(2014 07 15)

Geometric Representations of Graphs

Given a graph $G=(V,E)$, a unit distance representation of G is $v: V \to \mathbb{R}^d$ ~~then~~ such that $\|v^{(i)}-v^{(j)}\|_2 = 1 \quad \forall \{i,j\} \in E$.

Every graph $G=(V,E)$ admits a unit distance representation in $\mathbb{R}^{|V|}$.

Theorem 6.2: For every graph $G=(V,E)$, the following SDP has an optimal solution with objective value equal to the square of the smallest radius of a Euclidean Ball which contains a unit distance representation of G.

$$\begin{aligned} &\text{Min } t \\ &\quad \operatorname{diag}(X) - t\bar{e} \le 0 \\ &\quad X_{ii} + X_{jj} - 2X_{ij} = 1 \quad \forall \{i,j\} \in E \\ &\quad X \succeq 0 \end{aligned}$$

Sketch of Proof: Call the SDP which has Slater point $\bar{X} := \frac{1}{2} I$, $\bar{t} := \frac{1}{2} + \underbrace{\varepsilon}_{>0}$

$$(D) \quad \begin{aligned} &\text{Max } \sum_{\{i,j\}\in E} z_{ij} \\ &\quad \operatorname{Diag}(y) + \sum_{\{i,j\}\in E} z_{ij}\left(e_i e_i^T + e_j e_j^T - e_i e_j^T - e_j e_i^T\right) \preceq 0 \\ &\quad -\bar{e}^T y = 1 \\ &\quad y \le 0 \quad (y \in \mathbb{R}^V_-) \end{aligned}$$

also has Slater points $\bar{y} := \frac{-1}{|V|}\bar{e}$

Optimal solution $X^* \succeq 0, t^*$ exists. $X^* =: BB^T$, $B^T =: [v^{(1)} \dots v^{(n)}]$, $n := |V|$

$X_{ii} + X_{jj} - 2X_{ij} = 1 \; \forall \{i,j\} \Rightarrow \{v^{(1)}, \dots, v^{(n)}\}$ is a unit distance rep. of G

$t^* = \max_{i \in V} \{\|v^{(i)}\|_2^2\}$

Let $\{u^{(1)}, \dots, u^{(n)}\}$ be an optimal unit distance representation of G. (Exists since cont. func. over a compact set.) We may assume Euclidean Ball is centered at the origin. Consider $M := [u^{(1)} \dots u^{(n)}]$, $\hat{X} = M^T M$, $\hat{X}$ is feasible in (P) ... ∎

In many cases due to ~~the~~ symmetries in the underlying problem SDP's may simplify.

Theorem 6.3: Suppose in SDP (P), the matrices $C, A_1, \ldots, A_m$ pairwise commute. Then (P) and (D) are equivalent to a pair of primal-dual LP problems.

(P) $\inf \langle C, X \rangle$
$\langle A_i, X \rangle = b_i \quad \forall i \in \{1, \ldots, m\}$
$X \succeq 0$

(D) $\sup b^T y$
$\sum_{i=1}^m y_i A_i \preceq 0$

$M_1, M_2 \in \mathbb{S}^n$ <u>commute</u> if $M_1 M_2 = M_2 M_1$
(iff M_1 and M_2 are simultaneously diagonalizable by an orthonormal Q
(ie $Q^T M_1 Q$, $Q^T M_2 Q$ are diagonal))

Proof idea: Note that

$$\sum_{i=1}^n y_i A_i \preceq C \iff Q^T \left(\sum_{i=1}^n y_i A_i \right) Q \preceq Q^T C Q \qquad \text{since } Q^T \cdot Q \in \mathrm{Aut}(\mathbb{S}^n)$$

$$\iff \underbrace{\sum_{i=1}^n y_i \underbrace{(Q^T A_i Q)}_{\text{diagonal}}}_{\text{diagonal}} \preceq \underbrace{Q^T C Q}_{\text{diagonal}}$$

$$\iff \sum_{i=1}^n y_i (Q^T A_i Q) \leqslant Q^T C Q.$$

Consider a restricted version of the unit distance representation, ~~with~~ where every node $v \in V$ is represented by a point on the hypersphere with radius $\sqrt{t}$. The corresponding SDP: 2014 07 17

