(2014 06 12)

Interior-Point Method.

Suppose $X^{(0)}$, $(y^{(0)}, S^{(0)})$ are given Slater points for (P) and (D) respectively. Will generate $X^{(k)}$, ~~$(y^{(k)}$~~ $S^{(k)}\}$ pair of primal-dual Slater points such that $\langle X^{(k)}, S^{(k)}\rangle \to 0$.

To remove the "difficult part" of SDP, we treat the constraint $X \succeq 0$ as a barrier function

$$f(X) := -\ln(\det(X)) = \Sigma_{j=1}^{n} -\ln(\lambda_j(X)),$$

domain of f will be

$$\mathrm{dom}(f) := \{X \in \mathbb{S}^n \,;\, X \succ 0\}.$$

If we have a sequence $\{X^{(k)}\} \subset \mathbb{S}^n_{++}$ such that $X^{(k)} \to \bar{X} \in \mathrm{bd}(\mathbb{S}^n_+)$ then $f(X^{(k)}) \to \infty$.

$f(X)$ is finite on $\mathbb{S}^n_{++}$

For every $\mu > 0$, consider

$$(P_\mu) \qquad \begin{array}{l} \inf \frac{1}{\mu}\langle C, X\rangle + f(X) \\ \mathcal{A}(X) = b. \end{array}$$

We hope as $\mu \to 0^+$, optimal solutions of (P_μ) converge to an optimal solution of (P).

Proposition 4.1: With the above definition for f, for every $X \in \mathbb{S}^n_{++}$ and every $H \in \mathbb{S}^n$, we have:

(a) $\langle f'(X), H\rangle = -\langle X^{-1}, H\rangle$;

(b) $f''(X)[H,H] = \mathrm{Tr}\left(\left(X^{-1/2} H X^{-1/2}\right)^2\right)$;

(c) $f'''(X)[H,H,H] = -2\,\mathrm{Tr}\left(\left(X^{-1/2} H X^{-1/2}\right)^3\right)$. (see textbook)

Think about LP:

$$\begin{array}{l} \min \frac{1}{\mu}c^T x + -\Sigma_{j=1}^{n} \ln(x_j) \\ Ax = b \end{array}$$

$$f(x_j) := -\ln(x_j)$$
$$f'(x_j) = -1/x_j$$
$$f''(x_j) = 1/x_j^2$$
$$f'''(x_j) = -2/x_j^3$$

By the previous proposition, $f(X)$ is strictly convex on $\mathbb{S}^n_{++}$ ($f''(X)$ is positive definite).

Under our assumptions, (P_μ) has a unique solution for every $\mu > 0$.

Apply KKT to (P_μ). The unique optimal solution is characterized by

$$\begin{cases} \mathcal{A}(X) = b, \quad X \succ 0 \\ \frac{1}{\mu}C + f'(X) - \mathcal{A}^*(y) = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \mathcal{A}(X) = b, \quad X \succ 0 \\ \mathcal{A}^*(\mu y) + \mu X^{-1} = C. \end{cases}$$

Redefine $y \leftarrow \mu y$

$$\Leftrightarrow \begin{cases} \mathcal{A}(X) = b, \quad X \succ 0 \\ \mathcal{A}^*(y) + S = C, \quad S = \mu X^{-1}. \end{cases}$$

If $\mu \to 0^+$, then

$$\langle X,S\rangle = \mu\langle X^{-1}, X\rangle = n\overset{\mu}{\cancel{}} \to 0 \quad ☆$$

Under the assumptions that both (P) & (D) have Slater points, and $\mathcal{A}$ is onto, for every $\mu > 0$,

2014 06 17

$$(P_\mu) \quad \inf \ \frac{1}{\mu}\langle C, X\rangle - \ln\det X$$

$$\mathcal{A}(X) = b, \quad (X \in \mathbb{S}^n_{++})$$

has a unique solution $X(\mu)$ which corresponds to a unique solution $(X(\mu), y(\mu), S(\mu))$ of

$$\boxed{\begin{aligned} \mathcal{A}(X) &= b, \quad X \succ 0 \\ \mathcal{A}^*(y) + S &= C \\ S &= \mu X^{-1} \end{aligned}}$$

$$\{(X(\mu), y(\mu), S(\mu));\ \mu > 0\} \leftarrow \text{primal-dual central path}$$

