Topics:
Trees, ...
Max Flow-Min Cut
Matching, T-joins, Weighted Matching
Matroids, Matroid Intersection
Disjoint Directed Cuts
TSP: example of how to deal with NP-hard problem

CO 450 - Combinatorial Optimization

2014 09 08

- searching for an optimal object in a finite set
- typically the set has a concise representation, but the number of objects is huge

(1) number of candidate solutions is not what makes a problem difficult

Traveling Salesmen Problem (TSP)

n locations "cities"

distance between each pair (a,b)

Find a shortest possible tour.

Newspaper: TSP cannot be solved for a modest number of cities, because the number of tours is $(n-1)!$.

Sorting n numbers is easy, but $n!$ candidate solutions.

(2) Good characterizations are key

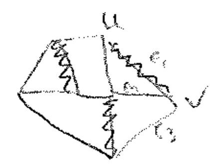
Graph $G=(V,E)$ undirected

$VG \equiv V$ = vertices or nodes

$EG \equiv G$ = edges, unordered pairs of nodes $\{u,v\}$, $u,v \in V$

$\delta_G(V) \equiv \delta(v) \equiv$ subset of E incident with node V

G



$\delta(v) = \{e_1, e_2, e_3\}$

<u>Matching</u> of G is a pairwise disjoint set of edges $M \subseteq E$.

Maximum Matching: max $|M|$ over all matchings of G.

M-augmenting path P: u ——— v, u,v not covered by M.

That is, $\delta(v) \cap M = \delta(u) \cap M = \emptyset \Rightarrow M' = M \Delta EP$ is a matching, $|M'| = |M|+1$.

Theorem (Peterson 1891): M is a maximal matching if and only if there is no M-augmenting path.

Proof: ($\Rightarrow$) easy/clear. ($\Leftarrow$): Larger matching N, look at $M \Delta N$.

Bipartite Graph

$V = V_1 \dot\cup V_2$

$\forall e \in E$ we have $|e \cap V_i| = 1$ $\forall i = 1,2$

$C \subseteq E$ is called a <u>cover</u> if $\forall e \in E$, $e \cap C \neq \emptyset$

Theorem (König)
If G is bipartite then the maximum size of a matching is equal to the minimum size of a cover.
$\nu(G)$ = size of max matching
$\tau(G)$ = size of min cover
easy: $\nu(G) \leq \tau(G)$

Formal structure:
Peterson: Allows you to certify that M is <u>not</u> a max matching. (trivial)
König: Allows you to certify that M <u>is</u> maximal. (good characterization)

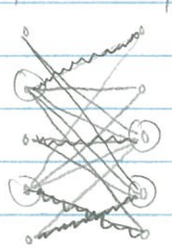
Complexity Classes: - yes/no questions
Is there a matching of size k?
NP - certify answer is yes
Co-NP - certify answer is no
certificate can be checked in polynomial number of steps, $O(n^t)$ t constant
Good characterization: NP $\cap$ Co-NP
P = problems solvable in a polynomial number of steps
P vs NP: is NP = P?
NP-hard - every problem in NP can be reduced to this problem X.
$Y \in NP \quad Y \rightarrow X$

ex TSP is NP-hard

③ Being NP-hard does <u>not</u> mean practical examples cannot be solved.

2014 09 10

Bipartite Graph



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"min-max thm"

Theorem [Konig]: If G is bipartite then $\nu(G) = \tau(G)$.

Proof: May assume G has an edge. Claim: There exists a node v such that v is hit by every maximum matching. Verification: Suppose not. Let $e = uv$ be an edge, let M be a max matching that misses v, and let N be a max matching that misses u. Consider the symmetric difference $M \Delta N$. The component of the graph $(V, M\Delta N)$ containing v is a path P:

The path P has even length because otherwise it would be an M-augmenting path, contradicting that M is maximal. Note: the end of the path cannot be u, because otherwise $P+e$ would be an odd circuit (but G is bipartite). Now since u is missed by N, we have that $e+P$ is an N augmenting path, a contradiction.

Hence the claim holds. Consider $G' = G - v$, obtained by deleting v. Thus $\nu(G') = \nu(G) - 1$. By induction, we have $\nu(G') = \tau(G')$. So we have a cover C of G' such that $|C| = \nu(G) - 1$. Thus $C \cup \{v\}$ is a cover of G. $\square$

<u>Bring in LP</u>

Matching $M \subseteq E$

Vector Representation $(x_e; e \in E)$, $\quad x^M_e = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{if } e \notin M \end{cases}$

Max matching:

$$\begin{aligned} \max \quad & \Sigma(x_e; e \in E) \\ \text{s.t.} \quad & \Sigma(x_e; e \in \delta(v)) \leq 1 \quad \forall v \in V \\ & x_e \geq 0 \quad \forall e \in E \end{aligned}$$

A 0-1 solution is a ~~max~~ matching.

Min cover:

$$\begin{aligned} \min \quad & \Sigma(y_v; v \in V) \\ & y_u + y_v \geq 1 \quad \forall e = uv \in E \\ & y_v \geq 0 \quad \forall v \in V \end{aligned}$$

A 0-1 solution is a cover.

General LP:

$$\max\ w^T x \qquad \text{s.t.: } Ax \le b,\quad x \ge 0$$

w is vector of all 1's

b "

A has $|E|$ columns and $|V|$ rows

$$A_{ev} = \begin{cases} 1 & \text{if } e \in \delta(v) \\ 0 & \text{if } e \notin \delta(v) \end{cases}$$

$\to \nu^*(G)$

Dual LP:

$$\min\ b^T y \qquad \text{s.t.: } A^T y \ge w,\quad y \ge 0$$

$\to \tau^*(G)$

$\nu^*(G) = \tau^*(G)$

$$\begin{array}{c} \\ \\ u \\ \\ v \\ \\ \end{array}\left[\begin{array}{c} uv \\ 0 \\ \vdots \\ 0\,1\,0 \\ \vdots \\ 0\,1\,0 \\ \vdots \\ 0 \end{array}\right] \xrightarrow{T} \begin{array}{c} \\ uv \end{array}\left[\begin{array}{c} u \qquad\quad v \\ 0\cdots 010\cdots 010\cdots 0 \\ \\ \end{array}\right]$$

Suppose (in general) $\bar{x}$ is an LP solution and $\bar{y}$ is a dual LP solution. Then

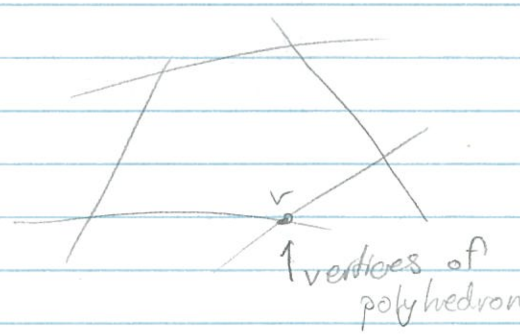
$$w^T\bar{x} \le \bar{y}^T A \bar{x} \le \bar{y}^T b = b^T \bar{y}. \qquad \text{"weak duality"}$$

Duality Theorem: If there exist solutions to LP and to dual LP, then there exist optimal solutions x^*, y^* and $w^T x^* = y^{*T} b$.

LP solution set is called a polyhedron.

v cannot be written as a combination of two other points $v = \frac{1}{2}(x^1 + x^2)$, $x^i \in$ polyhedron

Note: v is a vertex and v is a unique solution to $\bar{A}x = \bar{b}$ where $\bar{A}x \le \bar{b}$ is a subset of constraints.



To prove König's Theorem, show all vertices are 0,1-valued.

$$\bar{A}x = \bar{b}$$

Find unique $\bar{x}$ using Cramer's rule.

$$\bar{x}_j = \det(\hat{A}_j)/\det(\bar{A})$$

Obtain $\hat{A}_j$ by replacing the j^{th} column of $\bar{A}$ by $\bar{b}$.

Prove result by showing $\det(\bar{A}) \in \{\pm 1\}$.

2014 09 15

We have $\nu(G) \le \nu^*(G) = \tau^*(G) \le \tau(G)$. To prove König, we show $\nu(G) = \nu^*(G)$ and $\tau(G) = \tau^*(G)$. That is, the LPs have integer optimal solutions. (primal and dual)

$P = \{x ; Ax \le b, x \ge 0\}$
The LP $\max(w^T x ; Ax \le b, x \ge 0)$ has integer optimal solutions for every w if and only if all vertices of P are integral. Such a P is called an <u>integral polyhedron</u>.

How can we show P is integral?
Linear algebra approach: Hoffman and Kruskal (mid 1950s)

Cramer's Rule: $Ax = b$, $\det(A) \neq 0$, A is $m \times m$
$\bar{x} = A^{-1}b$, each component $\bar{x}_j$ can be written as

$$\frac{\det(A_j)}{\det(A)}$$

A_j is obtained from A by replacing the j^{th} column by b.

Lemma: Suppose A is integral. If $\det(A) = \pm 1$, then for each integral b, the system $Ax = b$ has an integral solution.

