CO 446: Matroid Theory

Typeset by Kevin Matthews

Spring 2015

Why study matroids

We will generalize:

- Menger's Theorem (disjoint paths)
- König's Theorem (matching)
- Tutte's Wheels Theorem (3-connected graphs)
- Kuratowski's Theorem (planarity)
- Matrix Tree Theorem (counting spanning trees)

We will prove:

- Tutte-Berge Formula (matching)
- Jaeger's 8-flow theorem
- Nash-William's Tree Theorem (disjoint spanning trees)

We will prove analogues of:

- Turan's Theorem
- Ramsey's Theorem
- Erdös-Stone Theorem

1 Introduction

Definition **1.** A *matroid* consists of a pair $(E, \mathcal{I})$ where E is a finite *ground set* and $\mathcal{I}$ is a collection of *independent* subsets of E such that:

(I0) the empty set is independent;

(I1) subsets of independent sets are independent; and

(I2) For $X \subseteq E$, all maximal independent subsets of X have the same size.

If $M = (E, \mathcal{I})$ is a matroid, then

- $E(M) := E$;
- $\mathcal{I}(M) := \mathcal{I}$;
- $|M| := |E|$; and
- $r(M) := \max\{|I|\,;\ I \in \mathcal{I}\}$.

Example **2.** [Graphic Matroids] For a graph $G = (V, E)$ we define $M(G) := (E, \mathcal{F})$ where $\mathcal{F} := \{F \subseteq E;\ G[V, F] \text{ is a forest}\}$. Note that $M(G)$ is a matroid; we call this the *cycle matroid* of G. A matroid is *graphic* if it is the cycle matroid of a graph.

Example **3.** [Representable matroids] Let $A \in \mathbb{F}^{r \times E}$ where $\mathbb{F}$ is a field. Define $M(A) := (E, \mathcal{I})$ where

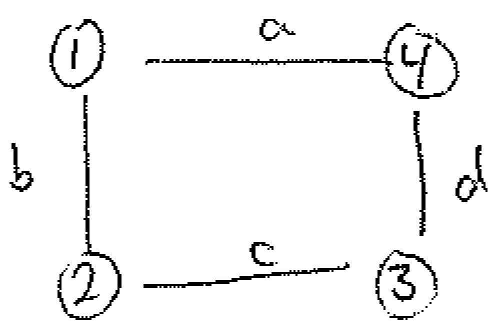
$$\mathcal{I} := \{I \subseteq E;\ A|I \text{ has linearly independent columns}\}.$$

Note that $M(A)$ is a matroid; we call this the *column matroid* of A. A matroid M is $\mathbb{F}$-*representable* if M is the column matroid of a matrix over $\mathbb{F}$. A GF(2)-representable matroid is called *binary*.

Example **4.** [Matching] Let $\mathcal{M}$ denote the set of all matchings of a graph $G = (V, E)$.

Claim**.** If G is a graph then $(E, \mathcal{M})$ is not necessarily a matroid.

Proof. Consider the following graph G:



Note $(E, \mathcal{M})$ satisfies (I0) and (I1). However, it does not satisfy (I2), as we show now. Let $X = \{a, b, c\}$, $I_1 = \{a, c\}$, and $I_2 = \{b\}$. Then I_1 and I_2 are maximal independent subsets of X but $|I_1| \neq |I_2|$. ∎

Definition **5.** In a graph G, a set $Z \subseteq V(G)$ is called *matchable* if there is a matching M of G such that each vertex in Z is incident with an edge of M. Let $\text{MSM}(G) := (V, \mathcal{I})$ where $\mathcal{I}$ is the collection of all matchable subsets of V. This is called the *matchable set matroid*.

Lemma **6.** [1.1] *If G is a graph then* $\text{MSM}(G)$ *is a matroid.*

Proof. Exercise. ∎

Remark **7.** Note that $\text{MSM}(G)$ is representable; the representation is the Tutte Matrix of G.

Example **8.** [Uniform matroids] We define $U_{r,n} := (E, \mathcal{I})$ where

$$E = \{1, \ldots, n\}$$

and

$$\mathcal{I} = \{S \subseteq E; |X| \leq r\}.$$

This is called a *uniform matroid*.

Note that $U_{r,n}$ is $\mathbb{R}$-representable and hence $U_{r,n}$ is a matroid. (Take an $r \times n$ matrix with algebraically independent entries.)

Lemma **9.** [1.2] *For any n, $U_{2,n}$ is $\mathbb{F}$-representable if and only if $|\mathbb{F}| \geq n - 1$.*

Proof. Suppose $U_{r,n} = M(A)$. By row operations and column scaling we may assume that

$$A = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}$$

where $a_2, \ldots, a_n \in \mathbb{F}$ are distinct. So $|\mathbb{F}| \geq n - 1$. The converse is similar. ∎

Open Problem **10.** Given a finite field $\mathbb{F}$, which uniform matroids are representable over $\mathbb{F}$.

Conjecture **11.** [1.3] We have $U_{r,n}$ is $\mathbb{F}$-representable if and only if either:

(i) $r \in \{0, 1, n-1, n\}$,

(ii) $n \leq |\mathbb{F}| + 1$, or

(iii) $|\mathbb{F}|$ is even, $r \in \{3, n-3\}$, and $n = |F| + 2$.

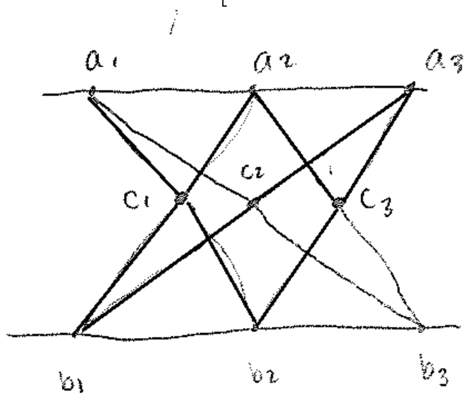
Remark **12.** The "if" direction holds. The "only if" direction holds for:

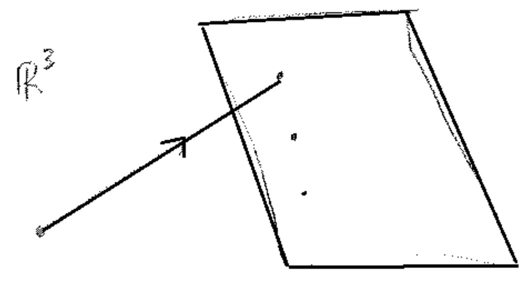
- $r \leq 5$ (1947-1970)

- $|\mathbb{F}|$ prime (Ball 2010)

EXERCISE **13.** Show that if $|\mathbb{F}|$ is odd and $n \geq |\mathbb{F}| + 2$, then $U_{3,n}$ is not $\mathbb{F}$-representable.

EXAMPLE **14.** [NON-PAPPUS MATROID]





The *Non-Pappus matroid* is $M = (E, \mathcal{I})$, where $E = \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\}$ and $I \in \mathcal{I}$ if and only if $|I| \leq 3$ and no three elements in I are on a common line in the diagram.

THEOREM **15.** [1.4 - PAPPUS CIRCA 340 AD] *The Non-Pappus matroid is not representable.*

Proof. Exercise. ∎

CONJECTURE **16.** [1.5] The proportion of n-element matroids that are representable tends to zero as $n \to \infty$.

DEFINITION **17.** A matroid M is *paving* if all subsets of $E(M)$ of size $\leq R(M) - 1$ are independent.

CONJECTURE **18.** [1.6] The proportion of n-element matroids that are paving tends to one as $n \to \infty$.

There are several other perspectives one may take to look at matroids.

1.1 Circuits

REMARK **19.** For a graph G, a minimal dependent set in $M(G)$ is the edge-set of a *circuit*. Graphs motivate the following definitions.

DEFINITION **20.** A *circuit* of a matroid is a minimal dependent set. A *loop* is a circuit of size 1 and a *parallel pair* is a circuit of size 2. A matroid is *simple* if it has no loops or parallel pairs.