(goes off to ∞ if feas. region unbd)

☆

We can follow this path approximately as $\mu \to 0^+$. If we have a point $(X(\mu), S(\mu))$ on the central path, then

$$S(\mu) = \mu X(\mu)^{-1}$$

and

$$\langle X(\mu), S(\mu)\rangle = n\mu.$$

We will extend this to arbitrary pairs $X, S \in \mathbb{S}^n_+$ and define "their μ" by

$$\mu := \frac{\langle X,S\rangle}{n}.$$

Given $X, S \in \mathbb{S}^n_{++}$, a measure of centrality is

$$\psi(X,S) := n\ln\left(\frac{\langle X,S\rangle}{n}\right) - \ln\det X - \ln\det S$$

Theorem 4.2: For every $X, S \in \mathbb{S}^n_{++}$, we have $\psi(X,S) \geq 0$. The equality holds if and only if $S = \mu X^{-1}$, where

$$\mu = \frac{\langle X,S\rangle}{n}.$$

(2014 06 17)

Proof: Note

$$\langle X,S\rangle - \ln\det X \geq \inf_{X\in\mathbb{S}^n_{++}} \{\langle S,X\rangle - \ln\det(X)\}$$

for all $X,S\in\mathbb{S}^n_{++}$. Hence $\qquad$ attained at $X=S^{-1}$

$$\langle X,S\rangle - \ln\det X \geq \langle S,S^{-1}\rangle - \ln\det(S^{-1})$$

for all $X,S\in\mathbb{S}^n_{++}$, and inequality holds if and only if $S=X^{-1}$. So

$$\langle X,S\rangle - \ln\det X - \ln\det S - n \geq 0$$

for all $X,S\in\mathbb{S}^n_{++}$. So

$$\langle \alpha_1 X, \alpha_2 S\rangle - \ln\det(\alpha_1 X) - \ln\det(\alpha_2 S) - n \geq 0$$

for all $X,S\in\mathbb{S}^n_{++}$, $\alpha_1,\alpha_2>0$, and equality holds iff $\alpha_1\alpha_2 X = S^{-1}$. So

$$\alpha_1\alpha_2\langle X,S\rangle - n\ln(\alpha_1\alpha_2) - \ln\det(X) - \ln\det S - n \geq 0$$

. Define $t=\alpha_1\alpha_2>0$, and minimize the LHS wrt t. Well $t\langle X,S\rangle - n\ln t$ is strictly convex, and is uniquely minimized at $t=\frac{n}{\langle X,S\rangle}$. Therefore (substituting $t=\frac{n}{\langle X,S\rangle}$),

$$\cancel{\frac{n}{\langle X,S\rangle}\langle X,S\rangle} + n\ln\left(\frac{\langle X,S\rangle}{n}\right) - \ln\det X - \ln\det S \cancel{-n} \geq 0,$$

and equality holds if and only if $tX=S^{-1}$ if and only if $S=\mu X^{-1}$. ∎

This result, among other things, gives a spectral generalization of arithmetic-geometric mean inequalities applied to eigenvalues of $X^{1/2}SX^{1/2}$.

To design some algorithms, we can focus on two criteria:

- We want $\langle X,S\rangle$ to be small.
- We want to be close to the central path, i.e. we want $\psi(X,S)$ small.

We consider putting them together in a single function

$$\varphi(X,S) := \sqrt{n}\ln(\langle X,S\rangle) + \psi(X,S),$$

potential function

Theorem 4.5: Suppose $X^{(0)}, S^{(0)}$ Slater points for (P) & (D) respectively are given such that for given $\varepsilon\in(0,1)$, $\psi(X^{(0)},S^{(0)}) \leq \sqrt{n}\ln(1/\varepsilon)$. Further assume that in each iteration, we decrease the value of $\varphi(X,S)$ by an absolute constant $\delta>0$, then for some $\bar{k}=O(\sqrt{n}\ln(1/\varepsilon))$, we have

$$\langle X^{(k)},S^{(k)}\rangle \leq \varepsilon\langle X^{(0)},S^{(0)}\rangle \quad \forall k\geq\bar{k}.$$