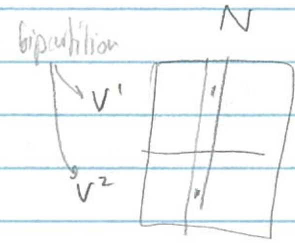
Connection to polyhedra.
$P = \{x ; Mx \le b\} \subseteq \mathbb{R}^n$
Then $v \in P$ is a vertex of P if and only if there exist n inequalities $\bar{M}x \le \bar{b}$ from $Mx \le b$ such that v is the unique solution to $\bar{M}x = \bar{b}$.

A matrix M is called <u>totally unimodular</u> if each square submatrix N we have $\det(N) \in \{0, \pm 1\}$.

Theorem: If M is totally unimodular, then for every integral b the polyhedron $\{x ; Mx \le b, x \ge 0\}$ is integral.

To prove König, show the bipartite matching matrix A is totally unimod.

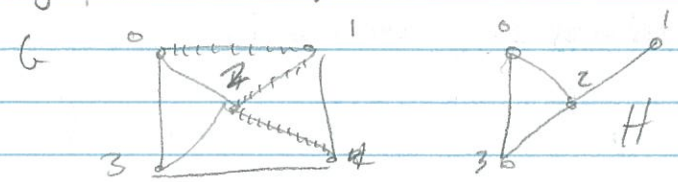
Proof: Show every $k \times k$ submatrix N has $\det(N) \in \{0, \pm 1\}$, by induction on k.
True when $k=1$. Suppose true for $k=1, \dots, t-1$. Let N be a $t \times t$ submatrix.
Case 1: If N has a column of all zeros. Then $\det(N) = 0$.
Case 2: If N has a column with exactly one 1, then compute det by expanding along this column and using induction.
Case 3: Every column has exactly two 1s.
Thus $\det(N) = 0$ (add rows in V^1, ~~add rows~~ subtract rows in V^2, obtain 0 row). $\square$



Minimum Spanning Tree (Book 2.1)

2014 09 17

Graph $G=(V,E)$, V - nodes/vertices, E - edges
$e \in E$ $e \equiv uv \equiv \{u,v\}$, $u,v \in V(G)$ ends of e
Subgraph $H=(W,F)$, $W \subseteq V$, $F \subseteq E$



H is called spanning if $V(H) = V(G)$
A path P is a sequence $v_0 e_1 v_1 \cdots e_k v_k$ where $v_0, \dots, v_k \in V(G)$, $e_1, \dots, e_k \in E(G)$, $e_i = v_{i-1} v_i$.
A graph is connected if every pair of nodes is joined by a path.
P is called simple if $v_0, \dots, v_k$ are distinct, edge-simple if $e_1, \dots, e_k$ are distinct, closed if $v_0 = v_k$, circuit if edge-simple and $v_0, \dots, v_{k-1}$ are distinct, closed.
TSP is find a minimum cost spanning circuit of G.

Def] A graph with no circuits is called a forest. A connected forest is called a tree.

Edge costs $(c_e; e \in E)$, the MST problem asks for a spanning tree T, minimizing $\sum (c_e; e \in E)$.

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Note: If $c_e \geq 0 \ \forall e \in E$, then MST $\leq$ TSP.

A spanning tree has $|E(T)| = |V(T)| - 1$.

Kruskal's Algorithm:

Sort the edges $e_1, \ldots, e_m$ so that $c_{e_1} \leq \cdots \leq c_{e_m}$. $H = (V, F)$, $F = \emptyset$

For $i \in \{1, \ldots, m\}$, if e_i has ends in different components of ~~H~~ H then add e_i to F.

Theorem: Kruskal finds an MST.

Proof: Via LP.

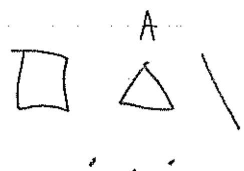
LP model: Variables $x = (x_e; e \in E)$

$$\min \sum (c_e x_e; e \in E)$$

Consider a set $A \subseteq E$. How large can $|T \cap A|$ be?

$K(A)$ = # of components of (V, A)

The max size of $|T \cap A|$ is $|V| - K(A)$.



LP constraint:

$$\sum (x_e; e \in A) \leq |V| - K(A), \ A \subset E \qquad K(A) = 6, |V| = 12$$

$$\sum (x_e; e \in E) = |V| - 1$$

$$x_e \geq 0 \ \forall e \in E$$

Note $A = \{e\} \longrightarrow x_e \leq |V| - (|V| - 1) = 1$

If T is a spanning tree, then x^T defined as

$$x^T_e = \begin{cases} 1 & \text{if } e \in E(T) \\ 0 & \text{otherwise} \end{cases}$$

is a feasible LP solution.

Suppose T is a tree found by Kruskal. Then x^T is an optimal solution to LP.

Thus T is a MST.

Recall LP:

$$\max w^T x \quad \text{st } Ax \leq b, \ x \geq 0 \qquad \qquad \min y^T b \quad y^T A \geq w^T, \ y \geq 0$$

$w^T x \leq y^T A x \leq y^T b$

opt. $\Longleftrightarrow w^T x = y^T b$

$\Longleftrightarrow w^T x = y^T A x$ ie $(w^T - y^T A) x =$

$\wedge \ y^T A x = y^T b$ ie $y^T (Ax - b) =$

ie $\underbrace{(y^T A - w^T)}_{\geq 0} \underbrace{x}_{\geq 0} = 0$

if $x_i > 0$, then $y^T A_i - w_i = 0$.

if $y_i > 0$, then $A^i x - b_i = 0$

Complementary Slackness

LP $\max \sum (-c_e x_e; e \in E)$

$$\sum (x_e; e \in A) \leq |V| - K(A) \quad \forall A \subset E$$

$$\sum (x_e; e \in E) = |V| - 1$$

$$x_e \geq 0 \quad \forall e \in E$$

y_A

y_e

Dual LP $\min \sum (y_A (|V| - K(A)); A \subseteq E)$

$$\sum (y_A; e \in A) \geq -c_e \quad \forall e \in E$$

$$y_A \geq 0 \quad \forall A \subset E$$

2014 09 22

Let T be the tree produced by Kruskal. Define LP solution x^0 by

$$x^0_e = \begin{cases} 1 & \text{if } e \in E(T), \\ 0 & \text{otherwise.} \end{cases}$$

We will define a dual LP solution y^0 such that the complementary slackness (CS) conditions hold:

$$\forall e \in E, \text{ if } x^0_e > 0 \text{ then } \sum(y^0_A ; e \in A) = -c_e,$$
$$\forall A \subset E, \text{ if } y^0_A > 0 \text{ then } \sum(x^0_e ; e \in A) = |V| - \kappa(A).$$

For $i \in \{1, \dots, m\}$ let $R_i = \{e_1, \dots, e_i\}$. For $A \subseteq E$,

$$y^0_A = 0 \text{ if } A \notin \{R_1, \dots, R_m\}$$
$$y^0_{R_i} = c_{e_{i+1}} - c_{e_i} \text{ if } i \in \{1, \dots, m-1\}$$
$$y^0_{R_m} = -c_{e_m}.$$

Since $c_{e_1} \le \dots \le c_{e_m}$, we have $y^0_{R_i} \ge 0 \ \forall i \in \{1, \dots, m-1\}$. Consider an edge e_i:

$$\sum(y^0_A ; e_i \in A) = \sum_{j=i}^{m} y^0_{R_j} = \sum_{j=i}^{m-1} (c_{e_{j+1}} - c_{e_j}) - c_{e_m} = -c_{e_i}.$$

Hence y^0 is dual feasible. Moreover, the first half of the CS conditions hold. We need to show the other half. Consider some $1 \le i \le m-1$. Must show that if $y^0_{R_i} > 0$, then

$$\sum(x^0_e ; e \in R_i) = |V| - \kappa(R_i). \quad (*)$$

Suppose $(*)$ is not true. Consider the edge set $T \cap R_i$. Thus, at the i^{th} step of Kruskal, we have $|T \cap R_i| < |V| - \kappa(R_i)$. This implies some e_j, such that $1 \le j \le i$, joins two different components of $(V, T \cap R_i)$. So Kruskal should have added e_j to T. Contradiction. Therefore CS conditions hold and x^0 is LP optimal. Therefore T is optimal.

Alternative MST algorithm: Prim's Algorithm

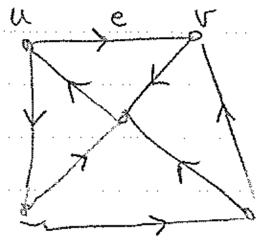
Grow a Tree: Choose $r \in V$, $V(T) = \{r\}$, $E(T) = \emptyset$.