Min t

$\mathrm{diag}(X) - t\bar{e} = 0$

$X_{ii} + X_{jj} - 2X_{ij} = 1 \quad \forall \{i,j\} \in E$

$X \succeq 0$

<u>Orthonormal Representations</u>

$u: V \to \mathbb{R}^d$ such that

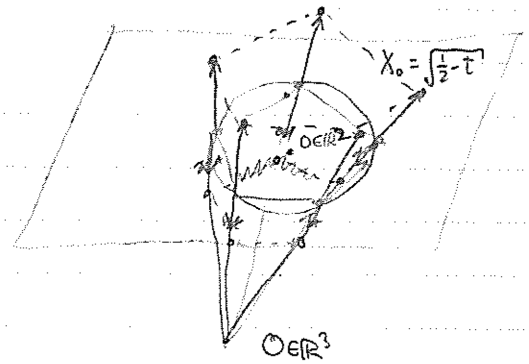
$\|u^{(i)}\|_2 = 1 \quad \forall i \in V$

$\langle u^{(i)}, u^{(j)} \rangle = 0 \quad \forall \{i,j\} \in \bar{E}$ (note $\{i,i\} \notin \bar{E}$)

Last two representations are closely related. We can obtain one from the other. E.g., let $\{v^{(i)}; i \in V\}$ be a hypersphere representation of G with $t < \frac{1}{2}$. Define

$$u^{(i)} := \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\frac{1}{2} - t} \\ v^{(i)} \end{bmatrix} \quad \forall i \in V$$

(2014 07 17)



$-2X_{ij} = 1 - X_{ii} - X_{jj} = 1-2t$

$\therefore X_{ij} = t - \frac{1}{2}$

$$\|u^{(i)}\|_2^2 = 2\Big(\tfrac{1}{2} - t + \underbrace{\|v^{(i)}\|_2^2}_{t}\Big) = 1 \quad \forall i \in V$$

$$\langle u^{(i)}, u^{(j)}\rangle = 2\Big(\tfrac{1}{2} - t + \underbrace{\langle v^{(i)}, v^{(j)}\rangle}_{t-1/2}\Big) = 0 \quad \forall \{i,j\} = 0$$

Therefore $\{u^{(i)};\ i \in V\}$ as constructed above is an orthonormal representation for $\bar{G} = (V, \bar{E})$.

Given a simple undirected graph $G = (V, E)$, $S \subseteq V$ is called a stable set in G if $\forall \{i,j\} \in S$, $\{i,j\} \notin E$.
"independent set"

$$\mathrm{STAB}(G) := \mathrm{conv}\{\underbrace{x \in \{0,1\}^V;\ x \text{ is the incidence vector of a stable set in } G}_{(*)}\}.$$

vector(i) = 1 if i is stable set, = 0 else

Optimizing linear functions over STAB(G) is $\mathcal{NP}$-hard in general.

$$\mathrm{FRAC}(G) := \{x \in [0,1]^V;\ x_i + x_j \le 1\ \forall \{i,j\} \in E\}.$$

Then $\mathrm{STAB}(G) = \mathrm{conv}(\underbrace{\mathrm{FRAC}(G) \cap \{0,1\}^V}_{=(*)})$.

Given an orthonormal representation $\{u^{(i)};\ i \in V\}$ of G and any $c \in \mathbb{R}^n$ such that $\|c\|_2 = 1$,

$$\sum_{i \in V} \langle c, u^{(i)}\rangle^2 x_i \le 1$$

is called an orthonormal representation constraint.

For every clique C in G, the linear inequality

$$\sum_{i \in C} x_i \le 1$$

is valid for STAB(G).

$$CLQ := \left\{ x \in \mathbb{R}^V_+ ;\ \underbrace{A_{clq}(x)}_{\text{matrix encoding all clique constraints}} \le \bar{e} \right\}$$

Optimizing a linear function over $CLQ(G)$ is also ~~is~~ NP-hard.

$$\underbrace{TH(G)}_{\text{Theta Body of } G} := \left\{ x \in \mathbb{R}^V_+ ;\ x \text{ satisfies all orthonormal representation constraints} \right\}$$

intractable

tractable

Theorem 6.6: For every graph G,

$$STAB(G) \subseteq TH(G) \subseteq CLQ(G) \subseteq FRAC(G)$$

Proof: Let S be a stable set in G. Then for every orthonormal representation $\{u^{(i)};\ i \in V\}$ of G, we have $\{u^{(i)};\ i \in S\}$ as an orthonormal system

$$U_S^T := [u^{(i)};\ i \in S]$$

we can extend these vectors to an orthonormal basis for the whole space $\mathbb{R}^V$; therefore,

$$\| U_S c \|_2^2 \le \|c\|_2^2 = 1, \quad \text{for all unit vectors } c \in \mathbb{R}^V.$$