Proof: Suppose $\varphi(X^{(k+1)}, S^{(k+1)}) \le \varphi(X^{(k)}, S^{(k)}) - \delta \;\; \forall k \ge 0$. After k iterations,

$$\varphi(X^{(k)}, S^{(k)}) \le \varphi(X^{(0)}, S^{(0)}) - k\delta$$

$$\iff \quad \sqrt{n}\ln\left(\frac{\langle X^{(k)}, S^{(k)}\rangle}{\langle X^{(0)}, S^{(0)}\rangle}\right) + \underbrace{\psi(X^{(k)}, S^{(k)})}_{\ge 0 \text{ by thm4.2}} \underbrace{- \psi(X^{(0)}, S^{(0)})}_{\ge -\sqrt{n}\ln(1/\varepsilon)} \le -k\delta$$

$$\implies \quad \sqrt{n}\ln\left(\frac{\langle X^{(k)}, S^{(k)}\rangle}{\langle X^{(0)}, S^{(0)}\rangle}\right) \le -k\delta + \sqrt{n}\ln(1/\varepsilon)$$

For,

$$k \ge 3\frac{\sqrt{n}}{\delta}\ln(1/\varepsilon),$$

we obtain the desired ~~conclusion~~ result. ▨

Our task is reduced to finding a way of updating $(X^{(k)}, S^{(k)}) \to (X^{(k+1)}, S^{(k+1)})$ st $\varphi(\cdot,\cdot)$ decreases.

We have Slater points X, S for (P) & (D). We want an algorithm that will be invariant under the automorphisms of the cones $\mathbb{S}^n_+$ and under switching the roles of (P) & (D). Let $T \in \operatorname{Aut}(\mathbb{S}^n_+)$ (self-adjoint). Given (P), consider

Dual $\quad \mathbb{S}^n_+ \to T(\mathbb{S}^n_+) = \mathbb{S}^n_+$

Primal $\quad \mathbb{S}^n_+ \to T^{-1}(\mathbb{S}^n_+) = \mathbb{S}^n_+$

$$X \to T^{-1}(X), \quad \bar{A}(\cdot) = A(T(\cdot)), \quad \bar{C} := T(C)$$

(P) becomes $(\bar{P})$ $\qquad\qquad$ (D) becomes $(\bar{D})$

$$\inf \langle \bar{C}, X\rangle \quad \bar{A}(X) = b \quad X \in \mathbb{S}^n_+$$

$$\sup b^T y \quad \bar{A}^*(y) + S = \bar{C} \quad S \in \mathbb{S}^n_+$$

2014 06 19

Given any pair $X, S \in \mathbb{S}^n_{++}$, we want to construct $T: \mathbb{S}^n \to \mathbb{S}^n$ linear, self-adjoint (symmetric matrix), positive definite, and

(i) $T(S) = T^{-1}(X) =: V$

(ii) $T(X^{-1}) = T^{-1}(S^{-1}) = V^{-1}$

(iii) $T \in \mathrm{Aut}(\mathbb{S}^n_+)$

Such T exists!

Consider $T(\cdot) = W \cdot W$ for some $W \in \mathbb{S}^n_{++}$.

(iii) ✓

(i) $WSW = W^{-1}XW^{-1} \iff W^2SW^2 = X$

$$\iff S^{1/2}W^2SW^2S^{1/2} = S^{1/2}XS^{1/2}$$
$$\iff (S^{1/2}W^2S^{1/2})(S^{1/2}W^2S^{1/2}) = S^{1/2}XS^{1/2}$$
$$\iff S^{1/2}W^2S^{1/2} = (S^{1/2}XS^{1/2})^{1/2}$$
$$\iff W^2 = S^{-1/2}(S^{1/2}XS^{1/2})^{1/2}S^{-1/2}$$