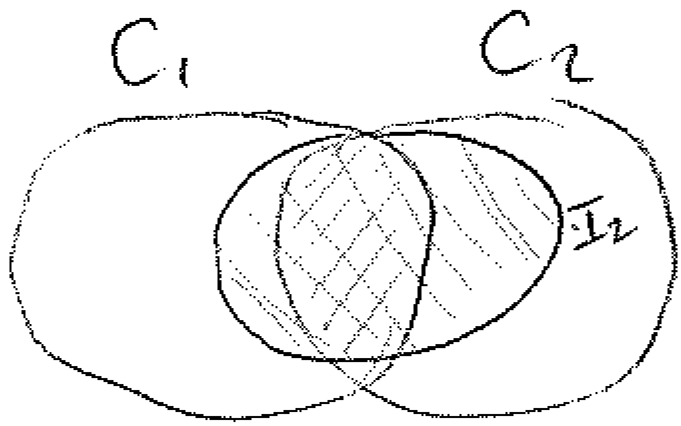
LEMMA **21.** [1.7] *Let $\mathcal{C}$ denote the collection of circuits of a matroid M. Then*

(C0) $\varnothing \notin \mathcal{C}$;

(C1) If $C_1, C_2 \in \mathcal{C}$ *are distinct, then* $C_1 \nsubseteq C_2$; *and*

(C2) If $C_1, C_2 \in \mathcal{C}$ *are distinct and* $e \in C_1 \cap C_2$, *then there exist* $C_3 \in \mathcal{C}$ *such that* $C_3 \subseteq (C_1 \cup C_2) - \{e\}$.

Proof. Note (C0) and (C1) are trivial. We show (C2). Consider distinct circuits C_1, C_2 and $e \in C_2 \cap C_2$. By (C1), $C_1 \cap C_2$ is independent. Let $X = C_1 \cup C_2$, let $I_1 = X - \{e\}$, and let I_2 be a maximal independent set such that $C_1 \cap C_2 \subseteq I_2 \subseteq X$.



Assume, for a contradiction, that I_1 is independent. By (I2), $|I_2| = |I_1| = |X| - 1$. But then either $C_1 \subseteq I_2$ or $C_2 \subseteq I_2$ — a contradiction. ∎

THEOREM **22.** [1.8] *Let $\mathcal{C}$ be a collection of subsets of a finite set E. Then $\mathcal{C}$ is the collection of circuits of a matroid if and only if $\mathcal{C}$ satisfies (C0), (C1), (C2).*

Proof. Exercise. ∎

1.2 Bases

DEFINITION **23.** For a matroid M, a maximal independent subset of $E(M)$ is a *basis.*

REMARK **24.** By (I2), each basis of a matroid M has size $r(M)$.

REMARK **25.** For a connected graph G, the bases of $M(G)$ are the edge sets of spanning trees.

LEMMA **26.** [1.9] *Let $\mathcal{B}$ denote the set of bases of a matroid M. Then*

(B0) $\mathcal{B} \neq \varnothing$

(B1) For each $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 - B_2$ there exists $f \in B_2 - B_1$ such that $(B_1 - \{e\}) \cup \{f\} \in \mathcal{B}$.

Proof. Note (B0) follows from (I0). We check (B1) now. Consider $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 - B_2$. Let $X = (B_1 - \{e\}) \cup B_2$. Now $B_1 - \{e\}$ and B_2 are independent subsets of X and $|B_1 - \{e\}| < |B_2|$. So by (I2), there exists $f \in X - (B_1 - \{e\})$ such that $(B_1 - \{e\}) \cup \{f\}$ is independent. Note that $f \in B_2 - B_1$ and $(B_1 - \{e\}) \cup \{f\}$ is a basis. ∎

THEOREM **27.** [1.10] *Let $\mathcal{B}$ be a collection of subsets of a finite set E. Then $\mathcal{B}$ is the collection of bases of a matroid if and only if it satisfies (B0) and (B1).*

Proof. Exercise. ∎

1.3 Rank

DEFINITION **28.** For a matroid M and $X \subseteq E(M)$ we define the *rank* of X in M to be

$$r_M(X) := \max(|I|;\, I \in \mathcal{I}(M),\, I \subseteq X).$$

We sometimes abbreviate $r_M(X)$ to $r(X)$.

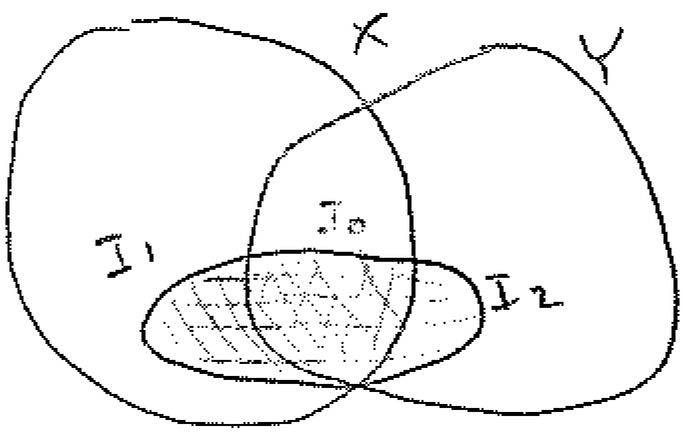
LEMMA **29.** [1.11] *For a matroid $M = (E, \mathcal{I})$*

(R0) $0 \le r(X) \le |X|$ for each $X \subseteq E$;

(R1) $r(X) \le r(Y)$ for each $X \subseteq Y \subseteq E$; and

(R2) $r(X) + r(Y) \ge r(X \cap Y) + r(X \cup Y)$ for each $x, y \subseteq E$.

Proof. Note (R0) and (R1) are obvious. We check (R2) now. Consider $X, Y \subseteq E$.



Let I_0, I_1, I_2 be maximal independent sets subject to $I_0 \subseteq X \cap Y$, $I_0 \subseteq I_1 \subseteq X$, and $I_1 \subseteq I_2 \subseteq X \cup Y$. Thus $|I_0| = r(X \cap Y)$, $|I_1| = r(X)$, $|I_2| = r(X \cup Y)$. Let $I_3 = Y \cap I_2$. Thus $|I_3| \le r(Y)$. Now $r(X \cap Y) + r(X \cup Y) = |I_0| + |I_2| = |I_1| + |I_3| \le r(X) + r(Y)$. ∎

REMARK **30.** Condition (R2) is called *submodularity*.

THEOREM **31.** [1.12] *Let E be a finite set and let $r : 2^E \to \mathbb{Z}$. Then r is the rank function of a matroid if and only if it satisfies (R0), (R1), (R2).*

Proof. Exercise. ∎

1.4 Ingleton's Inequality

EXAMPLE **32.** [INGLETON'S INEQUALITY] For $X_1, X_2, X_3, X_4 \subseteq E$ consider

$$\begin{aligned} & r(X_1) + r(X_2) + r(X_1 \cup X_2 \cup X_3) + r(X_1 \cup X_2 \cup X_4) + r(X_3 \cup X_4) \\ \leq & r(X_1 \cup X_2) + r(X_1 \cup X_3) + r(X_1 \cup X_4) + r(X_2 \cup X_3) + r(X_2 \cup X_4). \end{aligned} \tag{1}$$

Ingleton showed that:

(1) all representable matroids satisfy (1); and

(2) there are matroids that do not satisfy (1).