Step: Add to T the cheapest edge $\underbrace{\text{such that } T \text{ remains a tree.}}_{\text{of } \delta(V(T))}$

2014 09 24

Shortest Paths (2.2)

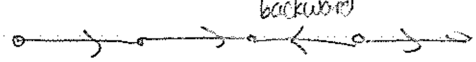
Directed Graphs $G=(V,E)$, edges sometimes called "arcs"



$\forall e = uv \in E$, v is called the head of e and u is the tail of e

$h(e)=v$, $t(e)=u$

Path $P = v_0 e_1 v_1 \cdots e_k v_k$, e_i is called forward if $t(e_i)=v_{i-1}$ and $h(e_i)=v_i$, Otherwise it is backward



P is a <u>directed path</u> (dipath) if all edges in P are forward.

Given $r \in V$ the shortest path problem is to find a min cost dipath from r to every other $v \in V$.

Edge costs: $(c_e; e \in E)$

cost of $P \equiv c(P) \equiv \sum (c_e;\ e \text{ an edge of } P)$

Note 1: Subpaths of shortest paths are also shortest paths.

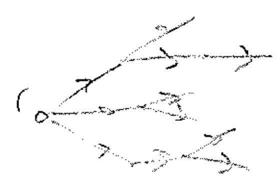
If P_1 is not a shortest path then we should replace it.

For each $v \in V \setminus \{r\}$, we only need the last edge on <u>one</u> shortest path from r to v.

If we look at the union of the r-v dipaths for all $v \in V \setminus \{r\}$, then each node s is the head of exactly one of the arcs.

$\Longrightarrow$ total of $n-1$ arcs

$\Longrightarrow$ spanning tree



Spanning tree <u>rooted</u> at r.

For each $v \in V$, there is a dipath in tree from r to v.

Note 2: There exists an answer to shortest path problem that is a rooted tree.

Trouble: Negative cost dicircuits.

Note 3: In general, shortest simple path is as hard as the TSP (HW).

Note 4: There exist algorithms that either solve the shortest path problem or find a negative dicircuit.

Dijkstra's Algorithm:

Applies when $c_e \geq 0 \ \forall e \in E$

Initial: $p(r) = \emptyset$, $p(v) = -1 \ \forall v \in V \setminus \{r\}$

$y_r = 0$, $y_v = \infty \ \forall v \in V \setminus \{r\}$

unscanned set $S = V$

while $(S \neq \emptyset)$

  choose $v \in S$ so that $y_v = \min\{y_t ; t \in S\}$

  scan v

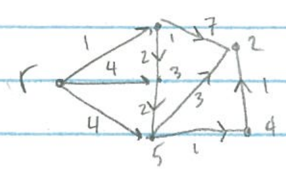
  $S \leftarrow S \setminus \{v\}$ (remove v from S)

<u>Def'l</u> Edge vw is called <u>incorrect</u> if $y_v + c_{vw} < y_w$

To <u>correct</u> vw we $y_w \leftarrow y_v + c_{vw}$, $p(w) \leftarrow v$

Scan(v)

For each vw, if vw is incorrect then correct it.



∞ ∞

0 ∞

∞ ∞

↓

1 ∞ p(1) = r

0 4 p(3) = r

4 ∞ p(5) = r

1 8 p(2) = 1

0 3 p(3) = r?

4 ∞

1 8

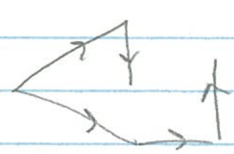
0 3

4 ∞

1 7 p(4) = 5

0 3 p(2) = 5

4 5



1 6 p(2) = 4

0 3

4 5

Dijkstra computes shortest paths

Lemma. After we scan a node $v \in V$, the value y_v never decreases.

Lemma: At termination, there are no incorrect arcs. That is, for each $vw \in E$ we have

$$y_v + c_{vw} \geq y_w$$

Proof: Consider $vw \in E$. After we scanned node v, the edge vw was correct. After we scan v, the value y_w can only decrease. Now since $c_{vw} \geq 0$, by lemma 1, the arc vw remains correct.

Theorem: Dijkstra is correct.
Proof: At termination, the p values construct a path of cost y_v from r to v. Consider any r-v dipath P, $r e_1 v_1 \cdots e_k v_k$, $v_k = v$.

$$c(P) = c_{e_1} + \cdots + c_{e_k} \geq (y_{v_k} - y_{v_{k-1}}) + \cdots + (y_{v_1} - y_r) = y_{v_k} - y_r = y_{v_k} = y_v.$$

2014 09 29

Handling negative costs

no shortest path!

By A1Q7, min simple path is hard as TSP.
Algorithms to find shortest dipath or detect negative circuit.

Def| Values $(y_v; v \in V)$ are called (feasible) potential if

$$y_v + c_{vw} \geq y_w \quad \forall vw \in E.$$

Proposition: If P is a r-s dipath and y a feasible potential, then

$$c(P) \equiv \sum (c_e ; e \text{ an edge of } P) \geq y_s - y_r.$$

Proof: Well,

$$c(P) = \sum_{i=1}^{k} c_{e_i} \geq \sum_{i=1}^{k} (y_{v_i} - y_{v_{i-1}}) = y_{v_k} - y_{v_0} = y_s - y_r. \qquad \square$$

So potential gives a stopping rule: if $c(P) = y_s - y_r$ then P is a shortest r-s dipath.

Ford's Algorithm

Initialize y, P
while y is not a potential
 choose an incorrect edge vw
 and correct it

Theorem: If no negative cost dicircuit, then Ford's Algorithm terminates after a finite number of steps. At termination, for each $v \in V$, P defines a least-cost dipath from r to v of cost y_v.

Specializations of Ford's Algorithm are efficient: Bellman-Ford.

LP connection

Look for shortest r-s dipath, $r, s \in V$

$$\text{(LP)} \qquad \begin{aligned} &\max\ y_s - y_r \\ &\text{s.t.: } y_w - y_v \le c_{vw} \quad \forall vw \in E \end{aligned}$$

$$\text{(Dual LP)} \qquad \begin{aligned} &\min\ \sum(c_e x_e ; e \in E) \\ &\text{s.t.: } \sum(x_{wv} ; wv \in E) - \sum(x_{vw} ; vw \in E) = b_v \quad \forall v \in V \quad \left(b_v = \begin{cases} -1 & v = r \\ 0 & \text{otherwise} \\ 1 & v = s \end{cases} \right) \\ &\qquad x_e \ge 0 \quad \forall e \in E \end{aligned}$$

Note: Any r-s dipath P gives a dual solution:

$$x_e = \begin{cases} 1 & \text{if } e \in P, \\ 0 & \text{if } e \notin P. \end{cases}$$

Ford's Algorithm terminates with a primal and dual LP solutions. So if $c_e \in \mathbb{Z}$ $\forall e \in E$, then LP solutions are integer.

LP constraints: $Ax \le c$ $\quad A:$ $\begin{array}{c|cc|c} & v & w & \\ \hline & -1 & 1 & vw \\ & & & \end{array}$ is totally unimodular. The same proof as before works here.

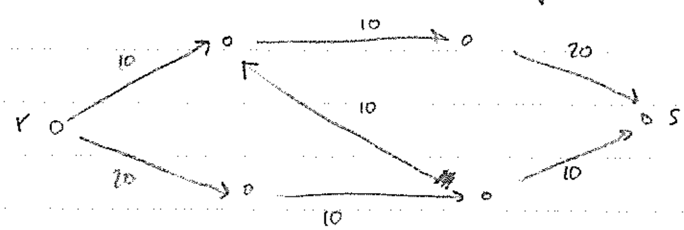
Max Flow Problem (Chap 3)

Directed Graph $G=(V,E)$.

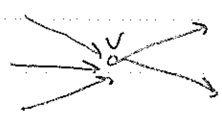
edge capacities $(u_e;\ e\in E)$

$u_e \geq 0\ \forall e$

Can send at most u_e "flow" along edge e.



Flow conservation



"flow in" = "flow out"

$$\sum(x_{wv};\ wv\in E) - \sum(x_{vw};\ vw\in E) = 0$$

Maximize flow arriving at s, value of flow

$$\sum(x_{ws};\ ws\in E) - \sum(x_{sw};\ sw\in E)$$

Max Flow Problem, $r,s\in V$

$$\begin{aligned} \max\ & \sum(x_{ws};\ ws\in E) - \sum(x_{sw};\ sw\in E) \\ \text{s.t.:}\ & \sum(x_{wv};\ wv\in E) - \sum(x_{vw},\ vw\in E) = 0 \quad \forall v\in V\setminus\{r,s\} \\ & 0\leq x_e \leq u_e \quad \forall e\in E \end{aligned}$$

Let $\delta(R) \equiv \{vw;\ v\in R,\ w\notin R\}$ for $R\subseteq V$. Such a set is called a cut. It is an r-s cut if $r\in R$, $s\notin R$. The capacity of $\delta(R)$ is

$$u(\delta(R)) \equiv \sum(u_e;\ e\in\delta(R)).$$

Lemma: If x is an r-s flow and R is an r-s cut, then the value of x is at most the capacity of $\delta(R)$.

Max Flow Min Cut Theorem:

maximum value of an r-s flow = minimum capacity of an r-s cut.

An x-augmenting path is an r-s path P such that if e is a forward edge then $x_e < u_e$, and if e is a reverse edge then $x_e > 0$.