Consider,

$$\cancel{\sum_{i \in S} \langle c, u^{(i)} \rangle^2} \quad \sum_{i \in V} \langle c, u^{(i)} \rangle^2 x_i = \sum_{i \in S} \langle c, u^{(i)} \rangle^2 = \| U_S \|_2^2 \le 1.$$

x is the incidence vector of S

This shows that incidence vectors of every stable set are in $TH(G)$. $TH(G)$ is a convex set (intersection of (closed) half-spaces) and $STAB(G)$ is the smallest convex set which contains all incidence vectors of ~~ever~~ all stable sets in G,

$$STAB(G) \subseteq TH(G).$$

Let C be a clique in G. Let $c \in \mathbb{R}^V$ an arbitrary unit norm vector. Let $u^{(i)} := c \ \ \forall i \in C$, for all other $i \in V \setminus C$, choose mutually orthogonal unit vectors orthogonal to c. This gives an orthonormal representation of G and the corresponding constraint is

$$\sum_{i \in V} \langle c, u^{(i)} \rangle^2 x_i = \sum_{i \in C} \langle c, c \rangle^2 x_i = \sum_{i \in C} x_i.$$

Thus, $TH(G) \subseteq CLQ(G)$. Since $A_{clq}(x) \le \bar{e}$ includes constraints for cliques of size one and two, $CLQ(G) \subseteq FRAC(G)$. ∎

if you had a list of all unit vectors you could compute it

2014 07 22

Recall

$$TH(G) := \{x \in \mathbb{R}^V_+ ;\ \sum_{i \in V} \langle c, u^{(i)} \rangle^2 x_i \le 1$$
$$\forall \text{ orthonormal representation } \{u^{(i)};\ i \in V\} \text{ of } G$$
$$\forall\ c \in \mathbb{R}^V \text{ with } \|c\|_2 = 1\},$$

$\forall w \in \mathbb{R}^V$, $\theta(G,w) := \max\{w^T x ;\ x \in TH(G)\}$. Given $w \in \mathbb{R}^V_+$, $W \in \mathbb{S}^n$, $W_{ij} := \sqrt{w_i w_j}$

Theorem 6.7: For every graph $G=(V,E)$ and every $w \in \mathbb{R}^V_+$, the following are all equal:

(i) $\theta(G,w)$

(ii) $\displaystyle \underset{\substack{\text{orthonormal rep.} \\ \text{of } G \text{ and } c \in \mathbb{R}^V,\ \|c\|_2 = 1}}{\text{Minimum}} \left\{ \underset{i \in V}{\text{Max}} \left\{ \frac{w_i}{\langle c, u^{(i)} \rangle^2} \right\} \right\}$

(iii) $\text{Min}\{\eta ;\ \eta I - S \succeq W,\ \text{diag}(S) = 0,\ S_{ij} = 0\ \forall \{i,j\} \in \bar{E}\}$

(iv) $$\begin{aligned} \text{Max} \quad & \langle W, X \rangle \\ & X_{ij} = 0 \quad \forall \{i,j\} \in E \\ & \langle I, X \rangle = 1 \\ & X \succeq 0 \end{aligned}$$

This theorem shows that $\theta(G,w)$ can be computed approximately to any precision in polynomial time by solving an SDP.

$TH(G)$ and $\theta(G,w)$ have very special properties in particular for perfect graphs.

A graph G is perfect if for every node induced subgraph H of G, the cardinality of max cardinality clique in H is equal to the chromatic number $\chi(H)$ of H.

An odd hole is a chordless odd cycle on at least five nodes.

The complement of an odd-hole is called an odd-anti hole.

Theorem 6.8: For every graph G, TFAE:

(i) G is perfect;

(ii) $STAB(G) = CLQ(G)$;

(iii) $TH(G)$ is a polytope;

(iv) $TH(G) = CLQ(G)$;

(v) $TH(G) = STAB(G)$;

(vi) G does not contain an odd-hole or odd anti-hole.

This theorem implies that max. weight ~~problem~~ stable set problem can be solved in polynomial time on perfect graphs.

Theorem 6.9: For all graphs G,

$$(TH(G))^\circ \cap \mathbb{R}^V_+ = TH(\bar{G}).$$

(For $K \subseteq \mathbb{R}^n$, $K^\circ := \{s \in \mathbb{R}^n ;\ \langle s, x \rangle \le 1 \ \forall x \in K\}$, polar of K.)