1.5 Local Connectivity

DEFINITION **33.** For $X, Y \subseteq E(M)$, we define the *local connectivity* of X and Y to be

$$\sqcap_M(X, Y) := r(X) + r(Y) - r(X \cup Y).$$

REMARK **34.** Note that submodularity, (R2), can be written as

$$\sqcap_M(X, Y) \geq r(X \cap Y).$$

LEMMA **35.** [1.13] *Let $A \in \mathbb{F}^{r \times E}$ and for each $X \subseteq E$ let $\langle X \rangle$ denote the subspace of $\mathbb{F}^r$ spanned by the columns of $A|X$. Then, for $X, Y \subseteq E$,*

$$\dim \langle X \rangle \cap \langle Y \rangle = \sqcap_M(X, Y).$$

Proof. By the dimension theorem

$$\dim \langle X \rangle + \dim \langle Y \rangle = \dim \langle X \rangle \cap \langle Y \rangle + \dim \langle X \rangle \cup \langle Y \rangle .$$

Now $\dim \langle X \rangle = r(X)$, $\dim \langle Y \rangle = r(Y)$, and $\dim \langle X \rangle \cup \langle Y \rangle = r(X \cup Y)$. So

$$\dim \langle X \rangle \cap \langle Y \rangle = r(X) + r(Y) - r(X \cup Y) = \sqcap_M(X, Y).$$

∎

REMARK **36.** Note that $\dim \langle X \rangle \cap \langle Y \rangle \geq r(X \cap Y)$. So representable matroids satisfy $\sqcap_M(X, Y) \geq r(X \cap Y)$ (as do all matroids — this is just another way to see this inequality in the particular case of representable matroids).

1.6 Closure

LEMMA **37.** [1.14] *For a matroid M let $X, X_1, X_2 \subseteq E(M)$ with $X \subseteq X_1$ and $X \subseteq X_2$. If $r(X_1) = r(X_2) = r(X)$, then $r(X_1 \cup X_2) = r(X)$.*

Proof. By submodularity,

$$2r(X) = r(X_1) + r(X_2) \geq r(X_1 \cap X_2) + r(X_1 \cup X_2) \geq 2r(X).$$

Equality holds throughout; in particular, $r(X_1 \cup X_2) = r(X)$. ∎

DEFINITION **38.** Given a matroid M and $X \subseteq E(M)$, by Lemma 1.14 there is a unique maximal set $\hat{X} \subseteq E$ such that $X \subseteq \hat{X}$ and $r\big(\hat{X}\big) = r(X)$; $\hat{X}$ is the called the *closure* of X and is denoted $\mathrm{cl}_M(X)$.

REMARK **39.** Note that:

(1) for $e \in E - X$, $e \in \mathrm{cl}_M(X)$ if and only if there is a circuit C such that $e \in C$ and $C - X = \{e\}$; and

(2) if $X \subseteq Y$, then $\mathrm{cl}_M(X) \subseteq \mathrm{cl}_M(Y)$.

DEFINITION **40.** For a matroid M, a set $F \subseteq E(M)$ is called a *flat* if $\mathrm{cl}(F) = F$.

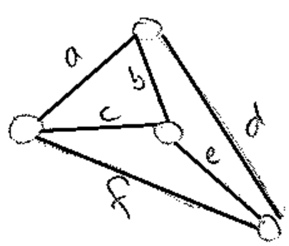
EXERCISE **41.** Show that the intersection of two flats is a flat.

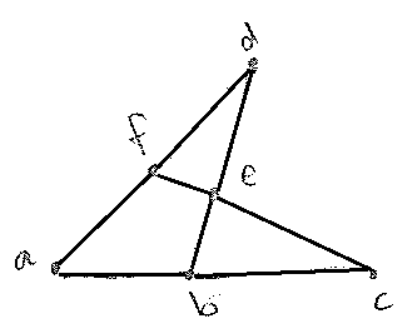
DEFINITION **42.** Let M be a matroid. A *point* is a rank-one flat. A *line* is a rank-two flat. A *plane* is a rank-three flat. A *hyperplane* is a rank-$(r(M) - 1)$ flat.

1.7 Geometric drawings of matroids

$U_{2,4}$

$M(K_4)$





Definition **43.** Let M be a matroid. For each $k \in \{0, \ldots, r(M)\}$, let W_k denote the number of rank-k flats in M.

Conjecture **44.** [1.15 - Rota, 1971] For a matroid M and $k \in \{1, \ldots, r(M) - 1\}$,

$$W_k \geq \min(W_{k-1}, W_{k+1}) \, .$$

Conjecture **45.** [Points-Lines-Planes Conjecture - Mason 1972] In a rank-4 matroid we have

$$W_2^2 \geq W_1 W_3.$$

1.8 Simplification

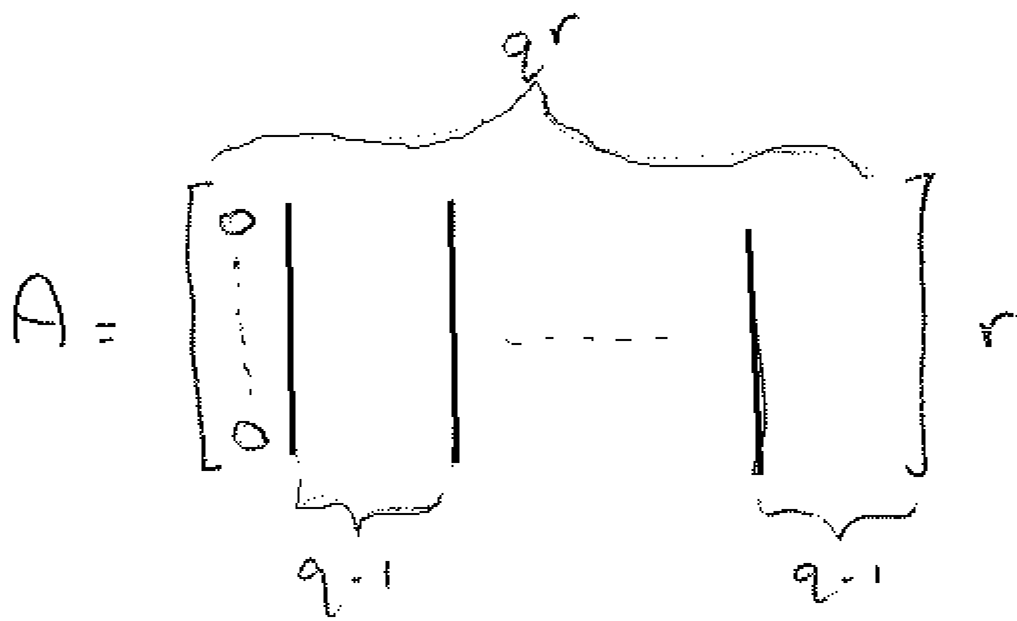
Definition **46.** Let L be the collection of loops of a matroid M. Now let $(P_1, \ldots, P_k)$ be the points of M. Note that $L \subseteq P_i$ $(i = 1, \ldots, k)$ and that $(P_1 - L, \ldots, P_k - L)$ is a partition $E(M) - L$. Note that $\{e, f\}$ is a parallel-pair if and only if $e, f \in P_i - L$ for some i. The sets $(P_1, \ldots, P_k)$ are called the *parallel* classes of M. If $e_i \in P_i - L$, for $i \in \{1, \ldots, k\}$, then we call the restriction N of M to $\{e_1, \ldots, e_k\}$ "the" *simplification* of M.

Exercise **47.** (1) Show that the simplification of a matroid is uniquely determined up to isomorphism.

(2) Show that a matroid is $\mathbb{F}$-representable if and only if its simplification is.

1.9 Projective Geometries

Definition **48.** Let $r \in \mathbb{Z}_+$ and $\mathbb{F}$ be a finite field with $q = |\mathbb{F}|$. Now let $A \in \mathbb{F}^{r \times q^r}$ have distinct columns.



We denote the simplification of $M(A)$ by $\mathrm{PG}(r-1,\mathbb{F})$ or $\mathrm{PG}(r-1,q)$. This is called a *projective geometry*.