Augmenting Path Algorithm

Initial: Any feasible flow, for example $x_e = 0$ $\forall e \in E$.

Repeat: Find an x-augmenting path.

$\varepsilon_1 = \min\{u_e - x_e ;\ e \text{ forward}\}$

$\varepsilon_2 = \min\{x_e ;\ e \text{ backward}\}$

$\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ is called the x-width of the path

For each forward edge e

$$x_e \leftarrow x_e + \varepsilon$$

For each backward edge e

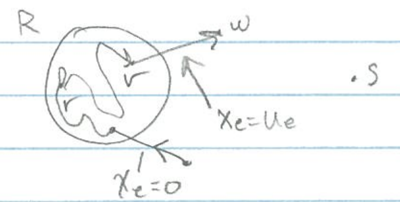
$$x_e \leftarrow x_e - \varepsilon$$

Stop when no x-augmenting path

Search for an augmenting path.

Start at r, move to nodes v such that there exist an r-v path P of x-width >0. If we reach s, we have an augmenting path.

If we don't reach s, let R be the set of vertices $\{v ;\ \exists$ an x-width >0 path from r to $v\}$.



We have flow $u(\delta(R))$ leaving R. The value of flow x is equal to $u(\delta(R))$.

Consequences

1) If $u_e \in \mathbb{Z}$ $\forall e$, then $\exists$ a max flow that is integer valued.

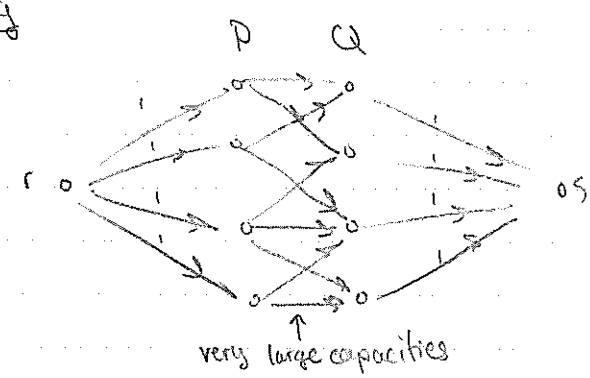
2) If $u_e \in \mathbb{Q}$ $\forall e$, then the algorithm is finite.

3) Edmonds-Karp: if always take shortest augmenting paths then $O(nm^2)$

4) Max Flow Min Cut Theorem

2014 10 01(2)

König



An Integer flow gives a matching,

$$M = \{pq;\ p\in P,\ q\in Q,\ x_{pq}=1\}.$$

Max flow gives max matching. Let $\delta(R)$ be a min cut. Note that if $pq \in E$ and $p \in R$, then $q \in R$. Let

$$C = (Q\cap R)\cup(P\setminus R).$$

Then C is a cut.

LP

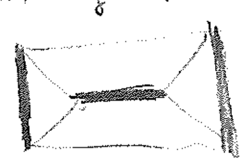
$$\min \sum(u_e y_e;\ e\in E)$$
$$\text{s.t.: } \sum(y_e;\ e\in E(P)) \ge 1 \quad \forall \text{ simple } r\text{-}s \text{ dipaths}$$
$$y_e \ge 0$$

$$\max \sum(w_P;\ P \text{ simple } r\text{-}s \text{ dipaths})$$
$$\text{s.t.: } \sum(w_P;\ e\in P) \le u_e \quad \forall e\in E$$
$$w_P \ge 0 \quad \forall P \text{ simple } r\text{-}s \text{ dipath}$$

2014 10 06

Matchings

$G=(V,E)$, $M \subseteq E$ disjoint, M is a perfect matching if $|M| = |V|/2$



perfect matching

Problems
* maximum matching
  - find M with $|M|$ as large as possible
* max weight matching
  - edge weights $w = (w_e;\ e\in E)$
  - max $\sum(\cancel{M_1}\, w_e;\ e\in E)$
* min-weight perfect matching
  - find perfect matching M that minimizes $\sum(w_e;\ e\in M)$

General Framework - Polyhedral Combinatorics

$S \subseteq \mathbb{R}^n$, $w \in \mathbb{R}^n$, minimize $(w^T x ; x \in S)$

S finite, but possibly very large

ex S - vectors of spanning trees

S - cuts in graph

S - perfect matchings

<u>Def</u> Convex hull of $S \equiv$ conv.hull$(S) \equiv$ smallest convex set containing S

$\equiv \{\lambda_1 x_1 + \cdots + \lambda_k x_k ; k \geq 1, \lambda_1, \ldots, \lambda_k \geq 0, \sum \lambda_i = 1, s_1, \ldots, s_k \in S\}$.

Suppose we have a convex combination

$$\bar{s} = \lambda_1 s_1 + \cdots + \lambda_k s_k.$$

Then

$$w^T \bar{s} = \lambda_1 w^T s_1 + \cdots + \lambda_k w^T s_k,$$

so for some $i \in \{1, \ldots, k\}$, we have $w^T s_i \leq w^T \bar{s}$. Therefore

$$\min(w^T x ; x \in S) = \min(w^T x ; x \in \text{conv.hull}(S)).$$

P is called a <u>polytope</u> if $P = \text{conv.hull}(S)$ for some finite set S.

Minkowski Theorem: If P is a polytope, then there exists a system of linear inequalities $Ax \leq b$ such that $P = \{x ; Ax \leq b\}$.

Thus

$$\begin{aligned} \min(w^T x ; x \in S) &= \min(w^T x ; x \in \text{conv.hull}(S)) \\ &= \min(w^T x ; Ax \leq b) \\ &= \max(y^T b ; y^T A = w^T, y \geq 0). \end{aligned}$$

So LP duality can be used in solving combinatorial problems.

If we can certify inequalities in $Ax \leq b$, then the original problem is in co-NP.

<u>Perfect Matching LP</u>

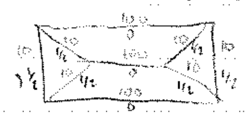
For any $F \subseteq E$, let $x(F) = \sum (x_e ; e \in F)$.

$$\min \sum (w_e x_e \,;\, e \in E) \qquad\qquad \min\ w^T x$$
$$x(\delta(v)) = 1 \quad \forall v \in V \qquad\qquad Ax = \mathbb{1}$$
$$x_e \geq 0 \quad \forall e \in E \qquad\qquad x \geq \mathbb{0}$$

G bipartite $\Rightarrow$ A totally unimodular $\Rightarrow$ all vertices of $\{x ; Ax = \mathbb{1},\ x \geq \mathbb{0}\}$ are integer $\Rightarrow$ conv.hull({perfect matchings}) $= \{x ; Ax = \mathbb{1}, x \geq \mathbb{0}\}$

Non-bipartite $\Rightarrow$ G has an odd circuit

x =



min cost perfect matching has to have 120

But x has value 30 and is feasible for the LP

<u>Edmonds</u>

For $U \subseteq V$, $\delta(U) = \{e \in E ; |e \cap U| = 1\}$.

If $|U|$ is odd, then every perfect matching must contain an edge in $\delta(U)$.

Blossom inequality: Every perfect matching satisfies
$$x(\delta(U)) \geq 1.$$

Consider the system

$$(*) \begin{cases} x(\delta(v)) = 1 & \forall v \in V \\ x(\delta(U)) \geq 1 & \forall U \subseteq V,\ |U| \text{ odd} \\ x_e \geq 0 & \forall e \in E \end{cases}$$

<u>Def</u> $PM(G) \equiv$ perfect matching polytope $\equiv$ conv. hull$(\chi^M ; M$ a perfect matching$)$

Theorem (Edmonds): $PM(G)$ is the solution set of $(*)$.

Proof: Let Q be the solution set of $(*)$. Clearly $PM(G) \subseteq Q$. Suppose $Q \not\subseteq PM(G)$, and let x be a vertex of Q with $x \notin PM(G)$. Choose this counter-example with $|V| + |E|$ as small as possible.

Claim: $0 < x_e < 1$ $\forall e \in E$.

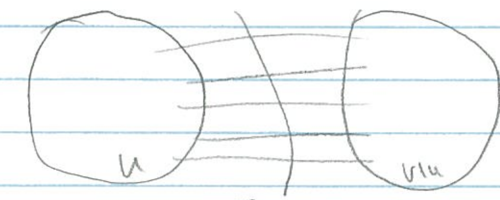
Verification: Otherwise, if $x_e = 0$, then delete e. If $x_e = 1$, then delete e and the ends of e. $\square$

Claim 2: We may assume $|E| > |V|$.

Verification: By Claim, each node of G has degree at least 2. So $|E| \geq |V|$. But if $|V| = |E|$ then ~~G is a circuit, and the theorem is true for circuits.~~

G is a disjoint ~~union~~ collection of circuits, but we know the theorem is true in this case. $\square$

Now since x is a vertex of Q, there are $|E|$ constraints of $(*)$ satisfied with equality by x. Then there exists and odd U with $3 \le |U| \le |V|-3$ and $x(\delta(U)) = 1$.