To verify (ii), note

$$\underset{\overset{\|}{V}}{WSW} = W^{-1}XW^{-1} \iff \underbrace{W^{-1}S^{-1}W^{-1}}_{T^{-1}(S^{-1})} = \underbrace{WX^{-1}W}_{T(X^{-1})} = V^{-1}$$

$$(X,S) \to (T^{-1}(X), T(S)) = (V,V)$$

We have $(X^{(k)}, S^{(k)})$ we want to compute $(X^{(k+1)}, S^{(k+1)})$,

$$D_X := X^{(k+1)} - X^{(k)}, \quad D_S := S^{(k+1)} - S^{(k)}$$

$$\mathcal{A}(X^{(k)}) = \mathcal{A}(X^{(k+1)}) = b \iff D_X \in \mathrm{Null}(\mathcal{A})$$

$$\mathcal{A}^*(y^{(k)}) + S^{(k)} = \mathcal{A}^*(y^{(k+1)}) + S^{(k+1)} = C \iff D_S \in \mathrm{Range}(\mathcal{A}^*)$$

$$\Rightarrow \langle D_X, D_S \rangle = 0$$

We want $(X^{(k+1)}, S^{(k+1)})$ to have a smaller potential function value.

(i) we want smaller $\langle X, S \rangle$

$$X(\alpha) := X^{(k)} + \alpha D_X, \quad S(\alpha) := S^{(k)} + \alpha D_S$$

$$\langle X(\alpha), S(\alpha) \rangle + \langle X, S \rangle + \alpha(\langle D_X, S \rangle + \langle X, D_S \rangle) + \alpha^2 \cancel{\langle D_X, D_S \rangle}^{0}$$

Consider T. Primal Space is mapped under T^{-1}

Dual Space " " T

$$\bar{D}_x := T^{-1}(D_x), \quad \bar{D}_S := T^{-1}(D_S)$$

$$\langle X(\alpha), S(\alpha)\rangle = \langle X, S\rangle + \alpha\Big(\langle T^{-1}(D_x), \underbrace{T(S)}_{=V}\rangle + \langle \underbrace{T^{-1}(X)}_{=V}, T(D_x)\rangle\Big)$$

$$= \langle X, S\rangle + \alpha\langle V, \bar{D}_x + \bar{D}_S\rangle$$

Largest decrease in the duality gap is obtained by setting $\bar{D}_x + \bar{D}_S = -V$.

(ii) The potential function has a "barrier part" $f(X) + f(S)$

Lemma 4.6: Let $X \in \mathbb{S}^n_{++}$. Suppose $D \in \mathbb{S}^n$ satisfies

$$\|D\|_X := \langle D, X^{-1}DX^{-1}\rangle^{1/2} < 1.$$

Then

$$f(X) + \langle f'(X), D\rangle \le f(X+D) \le f(X) + \langle f'(X), D\rangle + \frac{\|D\|_X^2}{2(1-\|D\|_X)^2}.$$

↑ is in dom(f) by below

Note that

$$\|D_X\| \le 1 \Rightarrow X \pm D \in \mathbb{S}^n_+$$

$$\|D_X\| < 1 \Rightarrow X \pm D \in \mathbb{S}^n_{++}$$

$$\|D_X\|^2 = \|X^{-1/2}DX^{-1/2}\|_F^2$$

$$\|D_X\|^2 \le 1 \Rightarrow \|X^{-1/2}DX^{-1/2}\|_2 \le 1 \Rightarrow I \mp X^{-1/2}DX^{-1/2} \succeq 0$$

$$\Leftrightarrow X \mp D \succeq 0 \quad \text{as } X^{1/2} \cdot X^{1/2} \in \mathrm{Aut}(\mathbb{S}^n_+)$$

Let's analyze how the barrier terms change as we move $(X,S) \to (\underbrace{X(\alpha)}_{?}, \underbrace{S(\alpha)}_{?})$.

Let's use the first-order estimate given by Lemma 4.6: see end of front of this sheet

$$\langle f'(X), D_x\rangle + \langle f'(S), D_S\rangle = \langle -X^{-1}, D_x\rangle + \langle -S^{-1}, D_S\rangle$$

$$= -\langle \underbrace{T(X^{-1})}_{V^{-1}}, \underbrace{T^{-1}(D_x)}_{\bar{D}_x}\rangle - \langle \underbrace{T^{-1}(S^{-1})}_{V^{-1}}, \underbrace{T(D_S)}_{\bar{D}_S}\rangle$$

$$= \langle -V^{-1}, \bar{D}_x + \bar{D}_S\rangle$$

To have the largest rate of decrease with respect to first-order estimate of the change in the barrier part, we pick $\bar{D}_x + \bar{D}_S = V^{-1}$!