REMARK **49.** Note that:

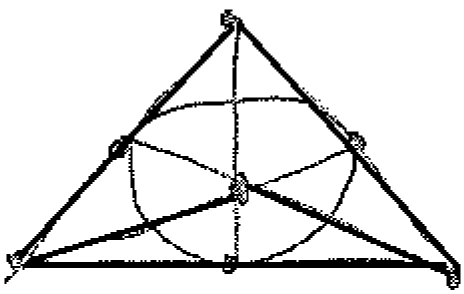
(1) $|\mathrm{PG}(r-1,q)| = \frac{q^r-1}{q-1}$;

(2) every simple rank-r $\mathbb{F}$-representable matroid is isomorphic to a restriction of $\mathrm{PG}(r-1,\mathbb{F})$;

(3) each line of $\mathrm{PG}(r-1,\mathbb{F})$ has length $q+1$; and

(4) each pair of distinct lines in $P(2,\mathbb{F})$ intersect in a point.

EXAMPLE **50.** The *Fano matroid* is $F_7 := \mathrm{PG}(2,2)$.



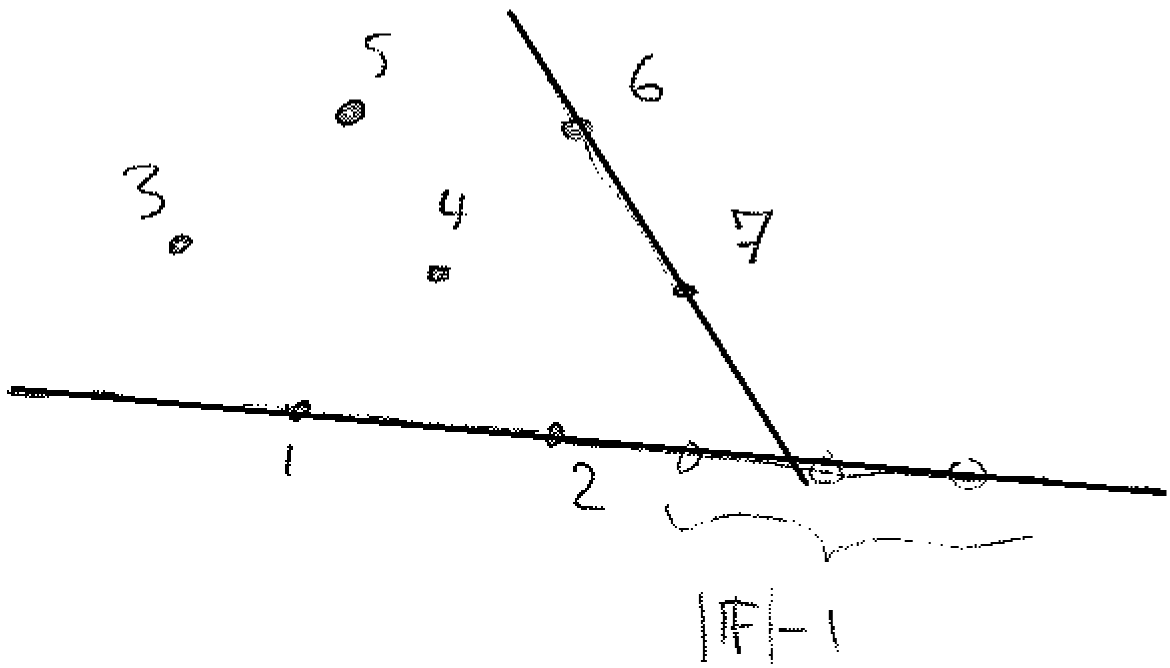
So

$$F_7 = M\left(\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}\right).$$

Recall the following exercise.

EXERCISE **51.** Show that if $|\mathbb{F}|$ is odd and $n \geq |\mathbb{F}|+2$, then $U_{3,n}$ is not $\mathbb{F}$-representable.

Hint: Consider $U_{3,n}$ as a restriction of $\mathrm{PG}(2,\mathbb{F})$ and consider a line through two points of $U_{3,n}$.



1.10 Duality

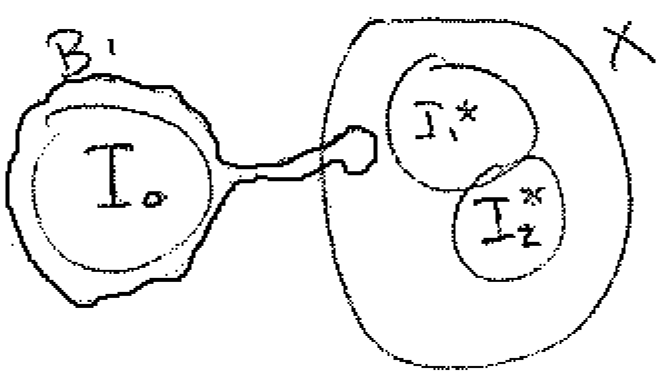
Definition **52.** Given a matroid $M = (E,\mathcal{I})$ we define $M^* := (E,\mathcal{I}^*)$ where

$$\mathcal{I}^* := \{I^* \subseteq E;\, r(E - I^*) = r(M)\}\,.$$

This is called the *dual* of M.

Theorem **53.** [1.16] *If M is a matroid then M^* is a matroid.*

Proof. Note M^* clearly satisfies (I0), (I1). So we need only check (I2). Let $X \subseteq E$ and let I_1^*, I_2^* be maximal M^*-independent subsets of X.



Let I_0 be a maximal M-independent subset of $E - X$. Let B_1 and B_2 be bases of M such that $I_0 \subseteq B_1 \subseteq E - I_1^*$ and $I_0 \subseteq B_2 \subseteq E - I_2^*$. Since $I_0 \subseteq E - X$ is maximal:

$$B_1 - X = I_0 \text{ and } B_2 - X = I_0.$$

Moreover, by the maximality of I_1^* and I_2^*:

$$I_1^* = X - B_1 \text{ and } I_2^* = X - B_2.$$

Note that $|B_1| = |B_2|$. So

$$|I_1^*| = |X - B_1| = |X - B_2| = |I_2^*|$$

as required. ∎

Remark **54.** Note that:

(1) B is a basis of M if and only if $E - B$ is a basis of M^*. Hence $M^{**} = M$.

(2) C^* is a circuit of M^* if and only if $E - C^*$ is a hyperplane of M.

(3) $r(M) + r(M^*) = |E|$.

(4) For $X \subseteq E$,

$$r_{M^*}(X) = |X| - (r(M) - r_M(E - X)).$$

Definition **55.** Let M be a matroid. Circuits of M^* are called *cocircuits* of M. Independent sets of M^* are called *co-independent* sets of M. More generally, we can append "co" in front of other terms to get analogous definitions.

1.11 Representability and duality

Definition **56.** Let $\mathbb{F}$ be a field, E be a finite set, and U be a subspace of $\mathbb{F}^E$. We define

$$U^\perp := \left\{x \in \mathbb{F}^E;\ x^t y = 0 \text{ for each } y \in E\right\}.$$

We call $U^\perp$ the *orthogonal space* of U.

Remark **57.** Note that:

(1) $\dim(U) + \dim(U^\perp) = |E|$; and

(2) U and $U^\perp$ need not be disjoint.

Exercise **58.** Let $B \subseteq E$, $A \in \mathbb{F}^{B \times E - B}$, $A_1 = [I, A]$, and $A_2 = [-A^t, I]$. Prove that $\text{rowspace}(A_1)^\perp = \text{rowspace}(A_2)$.

Theorem **59.** [1.17] *Let* $A_1 \in \mathbb{F}^{r_1 \times E}$ *and* $A_2 \in \mathbb{F}^{r_2 \times E}$. *If* $\text{rowspace}(A_1)^\perp = \text{rowspace}(A_2)$, *then* $M(A_1)^* = M(A_2)$.