$\Sigma = 1$

Let G' and G'' be obtained by shrinking U and $V \setminus U$ to be single nodes, resp.

G' G''

Let x' and x'' be obtained by shrinking x.

x' and x'' satisfy $(*)$ for G' and G'' resp. So by induction $x' \in PM(G')$ and $x'' \in PM(G'')$.

Since x is rational, we have x' and x'' are rational convex combinations of perfect matchings in G' and G'':

$$x' = \frac{1}{k}\sum_{i=1}^{k} \chi^{M_i'} \quad , \quad x'' = \frac{1}{k}\sum_{i=1}^{k} \chi^{M_i''} .$$

For each $e \in \delta(U)$, the number of i with $e \in M_i'$ is

$$k x_e' = k x_e = k x_e'' = \# \text{ of } i \text{ with } e \in M_i''.$$

Hence we can assume for each i that M_i' and M_i'' have an edge in $\delta(U)$ in common. So for each i we have

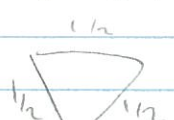
$$M_i = M_i' \cup M_i''$$

is a perfect matching of G. Thus

$$x = \frac{1}{k}\sum_{i=1}^{k} \chi^{M_i}$$

and $x \in PM(G)$, a contradiction. $\blacksquare$

<u>Def</u>] Matching Polytope $M(G) \equiv$ conv. hull $\{\chi^M ; M \text{ a matching}\}$



satisfies $x(\delta(v)) \le 1 \quad \forall v \in V$

$x_e \ge 0 \quad \forall e \in E$

2014 10 08C

so we need extra inequalities

Define

$$\gamma(U) = \{e \in E ; e \text{ has both ends in } U\}.$$

Every matching satisfies

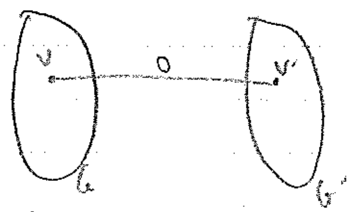
$$x(\gamma(U)) \leq \frac{|U|-1}{2} = \lfloor \tfrac{1}{2}|U| \rfloor \quad \forall U \text{ odd}$$

Theorem: $M(G)$ is defined by

$$(**) \begin{cases} x(\delta(v)) \leq 1 & \forall v \in V \\ x(\gamma(U)) \leq \lfloor \frac{1}{2}|U| \rfloor & \forall \text{ odd } U \\ x_e \geq 0 & \forall e \in E \end{cases}$$

Proof: (Sketch by Schrijver)

Two copies of G:



Let $\tilde{G}$ be the union of G and G' ~~with~~ and edges $\{v, v'\}$ $\forall v \in G$.

Let $\tilde{x}_e = \tilde{x}_{e'} = x_e \ \forall e \in E$, $\quad \tilde{x}_{vv'} = 1 - x(\delta(v)) \ \forall v \in V$

Claim: $\tilde{x} \in PM(\tilde{G})$.

Show $\tilde{x}$ satisfies all blossom constraints (leaving out details)

Generalizations of Matchings

$G = (V, E)$ and $T \subseteq V$, $|T|$ even

Def) A T-join of G is a set J of edges such that $|J \cap \delta(v)|$ is odd if and only if $v \in T$.

ex. if $|V|$ is even and $T = V$, then a perfect matching is a T-join.

ex. if $T = \{r, s\}$, then a T-join contains an rs-path (undirected)

you wants an Euler tour.
like Gollum, you wants it.

Chinese Postman Problem

2014 10

Input: $G=(V,E)$ connected
$(c_e;\ e\in E)$, $c_e\geq 0$ $\forall e\in E$

Find: min-cost closed path that uses each edge of G at least once.

Def A closed simple path P with $E(P)=E$ is called an <u>Euler tour</u>.

Def degree$(v)=|\delta(v)|$

Theorem: A connected graph G has an Euler tour if and only if every node of G has even degree.

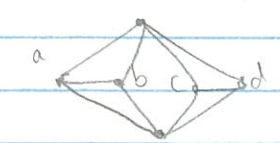
Note: • If G has an Euler tour T, then T is an optimal solution to the CPP.

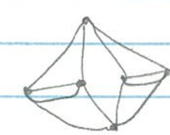
• If G has no Euler tour, then we must use some edges more than once.

• In an optimal solution to CPP, we never have to use an edge more than twice.

Proof: Let J be the edges we use more than once. Having an Euler tour means that the graph obtained by adding copies of the edges J has all even degrees. If we use e three times, then deleting two copies of e keeps all degrees even. $\blacksquare$ (He should note the graph would still be connected...)

• To solve CPP, we want to find a min-cost set $J\subseteq E$ such that duplicating every edge in J gives a graph with every node of even degree

ex 



$J=\{ab, cd\}$. By duplicating J, we ~~have~~ obtain all even degrees.

Let $T=\{v\in V;\ \text{degree}(v) \text{ is odd}\}$. To solve CPP, we must find a min-cost T-join.

(recall from front) **Def** Given $T\subseteq V$ with $|T|$ even, a <u>T-join</u> of G is a set $J\subseteq E$ st
$$|J\cap\delta(v)| \text{ is odd} \iff v\in T.$$

2014 10 15

T-join Problem

Find a min-cost T-join.

Connection to Matchings: If $T = V$ and $c_e = 1$ $\forall e \in E$, then a perfect matching is a min-cost T-join.

Connection to undirected Shortest Paths: If $T = \{r, s\}$ and $c_e \geq 0$ $\forall e$, then a min-cost T-join contains a shortest r-s path.

Note: If we permit negative cost edges, then T-join problem is as hard as the TSP.

LP Model

$S \subseteq V$ is called T-odd if $|S \cap T|$ is odd.

If $J \subseteq E$ is a T-join, then $J \cap \delta(S) \neq \emptyset$

Def) $\delta(S)$ is called a T-cut.

Every T-join satisfies

$$\sum (x_e ; e \in \delta(S)) \geq 1 \text{ , ie } x(\delta(S)) \geq 1$$

T-Join LP

$$\begin{aligned} \min \quad & \sum (c_e x_e ; e \in E) \\ \text{s.t.:} \quad & x(D) \geq 1 \quad \forall \text{ T-cut } D \\ & x_e \geq 0 \quad \forall e \in E \end{aligned}$$

Note: If $c_e < 0$ for some $e \in E$ and G has a T-join, then the LP is unbounded.

Theorem (Edmonds + Johnson): If $c_e \geq 0$ $\forall e \in E$, then the min cost of a T-join is equal to the optimal value of the LP.

$\mathbb{R}^n_+$ ~?

Geometry: If $P = \text{conv.hull}(x^J ; J \text{ a T-join})$, the polyhedron that is the feasible region of (LP) defines $P + \mathbb{R}^n_+$, called the <u>dominant</u> of P.

Polyhedral Combinatorics

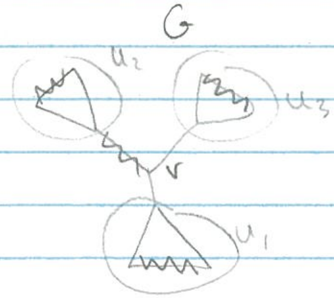
$$M(G) \equiv \text{conv.hull}\{\chi^M ; M \text{ a matching}\}$$

Edmonds LP

$$\max \sum (w_e x_e ; e \in E)$$
$$x(\delta(v)) \leq 1 \quad \forall v \in V$$
$$x_e \geq 0 \quad \forall e \in E$$
$$x(\gamma(U)) \leq \lfloor \tfrac{1}{2}|U| \rfloor \quad \forall U \subseteq V \text{ st } |U| \text{ odd}$$

Dual LP

$$\min \sum (y_v ; v \in V) + \sum (\lfloor \tfrac{1}{2}|U| \rfloor Y_U ; U \subseteq V, |U| \text{ odd})$$
s.t.:
$$y_v \geq 0 \quad \forall v \in V$$
$$Y_U \geq 0 \quad \forall \text{ odd } U \subseteq V$$
$$y_z + y_v + \sum (Y_U ; e \in \gamma(U), U \text{ odd}) \geq w_e \quad \forall e = (z,v) \in E$$



Suppose $w_e = 1 \ \forall e$, $|M| = 4$

$y_v = 1$, $Y_{U_i} = 1 \ \forall i \in \{1,2,3\}$

has dual objective value 4

2014 10 20

Def $C \subseteq V$ and $\mathcal{W} = \{U_1, \ldots, U_k\}$ with $|U_i|$ odd $\forall i$ is called an <u>odd-set cover</u> if $\forall e \in E$, either an end of e is in C or $e \in \gamma(U_i)$ for some i.

If M is any matching then
$$|M| \leq |C| + \sum_{i=1}^{k} \frac{|U_i| - 1}{2}.$$

Theorem [Cunningham + Marsh]: If w_e is integer $\forall e \in E$, then the matching dual LP has an integer optimal solution.

2014 10 20

Proof: Suppose not. Choose a counter-example with $|E|+\sum(w_e\, ; e\in E)$ as small as possible.

