_i \prod_{j \in J_2} (1 - x_j) \geq 0 \quad \forall x \in [0,1]^n$$

where $J_1, J_2 \subseteq \{1, 2, \ldots, n\}$, $J_1 \cap J_2 = \emptyset$. $|J_1| + |J_2| \leq \min\{k+1, n\}$.

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We obtain a system of poly inequalities.
Replace $x_i^2$ by $x_i$ $\forall i$ $\forall$ monomials ($x_i^2 = x_i$ $\forall x \in \{0,1\}^n$)
Replace $\prod_{i \in J} x_j$ by $y_J$ (linearization) for $|J| \geq 2$.
$\rightsquigarrow$ a relaxation of $P_I$ $\underbrace{\{x \in \mathbb{R}^n ; \hat{A}x + By \leq d\}, \exists y \in \mathbb{R}^N\}}_{= SA^k(P)}$

This kind of approach can be generalized to solving the problems of optimizing a continuous function over a compact set.
Let $F \subset \mathbb{R}^n$ be a compact set, let $f$ be a cont. function over $F$.

$$\min_{x \in F} f(x) \equiv \min x_0 \quad \left.\begin{array}{l} f(x) \leq x_0 \\ x \in F \\ l.b \leq x_0 \leq u.b \end{array}\right\} \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \underset{\text{compact}}{\tilde{F}} \subset \mathbb{R}^{n+1}$$

So, our problem is $\min_{x \in F \leftarrow \text{compact}} c^T x = \min_{x \in \text{conv}(F)} c^T x$

For almost all $c$, $\exists$ a unique solution for the second problem (when $F \neq \emptyset$) and such solution(s) are optimal for the first problem.
Every compact set $F$ in $\mathbb{R}^n$ admits a representation as the solution set of a system of (possibly infinite) quadratic inequalities.
Suppose $F$ is given by

$$F = \{x \in \mathbb{R}^n ; \underbrace{x^T Q x + 2q^T x + \gamma}_{= \left\langle \begin{bmatrix} \gamma & q^T \\ q & Q \end{bmatrix}, \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} \right\rangle} \leq 0 \ \forall (Q, q, \gamma) \in \mathcal{P}^{\text{potentially infinite}}\}$$

Suppose $\mathcal{P}$ contains $(I, 0, M)$, $M \in \mathbb{R}_{--}$.
We may assume $\mathcal{P}$ is a convex cone. We may also focus on $\text{ext}(\mathcal{P})$, if we like.

$$\{x \in \mathbb{R}^n ; x^T Q x + 2q^T x + \gamma \leq 0\} \text{ is convex} \iff Q \succeq 0$$

Let $\mathcal{Q}_+ = \left\{ \begin{bmatrix} \gamma & q^T \\ q & Q \end{bmatrix} \in \mathbb{S}^{n+1} ; Q \succeq 0 \right\}$

Theorem 9.4: With the above notation and assumptions,

$$\text{conv}(F) \subseteq \{x \in \mathbb{R}^n ; x^T Q x + 2q^T x + \gamma \leq 0 \ \forall (Q, q, \gamma) \in \text{cone}(\mathcal{P}) \cap \mathcal{Q}_+\}$$
$$= \left\{x \in \mathbb{R}^n ; \left\langle \begin{bmatrix} \gamma & q^T \\ q & Q \end{bmatrix}, \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \right\rangle \leq 0 \ \forall (Q, q, \gamma) \in \text{ext}(\mathcal{P}), \ \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \in \mathbb{S}^{n+1}_+ \right\}.$$