Proof. It suffices to prove that, if B is a basis of $M(A_1)$, then $E - B$ is a basis of $M(A_2)$. By row operations we may assume that $A_1 = [I, A]$. Now consider the matrix $A_2' = [-A^t, I]$. By the exercise,

$$\text{rowspace}(A_2') = \text{rowspace}(A_1)^\perp = \text{rowspace}(A_2).$$

Thus $M(A_2') = M(A_2)$. Now $E - B$ is a basis of $M(A_2')$ and hence also of $M(A_2)$. ∎

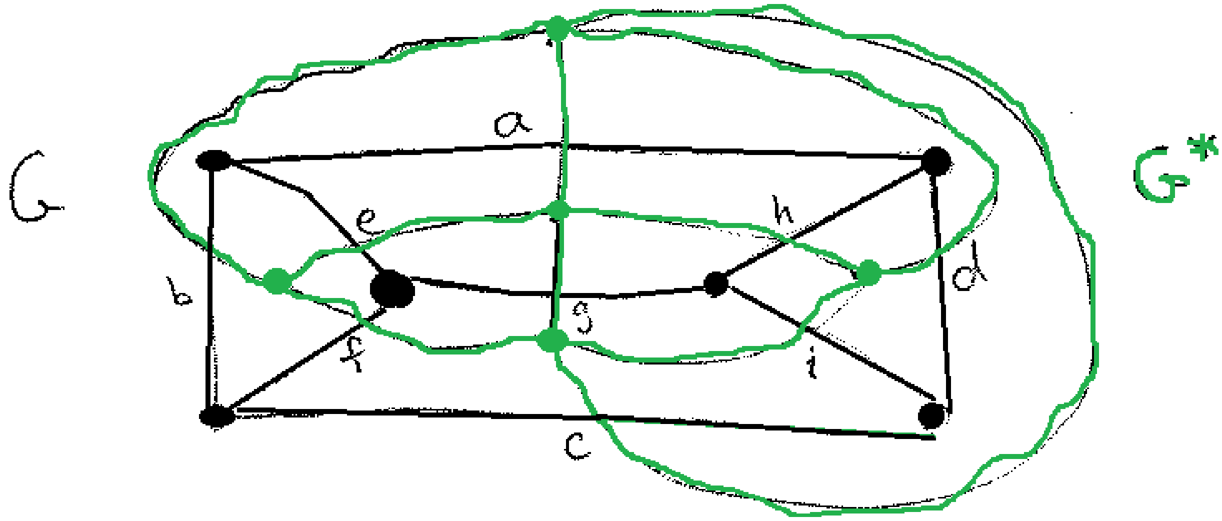
COROLLARY **60.** [1.18] *A matroid M is $\mathbb{F}$-representable if and only if M^* is $\mathbb{F}$-representable.*

1.12 Graphicness and duality

THEOREM **61.** [1.19] *If G is a plane graph, then $M(G^*) = M(G)^*$.*

REMARK **62.** We prove Theorem 1.19 after the Lemma 1.20.

Note that $E(G^*) = E(G)$.



LEMMA **63.** [1.20] *For a graph G and $X \subseteq E(G)$, $G[X]$ is a forest if and only if $G^* - X$ is connected.*

Proof sketch. If X contains a circuit, then $\mathbb{R}^2 - X$ is not connected and hence $G^* - X$ is not connected.

On the other hand, if $G[X]$ is a forest, then $\mathbb{R}^2 - X$ is connected. ∎

Proof of Theorem 1.19. By lemma 1.20, $M(G) = M(G^*)^*$. ∎

1.13 Euler's Formula

THEOREM **64.** [EULER'S FORMULA] *Let F be the set of faces of a connected plane graph $G = (V, E)$. Then*

$$|E| = |V| + |F| - 2.$$

Proof. Let T be the edge set of a spanning tree of G. Then $E-T$ is the edge-set of a spanning tree in G^*. So $|E| = |T| + |E-T| = |V| - 1 + |F| - 1$. ∎

Claim. The matroid $M(K_5)^*$ is not graphic.

Proof. Suppose that $M(K_5)^* = M(G)$. We may assume that G is connected. Now

$$|V(G)| = r\big(M(K_5)^*\big) + 1 = |E(K_5)| - r(M(K_5)) + 1 = 10 - 4 + 1.$$

Now $3|V(G)| > 2|E(G)| = \sum_{v\in V(G)} \deg(v)$. So G has a vertex v with degree at most two. So $M(G)$ has a cocircuit $C \subseteq \delta_G(v)$ of size at most two. Hence $M(K_5) = M(G)^*$ has a circuit of size at most two — contradiction. ∎

Exercise **65.** Prove that $M(K_{3,3})^*$ is not graphic.

Theorem **66.** [1.21 - Whitney] *A graph G is planar if and only if $M(G)^*$ is graphic.*

Remark **67.** We will return to the proof of Theorem 1.21 later in the course. We could prove it now but it would take considerable time.

1.14 Minors

Definition **68.** For a matroid M and $D \subseteq E(M)$ we define

$$M \setminus D := (E - D, \mathcal{I}')$$

where

$$\mathcal{I}' := \{I \subseteq E - D; I \in \mathcal{I}(M)\}.$$

This is called *deletion*.

Remark **69.** Note that:

(1) $M \setminus D$ is a matroid;

(2) If M is $\mathbb{F}$-representable, then so is $M \setminus D$; and

(3) $M(G \setminus D) = M(G) \setminus D$.

Definition **70.** For a matroid M and $C \subseteq E(M)$ we define

$$M/C := (M^* \setminus C)^*.$$

This is called *contraction*.

Remark **71.** Note that:

(1) M/C is a matroid; and

(2) If M is $\mathbb{F}$-representable then so is M/C.

Lemma **72.** [1.22] *For a matroid M and $S \subseteq E - C$ we have*

$$r_{M/C}(X) = r_M(X \cup C) - r_M(C).$$

Proof. Exercise. ∎

Remark **73.** By Lemma 1.22 we see

$$r_{M/C}(X) = r_M(X) - \sqcap_M(X, C). \tag{2}$$

Exercise **74.** Let $X', X, Y \subseteq E(M)$ in a matroid M, with $X' \subseteq X$ and $r(X') = r(X)$. Then $\sqcap_M(X', Y) = \sqcap_M(X, Y)$.

By (2) and the exercise we get the following lemma.

Lemma **75.** [1.23] *Let $X \subseteq E$ and let I be a maximum independent set of C. Then*

$$M/C = M \setminus (C - I)/I.$$

Remark **76.** For $I \subseteq E - C$, I is independent in M/C if and only if $|I| = r_M(I) - \sqcap_M(I, C)$. That is, $r_M(I) = |I|$ and $\sqcap_M(I, C) = 0$.

Definition **77.** Let M be a matroid. We say that $X, Y \subseteq E(M)$ are *skew* if $\sqcap_M(X, Y) = 0$.

The above proves the following lemma.

Lemma **78.** [1.24] *In a matroid M, $I \subseteq E(M/C)$ is independent in M/C if and only if I is independent in M and I is skew to C.*

Exercise **79.** Let M be a matroid.

(1) Let $X, Y \subseteq E(M)$ be disjoint sets. Prove that X is skew to Y if and only if there is no circuit $C \subseteq X \cup Y$ such that $C \cap X$ and $X \cap Y$ are both non-empty.

(2) Prove that, if $C \subseteq E(G)$ then

$$M(G/C) = M(G)/C.$$

1.15 Minors

DEFINITION **80.** Let M and N be matroids. If $C, D \subseteq E(M)$ are disjoint, then we call $M \setminus D/C$ a *minor* of M. We say that M has an *N-minor* if M has a minor that is isomorphic to N.

LEMMA **81.** [1.25 - SCUM THEOREM] *If N is a minor of a matroid M, then there is a partition (C, D) of $E(M) - E(N)$ into an independent set C and a co-independent set D such that $N = M/C \setminus D$.*

Proof. Exercise. ∎

REMARK **82.** Consider a representable matroid $M(A) = (E, \mathcal{I})$. Let $C, D \subseteq E$ be disjoint sets such that C is independent and D is co-independent. So there is a basis B such that $C \subseteq B \subseteq E - D$. By row operations we may assume that $A = [I, A']$.

Now suppose that

$$A' = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

That is

$$A = \begin{bmatrix} I & 0 & A_1 & A_2 \\ 0 & I & A_3 & A_4 \end{bmatrix}.$$

Then $M(A) \setminus D/C = M([I, A_1])$.

1.16 Regular Matroids

DEFINITION **83.** A matroid is *regular* if it is representable over every field.