Theorem 6.10: For every graph G,

$$TH(G) = \{x \in \mathbb{R}^V ;\ Y \in \mathbb{S}_+^{\{0\}\cup V},\ \mathrm{diag}(Y) = Ye_0 = \begin{pmatrix} 1 \\ x \end{pmatrix},\ Y_{ij} = 0 \ \forall \{i,j\} \in E\}.$$

Note that by the above theorem,

$$\theta(G, w) = \max \left\langle \begin{bmatrix} 0 & & & 0 \\ & w_1 & & \\ & & \ddots & \\ 0 & & & w_n \end{bmatrix}, Y \right\rangle,$$

$$\mathrm{diag}(Y) = Ye_0$$

$$Y_{ij} = 0 \quad \forall \{i,j\} \in E$$

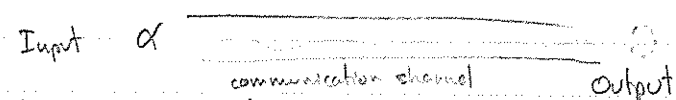
$$Y \in \mathbb{S}_+^{\{0\}\cup V}$$

$$Y = \begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & |V| \end{array} \\ \begin{bmatrix} 1 & & x^T & \\ & x_1 & & \\ x & & \ddots & \\ & & & x_{|V|} \end{bmatrix} \end{array}$$

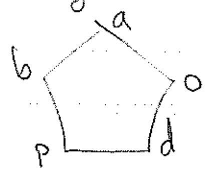
(2014 07 22

Shannon Capacity of a Graph

Suppose we are communicating through a channel but communication is not perfect:



Given a set of letters (alphabet) certain pairs of letters may be confused with each other. Make a graph: make a node for each letter and put an edge between two nodes if the underlying letters may be confused.



What is the maximum number of letters we may use to communicate without confusion?

$\alpha(G)$: the card. of max cardinality stable set

How about words?

Two words of the same length ~~may~~ will not be confused if $\exists$ a position at which the two words have unconfusable letters.

aapd obod may not be confused

Given graphs $G=(V,E)$, $H=(W,F)$, we define the strong product of G and H as follows:

$$G\otimes H = (V(G\otimes H), E(G\otimes H))$$

$$V(G\otimes H) := V\times W$$

$$E(G\otimes H) := \left\{ \{(i,u),(j,v)\} ; \begin{array}{l} (\{i,j\}\in E \wedge \{u,v\}\in F) \\ \vee (\{i,j\}\in E \wedge u=v) \\ \vee (i=j \wedge \{u,v\}\in F) \end{array} \right\}$$

$$G^k = \underbrace{G\otimes\cdots\otimes G}_{k \text{ copies}}$$

$\alpha(G^k)$ is the largest number of k letter words that can be communicated without confusion.

The Shannon capacity of G is

$$\Theta(G) := \lim_{k\to\infty} \left(\alpha(G^k)\right)^{1/k}$$

2014 07 24

If S_1 and S_2 are stable sets in G and H respectively then $S_1 \times S_2$ is a stable set in $G \otimes H$. Therefore
$$\alpha(G^k) \geq (\alpha(G))^k.$$
We also have
$$\alpha(G) \leq \Theta(G).$$
Define
$$\theta(G) := \theta(G, \bar{e})$$
How does $\theta(G)$ behave under strong graph products?
If $\{u^{(i)};\ i \in V(G)\}$ and $\{v^{(j)};\ j \in V(H)\}$ are ortho. rep. of G and H resp., then
$$\{[u^{(i)} \otimes v^{(j)}];\ (i,j) \in V(G) \times V(H)\}$$
is an o.n. rep for $G \otimes H$. Therefore
$$\theta(G \otimes H) \leq \theta(G)\,\theta(H)$$
and we can prove equality. For all graphs G,
$$\theta(G^k) = (\theta(G))^k.$$
We have also,
$$\Theta(G) \leq \theta(G).$$

$$\theta(\pentagon) = \sqrt{5}$$
$$\alpha(\pentagon) = 2 \quad \alpha(\pentagon^2) = 5 \longrightarrow \Theta(\pentagon) = \sqrt{5}$$

$$\Theta(\heptagon) = ?$$

(Sandwich) $\longrightarrow$ Theorem 6.13: For every graph G, we have

χ chromatic number

$$\theta(G) = \max \langle \bar{e}\bar{e}^T, X\rangle \qquad = \min t$$
$$X_{ij} = 0 \ \forall \{i,j\} \in E \qquad \operatorname{diag}(Z) = (t-1)\bar{e}$$
$$\langle I, X\rangle = 1 \qquad Z_{ij} = -1 \ \forall \{i,j\} \in \bar{E}$$
$$X \in \mathbb{S}_+^{V(G)} \qquad Z \in \mathbb{S}_+^{V(G)}$$
Moreover,
$$\alpha(G) \leq \Theta(G) \leq \theta(G) \leq \chi(\bar{G}).$$

Equality throughout if G is perfect.