(2014 06 19)

Recall $\bar{\mathcal{A}}(\cdot) = \mathcal{A}(T(\cdot))$. Then $(\bar{D}_x, \bar{D}_s)$ is the unique solution of

$$\bar{\mathcal{A}}(\bar{D}_x) = 0$$
$$\bar{\mathcal{A}}^*(d_y) + \bar{D}_s = 0 \qquad \text{for some } d_y$$
$$\bar{D}_x + \bar{D}_s = \tilde{\mathcal{U}},$$

where

$$\tilde{\mathcal{U}} := -\left(\frac{n+\sqrt{n}}{\langle X, S\rangle}\right)V + V^{-1}.$$

We need one more tool.

$$\tilde{\mu} := \frac{\langle X^{-1}, S^{-1}\rangle}{n}$$

Note

$$\mu = \frac{\langle X, S\rangle}{n} = \frac{\langle T^{-1}(X), T(S)\rangle}{n} = \frac{\langle V, V\rangle}{n}$$

$$\tilde{\mu} = \frac{\langle X^{-1}, S^{-1}\rangle}{n} = \frac{\langle T(X^{-1}), T^{-1}(S^{-1})\rangle}{n} = \frac{\langle V^{-1}, V^{-1}\rangle}{n}$$

Theorem 4.8: For every pair $X, S \in \mathbb{S}^n_{++}$, $\mu\tilde{\mu} \geq 1$. Moreover, equality holds if and only if $S = \mu X^{-1}$.

Proof:
$$0 \leq \frac{\langle T(S-\mu X^{-1}), T(S-\mu X^{-1})\rangle}{n\mu} = \frac{\langle V-\mu V^{-1}, V-\mu V^{-1}\rangle}{n\mu}$$
$$= \frac{1}{n\mu}\left(n\mu - 2n\mu + \mu^2 n\tilde{\mu}\right) = \mu\tilde{\mu} - 1.$$

Equality holds if and only if $T(S-\mu X^{-1}) = 0 \Leftrightarrow S = \mu X^{-1}$. ▨

$$\|\tilde{\mathcal{U}}\|_F^2 = \langle \tilde{\mathcal{U}}, \tilde{\mathcal{U}}\rangle = \left\langle -\frac{n+\sqrt{n}}{\langle X, S\rangle}V + V^{-1}, \quad '' \quad \right\rangle$$

$$= \underbrace{\frac{1}{\mu}}_{>0}\left(n\underbrace{(\mu\tilde{\mu}-1)}_{\geq 0 \text{ by } 4.8} + \underbrace{1}_{>0}\right) > 0.$$

$$\mathcal{U} := \frac{\tilde{\mathcal{U}}}{\|\tilde{\mathcal{U}}\|_F}, \quad \text{so that } \|\mathcal{U}\|_F = 1.$$

Redefine $(\bar{D}_x, \bar{D}_s)$ as the unique solution of

$$\begin{cases} \bar{D}_x \in Null(\bar{\mathcal{A}}),\ \bar{D}_s \in Range(\bar{\mathcal{A}}^*) \\ \bar{D}_x + \bar{D}_s = \mathcal{U} \end{cases}$$

Note $\|\mathcal{U}\|_F = 1 \Longrightarrow \|\bar{D}_x\|_F, \|\bar{D}_s\|_F \leq 1.$