DEFINITION **84.** A matrix A is *totally unimodular* if each square submatrix of A has determinant in $\{-1, 0, 1\}$.

THEOREM **85.** [TUTTE] *For a matroid M, the following are equivalent:*

(1) M is regular;

(2) M is representable over both GF(2) *and* GF(3)*; and*

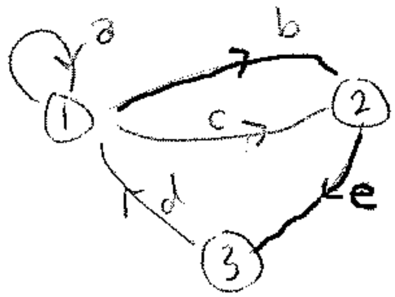
(3) $M = M_{\mathbb{R}}(A)$ where A is totally unimodular.

Proof. That (1) implies (2) is trivial. That (3) implies (1) is obvious. The proof that (2) implies (3) is straightforward, however it is omitted. ∎

1.17 Signed Incidence Matrix

DEFINITION **86.**

G



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

1.18 Incidence matrix

DEFINITION **87.**

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 2 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

LEMMA **88.** [1.26] *Let A be a signed incidence matrix of $G = (V, E)$ and let $\mathbb{F}$ be a field. Then*

$$r(M_{\mathbb{F}}(A)) = |V| - comps(G).$$

Proof. For $x \in \mathbb{F}^V$, $x^t A = 0$ if and only if $x_u = x_v$ for two vertices u, v in the same component of G. So

$$r(M_{\mathbb{F}}(A)) = \operatorname{rank} A = |V| - comps(G).$$

∎

LEMMA **89.** [1.27] *Let A be the signed incidence matrix of $G = (V, E)$ and let $\mathbb{F}$ be a field. Then $M_{\mathbb{F}}(A) = M(G)$.*

Proof. Apply Lemma 1.26 to $G[V, X]$ for each $X \subseteq E$. ∎

REMARK **90.** Note that over GF(2) we can ignore signs. That is, $M(G) = M_{\mathrm{GF}(2)}(\tilde{A})$ where $\tilde{A}$ is the incidence matrix of G.

THEOREM **91.** [1.28] *Graphic matroids are regular.*

Proof. ∎

EXERCISE **92.** Prove that signed incidence matrices are totally unimodular.

1.19 Counting bases

PROBLEM **93.** Given a TU matrix A, how many bases does $M(A)$ have?

PROPOSITION **94.** [1.29] *Let R, E be finite ordered sets and let $A \in \mathbb{F}^{R\times E}$. Then*

$$\det AB^t = \sum_{X\subseteq E,\,|X|=|R|} (\det A|X)(\det B|X)\,.$$

Proof sketch. Use:

(1)
$$\det\begin{bmatrix} I & B^t \\ -A & 0 \end{bmatrix} = \det AB^t;\text{ and}$$

(2) for $D, A \in \mathbb{F}^{V\times V}$, D diagonal, we have
$$\det(D+A) = \sum_{X\subseteq V} \det D[X,X]\det A[V-X, V-X].$$

∎

THEOREM **95.** [1.30] *If $A \in \{-1,0,1\}^{R\times E}$ is totally unimodular and has rank $|R|$, then the number of bases of $M(A)$ is $\det AA^t$.*

Proof. We have
$$\det AA^t = \sum_{X\subseteq E,\,|X|=|R|} \det(A|X)^2 = \sum_{X\text{ a basis of }M(A)} 1.$$

∎

1.20 Counting Spanning Trees

DEFINITION **96.** Let A be the signed incidence matrix of a graph G and let $L := AA^t$. Note that $L \in \mathbb{R}^{V\times V}$ and
$$L_{uv} = \begin{cases} \deg(u); & u = v \\ -n_{uv}; & u \neq v \end{cases}$$
where n_{uv} is the number of edges connecting u and v. We call L the *Laplacian* of G.

THEOREM **97.** [MATRIX TREE THEOREM] *Suppose G is a graph. Let $u \in V(G)$ and let $L_{\bar{u}}$ be obtained from the Laplacian L by deleting both the row and column indexed by u. Then the number of spanning trees of G is equal to* $\det(L_{\bar{u}})$.

Proof. ∎

EXERCISE **98.** For a graph G, let $u, v \in V(G)$ and let $L_{\bar{u},\bar{v}}$ be obtained from L by deleting the u^{th} row and the v^{th} column. Then the number of spanning trees of G is $\det L_{\bar{u},\bar{v}}$.

1.21 Excluded Minors

THEOREM **99.** [KURATOWSKI'S THEOREM] *A graph G is planar if and only if it has no minor isomorphic to $K_{3,3}$ or K_5.*

DEFINITION **100.** A class of matroids $\mathcal{M}$ is called *minor-closed* if for each matroid $M \in \mathcal{M}$, all minors of M are in $\mathcal{M}$.

EXAMPLE **101.** The following classes are examples of minor-closed classes of matroids:

(1) $\mathbb{F}$-representable matroids;

(2) graphic matroids;

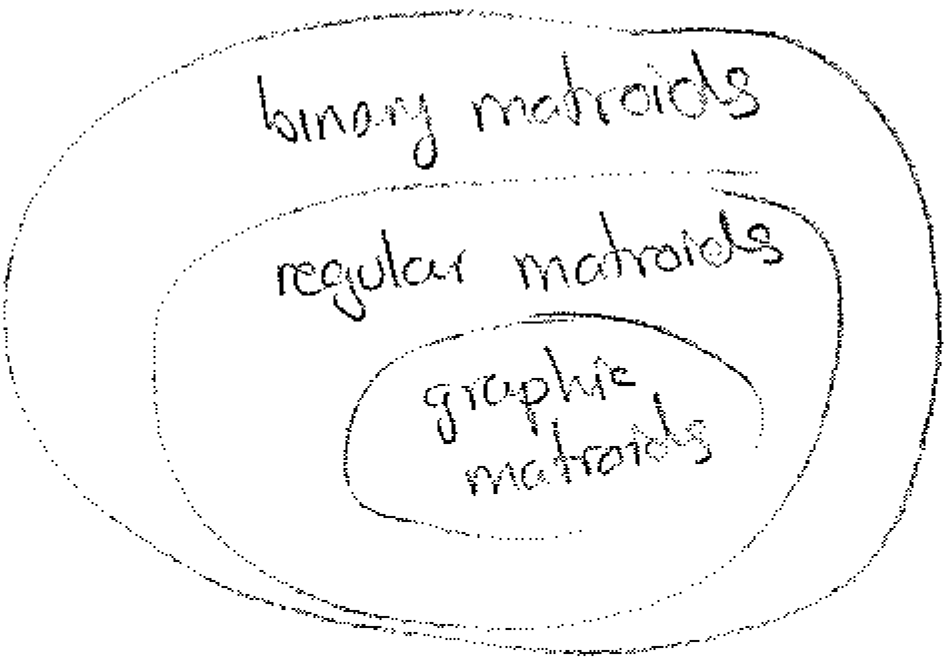
(3) regular matroids; and

(4) uniform matroids.

DEFINITION **102.** Let $\mathcal{M}$ be a minor-closed class of matroids. An *excluded-minor* for $\mathcal{M}$ is a matroid $M \notin \mathcal{M}$ whose proper minors are all contained in $\mathcal{M}$.

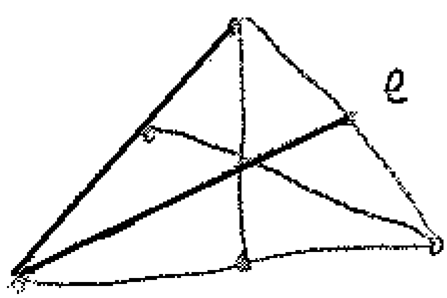
THEOREM **103.** [KURATOWSKI'S THEOREM] *The excluded-minors for the class of planar graphs are $K_{3,3}$ and K_5.*

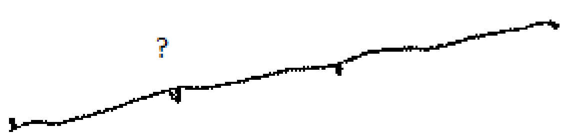
EXERCISE **104.** Determine the excluded minors for the class of uniform matroids.



