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Lemma: If y^{*i} is an optimal solution for dual LP^i, $i \in \{1,2\}$ then (y^{*1}, y^{*2}) is optimal for dual intersection LP.

Theorem: The matroid intersection system (*) is TDI.

Proof: Let $w = (w_e,\ e \in E)$ be integral. Find opt dual soln and create the split $w = w^1 + w^2$.

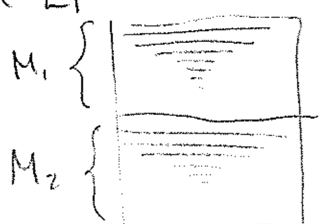
Take a greedy dual solution y^{*i} for each dual LP^i, $i \in \{1,2\}$.

So we have two nested families

$$T_1^1 \subset T_2^1 \subset T_3^1 \subset \cdots \subset T_{n_1}^1$$
$$T_1^2 \subset T_2^2 \subset T_3^2 \subset \cdots \subset T_{n_2}^2$$

of subsets such that these are the only positive dual variables.

In LP



- totally unimodular
- reduces to bipartite matching.

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<u>Travelling Salesman Problem (TSP)</u>

$G = (V,E)$ complete graph

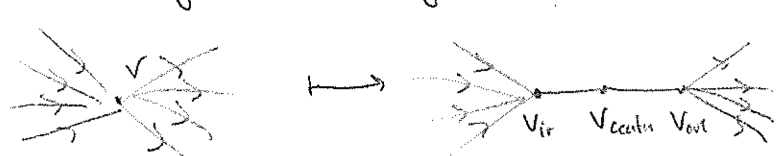
$c = (c_e;\ e \in E)$

$e = uv$, $c_e \equiv$ cost to travel ~~between~~ from u to v.

We study the symmetric TSP, cost is to travel from u to v is the same as from v to u.

TSP: find a min-cost spanning circuit $\equiv$ "Hamiltonian circuit" $\equiv$ "tour"

Reduction from asymmetric to symmetric



LP relaxations

Dantzig, Fulkerson, Johnson (1954)

Variables $(x_e; e \in E)$. - indicates which edges used in tour

$$\min \sum (c_e x_e; e \in E)$$
$$\text{s.t.: } x(\delta(v)) = 2 \quad \forall v \in V$$
$$0 \le x_e \le 1 \quad \forall e \in E$$

Integral solution is a family of circuits covering all nodes, that is, a 2-factor.

Let $\emptyset \subset S \subset V$. Any tour must use at least two edges from $\delta(S)$

Thus any tour satisfies $x(\delta(S)) \ge 2$ (subtour elimination constraint).

Note

$$\sum_{v \in S} x(\delta(v)) = 2|S|$$

and

$$\sum_{v \in S} x(\delta(v)) = 2x(\gamma(S)) + x(\delta(S)).$$

Thus

$$x(\delta(S)) = 2|S| - 2x(\gamma(S)).$$

Thus x satisfies

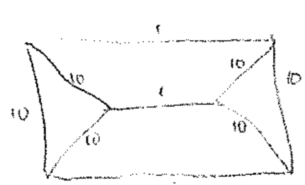
$$x(\delta(S)) \ge 2 \iff 2|S| - 2x(\gamma(S)) \ge 2$$
$$\iff |S| - x(\gamma(S)) \ge 1$$
$$\iff x(\gamma(S)) \le |S| - 1$$

Subtour Relaxation "Held-Karp"

$$\min \sum (c_e x_e; e \in E)$$

s.t.: subtour polytope
$$\begin{cases} x(\delta(v)) = 2 & \forall v \in V \\ x(\delta(S)) \ge 2 & \forall \emptyset \subset S \subset V \\ 0 \le x_e \le 1 & \forall e \in E \end{cases}$$

Integer solution is a tour. But for $|V| \ge 6$, subtour polytope has fractional vertices



opt tour = 42

SUB = 33 (Δ's $\frac{1}{2}$, —'s 1)

1:39.529

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Know: SUB $\leq$ TSP

Theorem: With Δ-inequality, TSP $\leq \frac{2}{3}$SUB

4/3'rds conjecture: Δ-inequality $\Rightarrow$ TSP $\leq \frac{4}{3}$SUB

<u>Solving the Subtour Relaxation</u>

Subtour separation:
Given $\bar{x}$ with $0 \leq \bar{x} \leq 1$, find a set $\emptyset \subset S \subset V$ such that $\bar{x}(\delta(S)) < 2$ if such S exists.
Recall, sets of form $\delta(S)$ are called cuts. Let $u_e = \bar{x}_e \; \forall e \in E$. Find a min ~~cut $\delta(S)$~~ capacity cut $\delta(S)$. If capacity <2, then $\bar{x}(\delta(S)) < 2$.
If capacity ≥ 2, then $\bar{x} \in$ subtour polytope.