Primal Dual Interior Point Algorithm

2014 07 03 (

```
Given $X^{(0)} \succ 0$, $S^{(0)} \succ 0$ feasible in (P) and (D) respectively, such that for given $\varepsilon \in (0,1)$
we have
        $\psi(X^{(0)}, S^{(0)}) \leq \sqrt{n}\ln(1/\varepsilon)$.
Set
        $k := 0$
WHILE
        $\langle X^{(k)}, S^{(k)} \rangle > \varepsilon \langle X^{(0)}, S^{(0)} \rangle$
DO                                                          side-notes:
        $X := X^{(k)}$
        $S := S^{(k)}$
        $W^2 := S^{-1/2}(S^{1/2} X S^{1/2})^{1/2} S^{-1/2}$
        $\bar{\mathcal{A}}(\cdot) = \mathcal{A}(W \cdot W)$                 $[\bar{A}_i = W A_i W]$
        $V := WSW$
        $\tilde{\mathcal{U}} := -\left(\frac{n+\sqrt{n}}{\langle X,S \rangle}\right) V + V^{-1}$
        $\mathcal{U} := \frac{1}{\|\tilde{\mathcal{U}}\|_F} \tilde{\mathcal{U}}$
    Solve the system
        $\begin{cases} \bar{A}(\bar{D}_x) = 0 \\ \bar{A}^*(d_y) + \bar{D}_s = 0 \\ \bar{D}_x + \bar{D}_s = \mathcal{U}. \end{cases}$      $\varphi_n = \varphi_- - \sqrt{n}\ln\langle X,S \rangle + \psi$
    Compute
        $\bar{\alpha} := \arg\min\{\varphi_{\sqrt{n}}(X + \alpha W\bar{D}_x W, S + \alpha W^{-1}\bar{D}_s W^{-1}); \alpha > 0\}$
        $X^{(k+1)} := X + \bar{\alpha} W \bar{D}_x W$
        $S^{(k+1)} := S + \bar{\alpha} W^{-1} \bar{D}_s W^{-1}$
        $k := k+1$
END WHILE
```

(2014 07 03)

(#) For the analysis of

$$\frac{\|D_x\|_x^2}{2(1-\|D_x\|_x)^2},$$

we estimate

$$\begin{aligned}\|D_x\|_x^2 &= \langle D_x, X^{-1}D_xX^{-1}\rangle \\ &= \langle W^{-1}D_xW^{-1}, (WX^{-1}W)(W^{-1}D_xW^{-1})(WX^{-1}W)\rangle \\ &= \langle \bar{D}_x, V^{-1}\bar{D}_xV^{-1}\rangle \\ &\le \|V^{-1}\cdot V^{-1}\|_2\|\bar{D}_x\|_F^2 \quad (\le \|V^{-1}\cdot V^{-1}\|_2) \\ &\le \frac{1}{\lambda_n(V)^2}\end{aligned}$$

and similarly

$$\begin{aligned}\|D_s\|_s^2 &= \langle D_s, S^{-1}D_sS^{-1}\rangle \\ &= \langle \bar{D}_s, V^{-1}\bar{D}_sV^{-1}\rangle \\ &\le \|V^{-1}\cdot V^{-1}\|_2\|\bar{D}_s\|_F^2 \\ &\le \frac{1}{\lambda_n(V)^2}\end{aligned}$$

Lemma 4.11: Let $\tilde{U}, V$ be as before. Then

$$\|\tilde{U}\|_F \ge \frac{\sqrt{3}}{2\lambda_n(V)}.$$

Proof: Note that

$$\tilde{U} = -\left(\frac{n+\sqrt{n}}{\langle X,S\rangle}\right)V + V^{-1} = \left(V^{-1} - \frac{1}{\mu}V\right) - \frac{1}{\sqrt{n}\mu}V$$

and

$$\langle V, V^{-1} - \tfrac{1}{\mu}V\rangle = 0.$$

Thus

$$\|\tilde{U}\|_F^2 = \|V^{-1} - \tfrac{1}{\mu}V\|_F^2 + \frac{1}{n\mu^2}\underbrace{\|V\|_F^2}_{n\mu},$$

$$\begin{aligned}\|V^{-1} - \tfrac{1}{\mu}V\|_F^2 &= \sum_{k=1}^{n}\left(\frac{1}{\lambda_k(V)} - \frac{1}{\mu}\lambda_k(V)\right)^2 \\ &\ge \left(\frac{1}{\lambda_n(V)} - \frac{\lambda_n(V)}{\mu}\right)^2 \\ &= \frac{1}{\lambda_n(V)^2} - \frac{2}{\mu} + \frac{\lambda_n(V)^2}{\mu^2}.\end{aligned}$$

So

$$
\begin{aligned}
\|\tilde{u}\|_F^2 &\geq \frac{1}{\lambda_n(V)^2} - \frac{2}{\mu} + \frac{\lambda_n(V)^2}{\mu^2} + \frac{1}{\mu} \\
&= \frac{1}{\lambda_n(V)^2} - \frac{1}{\mu} + \frac{\lambda_n(V)^2}{\mu^2} \\
&= \frac{1}{\lambda_n(V)^2}\left(\left(\frac{\lambda_n(V)^4}{\mu^2} - \frac{\lambda_n(V)^2}{\mu} + \frac{1}{4}\right) + \frac{3}{4}\right) \\
&= \frac{1}{\lambda_n(V)^2}\left(\left(\frac{\lambda_n(V)^2}{\mu} - \frac{1}{2}\right)^2 + \frac{3}{4}\right) \\
&\geq \frac{3}{4\lambda_n(V)^2}.
\end{aligned}
$$

Taking square-roots yields the desired result.