THEOREM **105.** [TUTTE'S EXCLUDED-MINOR THEOREMS] *(1) The matroid $U_{2,4}$ is the only excluded minor for the class of binary matroids;*

(2) The excluded-minors for the class of regular matroids are $U_{2,4}$, F_7, F_7^*; and





(3) The excluded minors for the class of graphic matroids are $U_{2,4}$, F_7, F_7^*, $M(K_5)^*$, $M(K_{3,3})^*$.

REMARK **106.** (a) We will prove (1) and (2).

(b) Item (2) has a short proof due to Gerards.

(c) Item (3) implies Kuratowski's theorem.

CONJECTURE **107.** [ROTA'S CONJECTURE] For each finite field $\mathbb{F}$, there are, up to isomorphism, only finitely many excluded minors for the class of $\mathbb{F}$-representable matroids.

REMARK **108.** (1) Rota's conjecture is claimed to be true (Geelen, Gerards, Whittle).

(2) Rota's conjecture is false for infinite fields (Lazarson).

THEOREM **109.** [GRAPH MINORS THEOREM - ROBERTSON, SEYMOUR] *Each minor-closed class of graphs has, up to isomorphism, only finitely many excluded minors.*

REMARK **110.** By (2) of the previous remark, the Graph Minors Theorem does not extend to all matroids.

REMARK **111.** The set $\{\mathrm{PG}(2,p);\, p \text{ prime}\}$ is an "infinite antichain" (that is, no one plane contains another as a minor).

1.22 Matching

DEFINITION **112.** Let G be a graph. We define $\nu(G)$ to be the size of a maximum matching of G, the *deficiency* of G to be $\operatorname{def}(G) \coloneqq |V(G)| - 2\nu(G)$, and $\operatorname{odd}(G)$ to be the number of components of G of odd size.

REMARK **113.** Note that

$$\operatorname{def}(G) \geq \operatorname{def}(G - X) - |X| \geq \operatorname{odd}(G - X) - |X|.$$

THEOREM **114.** [TUTTE-BERGE FORMULA] *In a graph G,*

$$\operatorname{def}(G) = \max_{X \subseteq V(G)} (\operatorname{odd}(G - X) - |X|).$$

DEFINITION **115.** A vertex v in a graph G is *avoidable* if $\nu(G - v) = \nu(G)$.

REMARK **116.** Note that

$$\operatorname{def}(G - v) = \begin{cases} \operatorname{def}(G) - 1, & v \text{ is avoidable} \\ \operatorname{def}(G) + 1, & v \text{ is not avoidable.} \end{cases}$$

DEFINITION **117.** A graph G is *hypomatchable* if

(1) $V(G) \neq \emptyset$;

(2) G is connected; and

(3) each vertex of G is avoidable.

LEMMA **118.** [GALLAI'S LEMMA] *If a graph G is hypomatchable then* $\operatorname{def}(G) = 1$ *and, hence, $|V(G)|$ is odd.*

1.23 Coloops and series-pairs

DEFINITION **119.** Let M be a matroid. We call e a *coloop* of M if $r(E - \{e\}) < r(M)$. We call $\{e, f\}$ a *series-pair* if neither e nor f is a coloop, but $r(E - \{e, f\}) < r(M)$.

REMARK **120.** (1) Thus e is a coloop of M if and only if e is a loop of M^*.

(2) Thus a series-pair of M is a parallel pair of M^*.

Recall the first problem on the first assignment: If $\{e, f\}, \{f, g\}$ are series-pairs, then so is $\{e, g\}$.

Proof of Gallai's Lemma. Recall that $\operatorname{MSM}(G) = (V, \mathcal{I})$ where $\mathcal{I}$ is the set of matchable sets. Since each vertex is avoidable, $\operatorname{MSM}(G)$ has no coloops. For an edge $e = uv$ of G, $\nu(G - u - v) < \nu(G)$. Hence $\{u, v\}$ is a series pair of $\operatorname{MSM}(G)$. However G is connected, so by Assignment 1 Problem 1, each pair of vertices of G is a series-pair of $\operatorname{MSM}(G)$. Hence $\operatorname{def}(G) = 1$. ∎

Proof of Tutte-Berge Formula. For each $X \subseteq V(G)$,

$$\operatorname{def}(G) \geq \operatorname{def}(G - X) - |X| \geq \operatorname{odd}(G - X) - |X|.$$

It remains to find $X \subseteq V(G)$ for which equality holds. Choose $X \subseteq V(G)$ maximal such that

$$\operatorname{def}(G) = \operatorname{def}(G - X) - |X|.$$

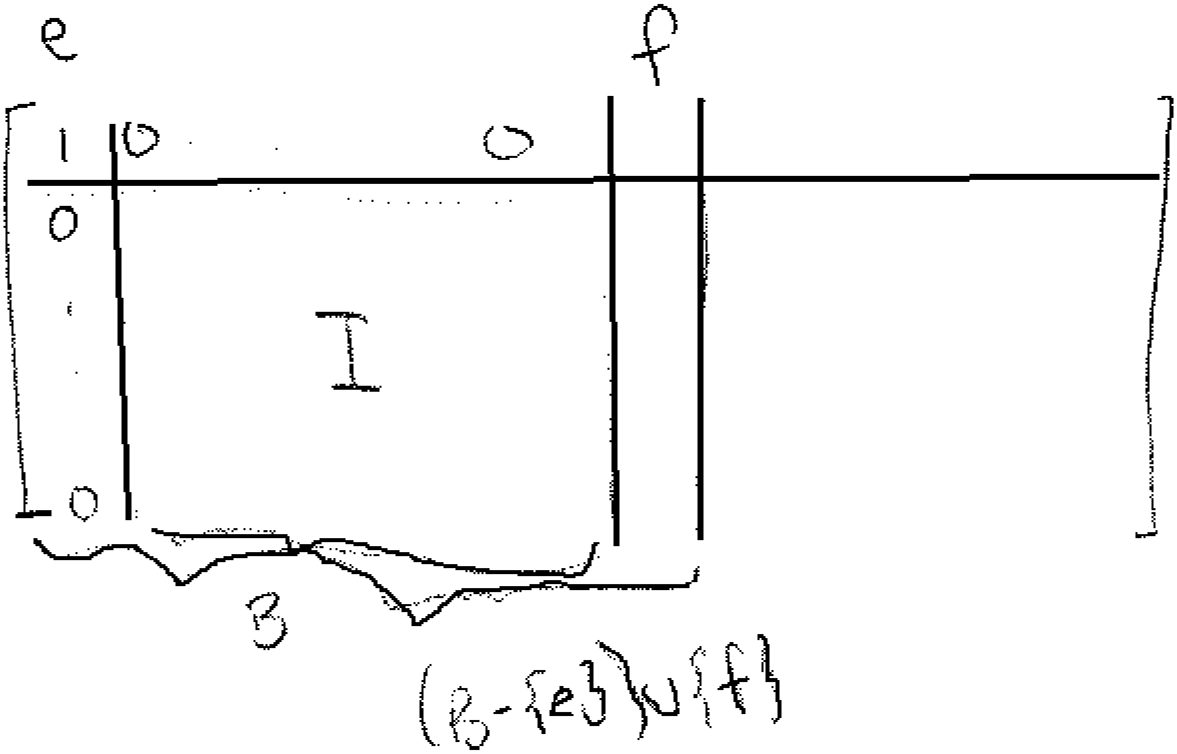
By our choice of X each component of $G - X$ is hypomatchable. Hence $\operatorname{def}(G) = \operatorname{odd}(G - X) - |X|$. ∎

