(2014 06 10)

Ellipsoid Method

An ellipsoid with centre $c \in \mathbb{R}^n$ and a shape-size matrix $A \in \mathbb{S}^n_{++}$ is

$$E(A,c) := \{x \in \mathbb{R}^n ; \langle A^{-1}(x-c), (x-c) \rangle \le 1\}.$$

$E(A,c)$ is the image of Euclidean Unit Ball $B_n(0,1)$.

$E(A,c) = c + A^{1/2} B_n(0,1)$.

$\mathrm{vol}(E(A,c)) = \sqrt{\det(A)}\ \mathrm{vol}(B_n(0,1))$

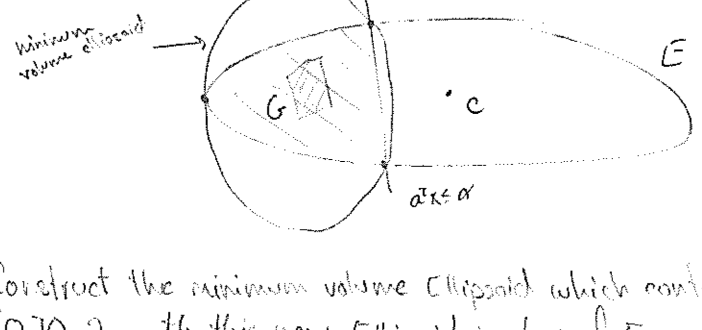
Ellipsoids are simple convex sets that are easy to deal with in mathematical analysis and in algorithms. Also, they are rich enough to approximate every convex set relatively well:

Theorem 3.1: For every compact convex set G in $\mathbb{R}^d$ with non-empty interior, $\exists$ a unique minimal volume ellipsoid which contains G. Moreover, shrinking the ellipsoid around its centre by a factor of d yields an ellipsoid contained in D.

(Proof hints in textbook.) (Löwner-John ellipsoid)

Sketch of finding a point a convex set G via Ellipsoid Method.

1. Find a possibly very large Ellipsoid E which contains G.
2. Ask a separation oracle for G: is the center of the Ellipsoid E in G?
3. If yes, STOP. Otherwise, we have a separating hyperplane



4. Construct the minimum volume Ellipsoid which contains $E \cap \{x ; a^T x \le \alpha\}$. GOTO 2 with this new Ellipsoid in place of E.

2014 06 12

To summarize, we will see a way of applying Ellipsoid Method for solutions of

$$\inf_{x \in G} c^T x$$

where $G \subset \mathbb{R}^d$ is convex and bounded with $\mathrm{int}(G) \neq \emptyset$.

(assume) → There are $r, R > 0$ such that for some $\hat{x} \in G$,

$$B_d(\hat{x}, r) \subseteq G \subseteq B_d(\hat{x}, R).$$

Given $\varepsilon > 0$, define

$$\mu_0 := \varepsilon + \sup\{c^T x ; x \in G\} - \inf\{c^T x ; x \in G\}.$$

Then Ellipsoid Method can be used to generate a feasible solution $\bar{x} \in G$ such that

$$c^T \bar{x} \leq \inf\{c^T x ; x \in G\} + \varepsilon$$

in

$$O\left(d^2\left(\ln\left(\frac{R}{r}\right) + \ln\left(\frac{\mu_0}{\varepsilon}\right)\right)\right)$$

interations, where each iteration makes a call to the separation oracle for G and does some $O(d^2)$ computations.

Let's apply the last result to SDP. Suppose (P) and (D) have Slater points $\bar{X}$ and $(\bar{y}, \bar{S})$ respectively. Consider

$$(\tilde{P}) \quad \begin{aligned} &\inf \langle C, X \rangle \\ &\mathcal{A}(X) = b \\ &\langle \bar{S}, X \rangle \leq 2 \langle \bar{S}, \bar{X} \rangle \\ &X \in \mathbb{S}^n_+. \end{aligned}$$

Theorem 3.9:

(a) (P) & $(\tilde{P})$ attain their optimal values.

(b) The optimal solution sets of (P) & $(\tilde{P})$ are the same.

(c) Let G denote the feasible region of $(\tilde{P})$. Then G is convex and

$$B_{\dim(G)}(\bar{X}, \lambda_n(\bar{X})) \subseteq G \subseteq B_{\dim(G)}\left(0, \frac{2\langle \bar{X}, \bar{S} \rangle}{\lambda_n(\bar{S})}\right).$$