<u>Algorithm</u>
Select $r \in V$.
For each $v \in V \setminus \{r\}$, use max-flow to find a min r-v cut $\delta(S_{rv})$.
If $u(\delta(S_{rv})) <$ BEST then BEST $= u(\delta(S_{rv}))$.

n-1 flows finds min cut
$\Rightarrow$ poly time
$\Rightarrow$ optimization over subtour polytope is poly time
$\Rightarrow$ theory, not practice.

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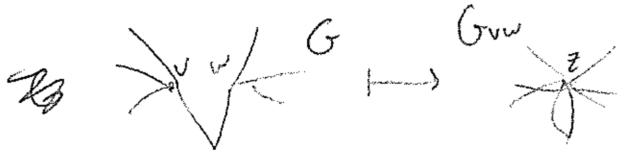
Karger's Random Contraction Alg (1993) (Text section 3.5)
Node identification:



<u>Alg</u>: While G has more than 2 nodes, choose an edge $e = vw$ of G with probability $u_e / u(E)$. Replace G by G_{vw}.
Return unique cut of G

Theorem: Let A be a min-cut of G. Then the alg returns A with probability at least $2/n(n-1)$.

Corollary: Let k be a positive integer, and let A be a min cut. The probability that the algorithm does not return A after kn^2 rounds is at most e^{-2k}

Note: This means the alg can find all min-cuts with high probability.

Corollary: A graph has at most $n(n-1)/2$ min-cuts.

Practical Use of Separation

Let $\mathcal{S}$ be a family of sets $\emptyset \subset S_1, \ldots, S_k \subset V$, Relaxation of Subtour LP

$$(*) \begin{cases} \min \Sigma(c_e x_e ;\ e \in E) \\ \text{s.t.:} \quad x(\delta(v)) = 2 \quad \forall v \in V \\ \qquad x(\delta(S)) \geq 2 \quad \forall S \in \mathcal{S}. \\ \qquad 0 \leq x_e \leq 1 \quad \forall e \in E \end{cases}$$

Let x^* be an optimal solution $(*)$. Then x^* satisfies all subtour inequalities, then x^* is an optimal solution to the subtour LP.

Cutting-Plane Alg

Initial $\mathcal{S} = \emptyset$

Let x^* be an optimal solution to $(*)$. Run subtour SEP on x^*. If x^* satisfies all subtour inequalities, STOP.

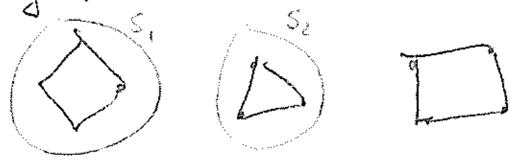
Else add to $\mathcal{S}$ one or more sets S such that $x^*(S) < 2$. Repeat.

Not ~~[illegible]~~ poly-time, but can run fast in practice.

Practice:

- simplex method is used
  - use basis for x^* as the starting point for new LP
- add rounds of cuts $\delta(S)$, not just one

Initially x^*



$$G^* = (V, E^*), \quad E^* = \{e;\ x^*_e > 0\}$$

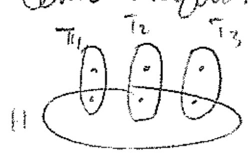
- connected components of G^*.

To solve TSP, we use SUB as a lower bound.

- use branch-and-bound
- alternatively, continue the cutting-plane algorithm

Need: Inequalities satisfied by all tours.
Heuristic idea: Use facet-defining inequality.

ex Comb inequalities



$H, T_i \subseteq V \quad i \in [3]$
$H \cap T_i \neq \emptyset \quad i \in [3]$
$T_i \setminus H \neq \emptyset \quad i \in [3]$
$T_i \cap T_j = \emptyset \quad i \in [3] \setminus \{j\}$

<u>Theorem</u>: Every tour satisfies

(*) $$x(\delta(H)) + x(\delta(T_1)) + x(\delta(T_2)) + x(\delta(T_3)) \geq 10.$$

Proof: Let T be a tour and let $\bar{x}$ be the tour vector. For any $\emptyset \subset S \subset V$, we know that $\bar{x}(\delta(S))$ is an even number. Thus all we need to show is that $(*) > 8$. We know $(*) \geq 8$.
Suppose $(*) = 8$. Now if $\bar{x}(\delta(S)) = 2$ then T restricted to S is a path. For $i \in [3]$, let P_i be the path obtained by restricting T to T_i. Thus for $i \in [3]$ we have $P_i \cap \delta(H) \neq \emptyset$.
Thus $\bar{x}(\delta(H)) \geq 3$. #

In general, with k teeth (odd) we have

$$x(\delta(H)) + \sum_{i=1}^{k} x(\delta(T_i)) \geq 3k+1$$

satisfied by all tours.
Separation for combs is open. If k is fixed then separation is poly-time solvable.