Claim 1: We may assume $w_e \geq 1\ \forall e\in E$.

Verification: Easy – otherwise delete e. □

Claim 2: For every $v\in V$, there is a max-weight matching missing v.

Verification: Suppose not.

$$w'_e = w_e - 1 \quad \forall e\in\delta(v)$$
$$w'_e = w_e \quad \forall e\in E\setminus\delta(v)$$

With weights w', the max-weight matching goes down by one. Now take an optimal integral solution for w' and increase y_v by 1. This gives an integral optimal dual solution for w, a contradiction. □

Claim 3: If y, Y is an optimal dual solution, then $y_v = 0\ \forall v\in V$. □

Verification: Complementary slackness and claim 2. □

Choose an optimal dual solution such that

$$\sum_{\text{odd } U\subseteq V}\left(\frac{|U|-1}{2}\right)^2 Y_U = (*)$$

is as large as possible. Let $\mathcal{F}=\{U \text{ odd}; Y_U>0\}$.

Claim 4: $\mathcal{F}$ is laminar (ie $U,W\in\mathcal{F} \wedge U\cap W\neq\emptyset \Rightarrow U\subseteq W \vee W\subseteq U$).

Verification: Suppose not, say U and W, with $v\in U\cap W$.

a) There exists a max-weight matching M missing v with $\frac{|U|-1}{2}$ edges in $\gamma(U)$.

b) M uses $\frac{|W|-1}{2}$ edges in $\gamma(W)$.

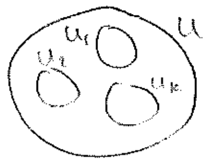
Hence every node in $(U\cap W)\setminus\{v\}$ is covered by an edge in M that is contained in $(U\cap W)\setminus\{v\}$.

Therefore $|U\cap W|$ is odd. Let $\varepsilon=\min\{Y_U, Y_W\}$. We will decrease Y_U and Y_W by ε and increase $Y_{U\cap W}$ and $Y_{U\cup W}$ by ε.

So we have a new optimal dual solution. Can check that $(*)$ increases, a contradiction.

Claim 5: Y is integral.

Verification: Suppose not. Let U be a maximal set in $\mathcal{F}$ with Y_U not integral.



Let $U_1, \ldots, U_k$ be the maximal sets in $\mathcal{F}$ properly contained in U.
Let $\varepsilon = Y_U - \lfloor Y_U \rfloor > 0$. Decrease Y_U by ε, increase Y_{U_i} by ε $\forall i$.
Now we have a new dual solution, with smaller objective value. Absurd. $\square$

General Min-Max LP 2014 10 22

$$\max(w^T x ; Ax \le b) = \min(y^T b ; y^T A = w^T, y \ge 0)$$

$\{x ; Ax \le b\}$ is an integer polyhedron if and only if $\max(w^T x ; Ax \le b)$ has an integer-valued optimal solution x^* for every w such that the optimum exists.

Def] $Ax \le b$ is called totally dual integral (tdi) if for every integer-valued w such that the minimum exists, there is an integer optimal dual solution y^*.

TDI Theorem [Hoffman, Edmonds-Giles]:
If $Ax \le b$ is TDI and b is integral, then $P = \{x ; Ax \le b\}$ is an ~~integral~~ integer ~~polynomial~~. polyhedron.

Cunningham-Marsh Theorem: Matching system (*) is TDI.

$$(*) \begin{cases} x(\delta(v)) \equiv \sum(x_e ; e \in \delta(v)) \le 1 & \forall v \in V \\ x(\gamma(U)) \equiv \sum(x_e ; e \in \gamma(U)) \le \frac{1}{2}(|U|-1) & \forall \text{ odd } U \subseteq V, |U| \ge 3 \\ x_e \ge 0 & \forall e \in E \end{cases}$$

These two theorems imply that (*) defines $M(G) \equiv \text{conv.hull}(\chi^M ; M \text{ a matching})$

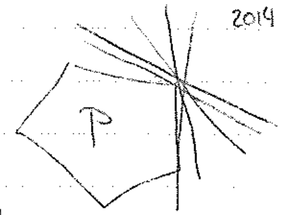
$$\chi^M = \begin{cases} 1 & e \in M \\ 0 & e \notin M \end{cases}$$

Giles-Pulleyblank: Let P be a rational polyhedron. Then there exists a TDI system $Ax \le b$ with A integral such that $P = \{x ; Ax \le b\}$.
And, if P is an integer polyhedron, then b can be chosen to be integral.
(Discussed in section 6.6 of text.)

General plan for proving a polyhedron is integral.
1) Find an appropriate system $Ax \le b$ with A, b integral.
2) Prove $Ax \le b$ is TDI.
3) Apply TDI theorem to conclude that $\{x ; Ax \le b\}$ is an ~~integer polyhedron~~ int. poly.

2014 10 22

Warning: The Giles-Pulleyblank theorem may add redundant (extra) inequalities to obtain a TDI system.



Note: For most graphs, the matching system $(*)$ contains redundant inequalities.

Theorem: If P is a rational polyhedron of full dimension, then there exists a unique (up to positive scalar multiples) minimal system $Ax \le b$ such that $P = \{x ; Ax \le b\}$. (Section 6.3)

$\Rightarrow$ each inequality in the minimal system corresponds to a <u>facet</u> of P (maximal proper face of P)

* exercise 5 ie A3Q5

<u>Pulleyblank + Edmonds</u>

Found the minimal defining system for $M(G)$.

$$x_e \ge 0 \quad \forall e \in E$$
$$x(\delta(v)) \le 1 \quad \forall v \in V'$$
$$x(\gamma(S)) \le \tfrac{1}{2}(|S|-1) \quad \forall S \in \mathcal{B}$$

$S \subseteq V$

$G[S] \equiv (S, \gamma(S))$

$S \in \mathcal{B}$ if (1) $G[S]$ is connected and $\nexists v \in S$ st $G[S \setminus \{v\}]$ is not connected (ie 2-connected)

or? (2) $G[S]$ is hypo-matchable, that is, $\forall v \in S$, the graph $G[S \setminus \{v\}]$ has a perfect matching

Theorem: The minimal system for $M(G)$ is TDI.

Perfect Matching Polyhedron $PM(G)_*$ - not of full dimension.

$PM(G)$ (3)
$$x(\delta(v)) = 1 \quad \forall v \in V$$
$$x(\delta(U)) \ge 1 \quad \forall U \subseteq V, |U| \ge 3 \text{ odd}$$
$$x_e \ge 0 \quad \forall e \in E$$

$M(G)$ (1)
$$x(\delta(v)) \le 1 \quad \forall v \in V$$
$$x(\gamma(S)) \le \tfrac{1}{2}(|S|-1) \quad \forall S \subseteq V, |S| \ge 3 \text{ odd}$$
$$x_e \ge 0$$

$(*)$ ———— $PM(G)$
$$x(\delta(v)) = 1 \quad \forall v \in V$$
$$x(\gamma(S)) \le \tfrac{1}{2}(|S|-1) \quad \forall S \subseteq V, |S| \ge 3 \text{ odd}$$
$$x_e \ge 0 \quad \forall e \in E$$

(2)

$$\sum_{v \in U} x(\delta(v)) = x(\delta(U)) + 2x(\gamma(U))$$

Adding $x(\delta(v)) = 1 \ \forall v \in U$, we get

$$\sum_{v \in U} x(\delta(v)) = |U|.$$

Therefore

$$x(\delta(U)) + 2x(\gamma(U)) = |U|$$

and so

$$2x(\gamma(U)) = |U| - x(\delta(U))$$

$$x(\gamma(U)) = \frac{|U| - x(\delta(U))}{2}$$

Thus $(\approx)$ shows

$$\frac{|U| - x(\delta(U))}{2} \leq \frac{|U| - 1}{2},$$

and so

$$x(\delta(U)) \geq 1.$$

Moreover, $x(\delta(U)) \geq 1 \Rightarrow (\approx)$ by reversing the last few steps.

(1) TDI $\Rightarrow$ (2) TDI (general theory)

(2) TDI $\Rightarrow$ (3) $\underbrace{\tfrac{1}{2}\text{-TDI}}_{\text{always exists a } \frac{1}{2}\text{-int opt sol}}$

2014 10 27

Matching Polytope

- conv.hull $\{\chi^M ; M \text{ a matching}\}$

$$x(\delta(v)) \le 1 \quad \forall v \in V$$
$$x(\gamma(S)) \le \tfrac{1}{2}(|S|-1) \quad \forall \text{ odd } S \subseteq V,\ |S| \ge 3$$
$$x_e \ge 0 \quad \forall e \in E$$

- TDI
- minimal defining system is TDI!