Let's prove that the potential function decreases:

$$
\begin{aligned}
\varphi(X(\alpha),S(\alpha)) - \varphi(X,S) &\overset{\text{lemma 4.6}}{\leq} (n+\sqrt{n})\ln\left(\frac{\langle X(\alpha),S(\alpha)\rangle}{\langle X,S\rangle}\right) - \alpha\langle U,V^{-1}\rangle \\
&\quad + \alpha^2\left(\frac{\|D_x\|_x^2}{2(1-\|D_x\|_x)^2} + \frac{\|D_s\|_s^2}{2(1-\|D_s\|_s)^2}\right)
\end{aligned}
$$

front of this page

$$
\begin{aligned}
&\overset{(\#)\ +\ \text{lemma 4.6}}{\leq} (n+\sqrt{n})\ln\left(\frac{\langle X,S\rangle + \alpha\langle U,V\rangle}{\langle X,S\rangle}\right) - \alpha\langle U,V^{-1}\rangle + \alpha^2\left(\frac{1/\lambda_n(V)^2}{(1-1/\lambda_n(V))^2}\right) \\
&\leq (n+\sqrt{n})\underbrace{\left(\alpha\frac{\langle U,V\rangle}{\langle X,S\rangle}\right)}_{\text{linear estimate of ln}} - \alpha\langle U,V^{-1}\rangle + \frac{\alpha^2}{(\lambda_n(V)-1)^2} \\
&\leq -\alpha\left\langle \underbrace{U, \underbrace{-\left(\frac{n+\sqrt{n}}{\langle X,S\rangle}\right)V + V^{-1}}_{\tilde{u}}}_{\|\tilde{u}\|_F} \right\rangle + \frac{\alpha^2}{(\lambda_n(V)-1)^2}
\end{aligned}
$$

So choosing $\alpha = \lambda_n(V)/8 > 0$ yields

$$
\varphi(X(\alpha),S(\alpha)) - \varphi(X,S) \leq -\alpha\left(\frac{\sqrt{3}}{2\lambda_n(V)}\right) + \frac{\alpha^2}{(\lambda_n(V)-1)^2} = \frac{-\sqrt{3}}{16} + \frac{1}{49} < -\frac{1}{12}.
$$

(2014 07 03+)

Now we have all the components for

Theorem 4.13: The Interior Point Algorithm terminates in at most

$$O(\sqrt{n}\ln(1/\varepsilon))$$

iterations with a primal-dual feasible pair X, S such that

$$\langle X, S\rangle \leq \varepsilon \langle X^{(0)}, S^{(0)}\rangle.$$

How do we find the initial Slater points?

1) Convert optimization problem to a feasibility problem.

$$\left\{\begin{array}{c} (P), (D) \text{ have optimal solutions and optimal objective values are the same} \\ \Longleftrightarrow \\ \left\{\begin{array}{l} \mathcal{A}(X)=b,\ X \succcurlyeq 0 \\ \mathcal{A}^*(y)+S=C,\ S \succcurlyeq 0 \\ \langle C, X\rangle - b^T y = 0 \end{array}\right. \quad \text{is feasible} \end{array}\right.$$

2) Given a feasibility problem, we can construct an optimization problem to solve it

$$\left\{\begin{array}{l} \mathcal{A}(X)=b,\ X \succcurlyeq 0 \\ \text{has feasible solutions} \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} \inf\ \gamma \\ \mathcal{A}(X)+\gamma(b-\mathcal{A}(I))=b \\ \quad X \succcurlyeq 0 \\ \quad \gamma \in \mathbb{R}_+ \\ \text{has optimal value } 0\text{, and it's attained} \end{array}\right.$$

Notice that $(I, \gamma=1)$ is a Slater point.