Perfect Matching Polytope

- conv.hull $\{\chi^M ; M \text{ a perfect matching}\}$

$$x(\delta(v)) = 1 \quad \forall v \in V$$
$$x(\delta(S)) \ge 1 \quad \forall \text{ odd } S \subseteq V,\ |S| \ge 3$$

- $\frac{1}{2}$TDI, means $\exists y^*$, an optimal dual solution, st $2y^*$ is integral
- minimal TDI system (?)
  open, but needs to be formulated carefully

T-joins

$T \subseteq V$, $|T|$ even

$J \subseteq E$ T-join if

$$|J \cap \delta(v)| \text{ is odd} \iff v \in T$$

$S \subseteq V$ is called T-odd if $|S \cap T|$ is odd, $\delta(S)$ is called a T-cut

$J \cap \delta(S) \ne \emptyset$ for any T-join J any any T-cut $\delta(S)$.

T-join system

$$x(D) \ge 1 \quad \forall \text{ T-cut } D$$
$$x_e \ge 0 \quad \forall e \in E$$

- dominant of convex hull of T-joins
- not always TDI.

ex $K_4 = G$

$T = V$, min cardinality T-join $= 2$

$c_e = 1 \ \forall e \in E$

LP min $\sum(c_e x_e ; e \in E)$
$x(D) \ge 1 \ \forall$ T-cut D
$x_e \ge 0 \ \forall e \in E$

Dual LP max $\sum(y_D ; D \text{ a T-cut})$
$\sum(y_D ; e \in D, \text{T-cut}) \le c_e \quad \forall e \in E$
$y_D \ge 0 \quad \forall$ T-cut D

In the case $c_e = 1$, $\forall e \in E$, a 0-1 solution to dual LP corresponds to disjoint T-cuts.

$\nexists$ two disjoint T-cuts

Dual opt. sol. $y_D = \frac{1}{2}$ $\forall D$ T-cut

Open research area: For what pairs (G,T) is the T-join system TDI?
A2: TDI if G is bipartite.
When is min T-join = max number of edge disjoint T-cuts?
  - "packing" of T-cuts

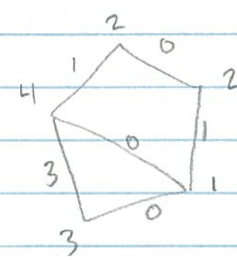
Packing T-joins

Given (G,T), when is the min cardinality T-cut equal to the max # of disjoint T-joins?

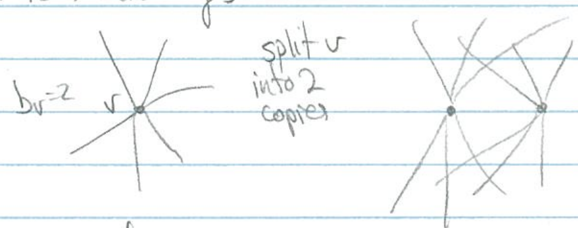
$$x(J) \geq 1 \quad \forall \text{ T-join } J$$
$$x_e \geq 0 \quad \forall e \in E$$

General problem area: "blocking" hypergraphs.

Generalization of Matching



b-matching $\quad b = (b_v ; v \in V), \quad b_v \geq 0, \; b_v$ int
Integer vector $x \geq 0$ st $x(\delta(v)) \leq b_v \;\; \forall v \in V$
Reduction to matchings



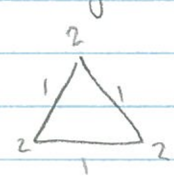
$b(S) = \sum (b_v ; v \in S)$

b-matching polytop

$$x(\delta(v)) \leq b_v \quad \forall v \in V$$
$$x(\gamma(S)) \leq \lfloor \tfrac{1}{2} b(S) \rfloor \quad \forall S \subseteq V, \; b(S) \text{ odd}$$
$$x \geq 0$$

- not always TDI

ex



TDI System (Pulleybank)

$$x(\delta(v)) \leq b_v \quad \forall v \in V$$
$$x(\gamma(S)) \leq \lfloor \tfrac{1}{2} b(S) \rfloor \quad \forall S \subseteq V$$
$$x_e \geq 0 \quad \forall e \in E$$

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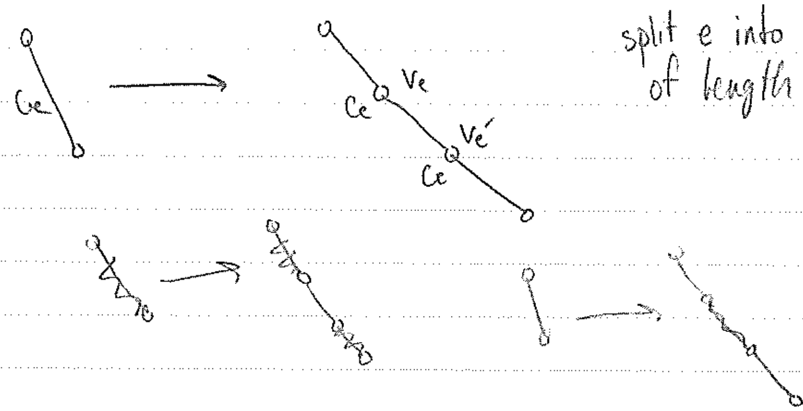
Capacitated b-matching

$(c_e; e \in E)$, b-matching satisfying $x_e \le c_e \ \forall e \in E$

Special case: $c_e = 1 \ \forall e \in E$, then cap. b-matching is a subset of edges

Reduction



split e into path of length 3

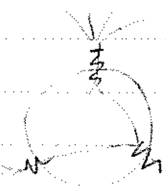
2014 10 29

2-factors ≡ perfect 2-matching with all ~~edges~~ capacities 1
≡ disjoint collection of circuits such that each node is in exactly one of the circuits

Relaxation of TSP: TSP is a 2-factor having exactly one circuit

<u>2-factor poly~~nomial~~tope</u>

$$x(\delta(v)) = 2 \quad \forall v \in V$$
$$x_e \ge 0 \quad \forall e \in E$$
$$x(\delta(S) \setminus F) - x(F) \ge 1 - |F|$$



$S \subseteq V$, $F \subseteq \delta(S)$
$|F|$ odd
"matching"

If every $e \in F$ is in our 2-factor M, then there must be some other edge in $\delta(S)$ that is also in M

<u>2-matching polytope (perfect)</u>

- edge values of 0, 1, or 2

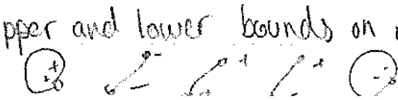
$$x(\delta(v)) = 2 \quad \forall v \in V$$
$$x_e \ge 0 \quad \forall e \in E$$

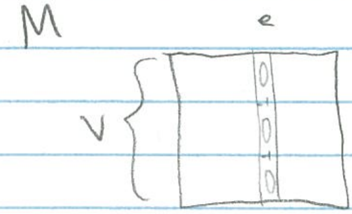
Δ-free 2-factor: 2-factor with no circuits of length 3.

Open Problem: min cost Δ-free 2-factor
- not known to be NP-hard
- no known polynomial algorithm

Known: NP-hard to forbid all circuits of length ≤ 5

Further Extensions: upper and lower bounds on edges and degrees





M has entries $0, \pm 1, \pm 2$, sum of absolute values of entries in each column is at most 2.

Problem
$$\max \sum (w_e x_e ; e \in E)$$
$$b_1 \le Mx \le b_2$$
$$c_1 \le x \le c_2$$
$$x \text{ integer-valued}$$

Edmonds and Johnson (1973) gave polyhedron and algorithm.
Challenge: No current code for alg.

Blossom Algorithm (Edmonds, 1965)
- min cost perfect matchings

$ODD \equiv \{S \subseteq V ; |S| \text{ odd}, |S| \ge 3\}$

LP
$$\min \sum (c_e x_e ; e \in E)$$
$$x(\delta(v)) = 1 \quad \forall v \in V$$
$$x(\delta(S)) \ge 1 \quad \forall S \in ODD$$
$$x_e \ge 0 \quad \forall e \in E$$

dual variables
$y \equiv (y_v ; v \in V)$
$Y \equiv (Y_S ; S \in ODD)$

Dual LP
$$\max \sum (y_v ; v \in V) + \sum (Y_S ; S \in ODD)$$
$$y_v + y_w + \sum (Y_S ; e \in \delta(S), S \in ODD) \le c_e \quad \forall e = vw \in E$$
$$Y_S \ge 0 \quad \forall S \in ODD$$

Complementary Slackness

$x^*, (y^*, Y^*)$ are optimal solutions $\Leftrightarrow$ satisfy all constraints and all CS ~~relations~~ conditions
$$x_e^* > 0 \Rightarrow y_v^* + y_w^* + \sum (Y_S^* ; e \in \delta(S), S \in ODD) = c_e$$
$$Y_S^* > 0 \Rightarrow x^*(\delta(S)) = 1.$$

Def Given dual solution $(\bar{y}, \bar{Y})$, the reduced cost of edge $e = vw$ is
$$\bar{c}_e \equiv c_e - \bar{y}_v - \bar{y}_w - \sum (\bar{Y}_S ; e \in \delta(S), S \in ODD).$$

Dual contraints $\Rightarrow \bar{c}_e \ge 0$. To satisfy CS conditions, we want a matching M such that $e \in M \Rightarrow \bar{c}_e = 0$ and $\bar{Y}_S > 0 \Rightarrow |\delta(S) \cap M| = 1$.
In words, only use edges of reduced cost 0, only use dual variable Y_S if $\delta(S)$ contains exactly one matching edge. Blossom alg finds M and (y, Y).

Blossom Algorithm

Maintain a valid dual solution (y, Y).

Maintain a matching M such that

$$\begin{cases} e \in M \Rightarrow \text{reduced cost } \bar{c}_e = 0 \\ S \in ODD \text{ and } Y_S > 0 \Rightarrow |\delta(S) \cap M| = 1 \end{cases}$$

Goal: Increase size of M. When M is perfect, stop.

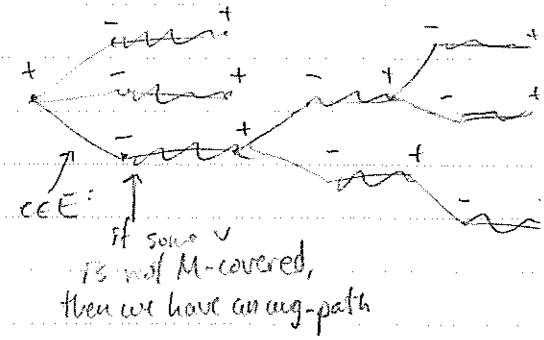
To increase $|M|$, look for an augmenting path.

Require all edges in augmenting path have reduced cost 0.

Search for augmenting path.

Choose r such that r is not M-covered

Grow an alternating tree from r. $\longrightarrow$



Def^n $E^= = \{e \in E, \bar{c}_e = 0\}$

$A(T) \equiv$ odd distance from r

$B(T) \equiv$ even distance from r

$|B(T)| = |A(T)| + 1$

Find ε and make a dual change

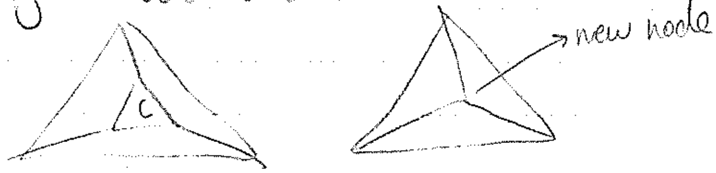
$$y_v \leftarrow y_v + \varepsilon \quad \forall v \in B(T)$$
$$y_v \leftarrow y_v - \varepsilon \quad \forall v \in A(T)$$

Restrictions:

$\varepsilon \le \bar{c}_e$ for each $e = uv$ st $u \in B(T)$ and $v \notin V(T)$

$\varepsilon \le \frac{1}{2}\bar{c}_e$ for $e = uv$ with $u, v \in B(T)$

Shrinking an odd circuit



G $\quad$ G×C

Proposition: C an odd circuit, $G' = G \times C$, M' a matching of G'. Then $\exists$ a matching M of G such that $M \subseteq M' \cup E(C)$ and the number of M-exposed nodes in G is the same as the number of M' exposed nodes in G'.

Blossom Alg: G' obtained from G by a sequence of shrinking odd circuits.

Suppose r is M-exposed and is an original node $(r \in V(G))$.
Let T be an alternating tree rooted at r.
Additional restriction on ε:

$\varepsilon \le \min(Y_S \, ; \, S$ is the node set of a pseudo-node in $A(T))$

- can't allow Y_S to become negative

2014 11 05

Suppose G' is derived from G by a sequence of odd-circuit shrinkings.
G' has two types of nodes
1) original nodes
2) pseudo nodes (obtained by shrinking)

Let $v \in V(G')$. Define

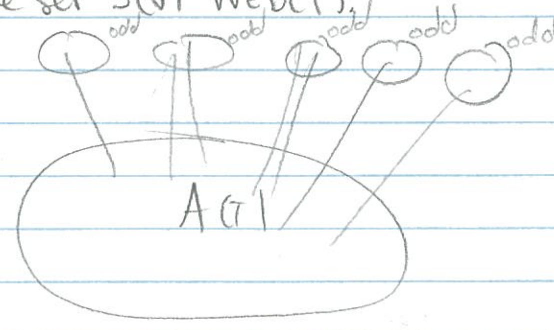
$S(v) \equiv \{v\}$ if $v \in V$

$S(v) \equiv \bigcup (S(w) \, ; \, w \in V(C))$

where C is the circuit we shrank to obtain v.

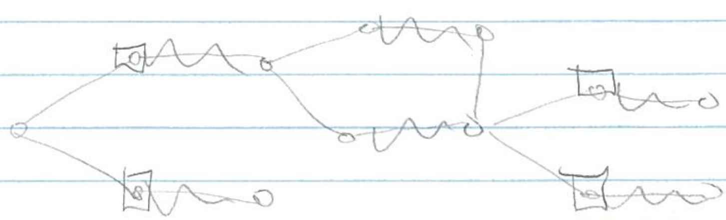
In any case, $|S(v)|$ is odd.
Alternating tree T in G'.
T is called __frustrated__ if every edge of G' that has one end in $B(T)$ has the other end in $A(T)$.
Lemma: If T is frustrated and no element of $A(T)$ is a pseudo node, then G' has no perfect matching.
Verification: When we delete $A(T)$ from G', we obtain a connected component with node set $S(v)$ $\forall v \in B(T)$.

$B(T)$ gives components



Since $|B(T)| = |A(T)|+1$, we know G' has no perfect matching. □

ex



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26

Theorem (Tutte) $G=(V,E)$ has a perfect matching $\iff$ $G \setminus U$ has at most $|U|$ odd components $\forall U \subseteq V$.

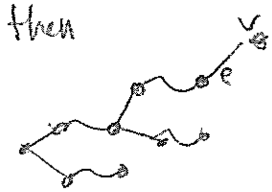
Either G has a perfect matching
or Tutte set to prove G has no perfect matching.
$\Rightarrow NP \cap co\text{-}NP$

Min-Cost Perfect Matching Algorithm

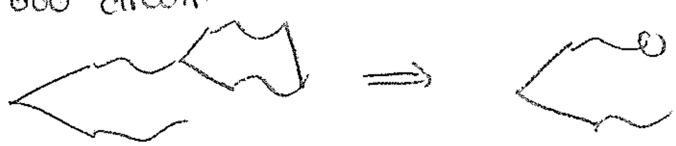
Dual solution (y, Y), $E^= \equiv \{e \in E;\ \bar{c}_e = 0\}$, $\bar{c}_e \equiv$ reduced cost of e.

Goal: Find perfect matching M in $E^=$ such that whenever $Y_S \geq 0$ then $|\delta(S) \cap M| = 1$.

1) Grow tree. If v is uncovered by M, have an augmenting path $\Rightarrow$ make M larger
Otherwise, add e to T and add the matching edge meeting v to T.



2) Shrink odd circuit



3) Dual Change

$\varepsilon_1 \equiv \min(\bar{c}_e;$ e joins a node in $B(T)$ to a node not in $T)$
$\varepsilon_2 \equiv \min(\bar{c}_e/2;$ e joins two nodes in $B(T))$
$\varepsilon_3 \equiv \min(Y_S;$ S gives a pseudo node in $A(T))$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.
Add ε to dual variables in $B(T)$.
Subtract ε from dual variables in $A(T)$.

Cases: $\varepsilon = \varepsilon_1$ grow tree
$\varepsilon = \varepsilon_2$ shrink odd circuit
$\varepsilon = \varepsilon_3$ unshrink pseudonode v

If $\varepsilon = \infty$, then we apply the lemma $\Rightarrow$ no perfect matching.

Running time $O(|E||V|^2)$
-improve $O(|V|^3)$

Alg $\Rightarrow$ Edmond's Polytope thm

Separation of Blossom Inequalities

$$PM(G) \qquad x(\delta(v)) = 2 \;\; \forall v \in V$$
$$x(\delta(S)) \geq 2 \;\; \forall S \in ODD$$
$$x \geq 0$$

Separation Problem

Given $\bar{x} = (\bar{x}_e ; e \in E)$, is $\bar{x} \in PM(G)$.
If $\bar{x} \notin PM(G)$, find an inequality satisfied by all matchings but not by $\bar{x}$.

Optimization $\underset{\text{poly time}}{\overset{\uparrow}{\equiv}}$ Separation

Familes of rational polytopes with some minor technical condition.

Edmonds + GLS (Grötschel-Lovász-Schrijver)
$\Rightarrow$ poly-time sep for $PM(G)$
Main task: Given $\underline{x}$, find an odd set S st $\bar{x}(\delta(S)) < 2$.
Let $T \equiv V$.
Find a min-weight T-cut where $w_e = \bar{x}_e \;\; \forall e \in E$
T-cut alg: Text section 6.8

1) Find a min cut in G, S.
   If $|T \cap S|$ is odd, then $\delta(S)$ is a min T-cut
   If not, then $\exists$ a min T-cut U with $U \subseteq S$ or $U \subseteq V \setminus S$.

G^1 $\qquad$ G^2