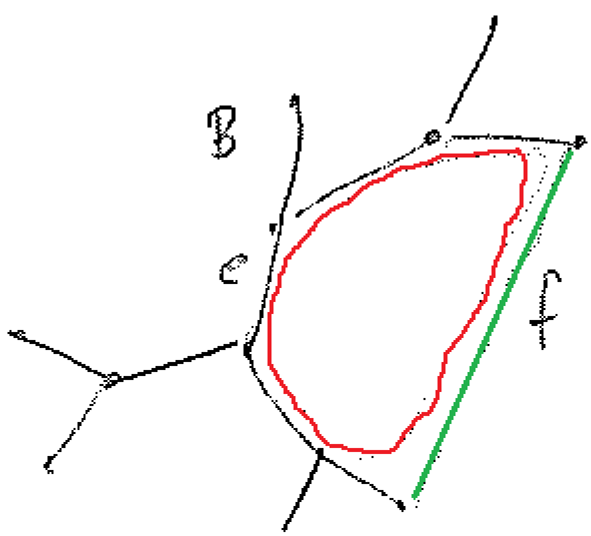
2 Binary Matroids

DEFINITION **121.** Let B be a basis of a matroid $M = (E, \mathcal{I})$. Define $F(M, B) \in \operatorname{GF}(2)^{B \times (E-B)}$ such that

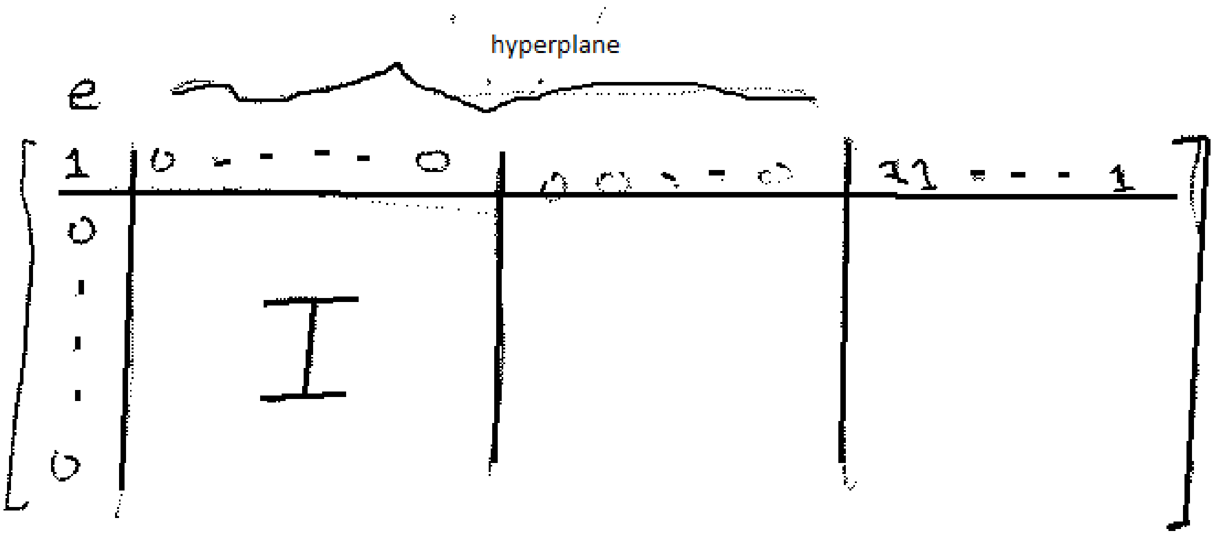
$$F(M, B)_{e,f} := \begin{cases} 1, & (B - \{e\}) \cup \{f\} \text{ is a basis} \\ 0, & \text{otherwise.} \end{cases}$$

We call $F(M, B)$ the *fundamental matrix* of (M, B).





Definition **122.** For $e \in B$, $E - cl_M(B - \{e\})$ is the unique cocircuit of M that is disjoint from $B - \{e\}$; we call $E - cl_M(B - \{e\})$ the *fundamental cocircuit* of e in (M, B).



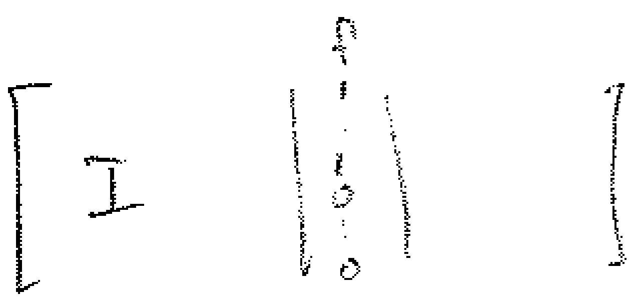
Exercise **123.** Prove that

$$E - cl_M(B - \{e\}) = \{e\} \cup \left\{f \in E - B;\ F(M,B)_{e,f} = 1\right\}.$$

Definition **124.** By duality, if $f \in E - B$ then there exists a unique circuit C in $B \cup \{f\}$; we call C the *fundamental circuit* of f in (M, B).

Remark **125.** By the exercise,

$$C = \{f\} \cup \left\{e \in B;\ F(M,B)_{e,f} = 1\right\}.$$



Lemma **126.** [2.1] *Let B be a basis of a binary matroid $M(A)$. Then*

$$\text{rowspace}(A) = \text{rowspace}\left(\begin{bmatrix} I & F(M(A),B)\end{bmatrix}\right).$$

Proof. Up to row operations, we may assume that

$$A = \begin{bmatrix} I & A' \end{bmatrix}.$$

For each $E \in B$ and $f \notin B$, $(B - \{e\}) \cup \{f\}$ is a basis of M if and only if $A'_{e,f} = 1$. Thus $A' = F$. ∎

LEMMA **127.** [2.2] *Let B be a basis of a matroid M. Then M is binary if and only if $M = M\big(\begin{bmatrix} I & F \end{bmatrix}\big)$.*

Proof. This follows from Lemma 2.1. ∎

LEMMA **128.** [2.3] *Suppose that $M(A_1)$ is a binary matroid and $M(A_1)^* = M(A_2)$. Then*

$$\text{rowspace}(A_1)^\perp = \text{rowspace}(A_2).$$

Proof. Note that $F(M(A_2), E - B) = F(M(A_1), B)^*$. By an assignment problem,

$$\text{rowspace}\big(\begin{bmatrix} I & F(M(A_1), B) \end{bmatrix}\big)^\perp = \text{rowspace}\big(\begin{bmatrix} F(M(A_1), B)^t & I \end{bmatrix}\big).$$

Hence, by Lemma 2.1,

$$\begin{aligned} \text{rowspace}(A_1)^\perp &= \text{rowspace}\big(\begin{bmatrix} I & F(M(A_1), B) \end{bmatrix}\big)^\perp \\ &= \text{rowspace}\big(\begin{bmatrix} F(M(A_1), B)^t & I \end{bmatrix}\big) \\ &= \text{rowspace}(A_2). \end{aligned}$$

∎

REMARK **129.** (1) Lemma 2.3 is a partial converse of Theorem 1.17.

(2) We see rowspace(A_1) and rowspace(A_2) are invariants of the binary matroid $M(A_1)$. That is, they are determined by the matroid and are independent of the representation. In particular, the following definition is well-defined.

2.1 Cycles and Cocyles

DEFINITION **130.** Let $M = M(A_1) = M(A_2)^*$ be a binary matroid. We call rowspace(A_1) the *cocycle space* of M and denote it by $\mathcal{C}^*(M)$. We call rowspace(A_2) the *cycle space* of M and denote it by $\mathcal{C}(M)$.

DEFINITION **131.** Let $M = M(A)$ be a binary matroid. We call $C \subseteq E(M)$ a *cycle* of M if the columns of $A|C$ sum to zero. Equivalently, $C \subseteq E(M)$ is a *cycle* of M if the characteristic vector of C is in $\mathcal{C}(M)$.

LEMMA **132.** [2.4] *In a binary matroid M, $C \subseteq E(M)$ is a cycle if and only if C is a disjoint union of circuits.*

Proof. We may assume that $C \neq \varnothing$. Now C is dependent, and, hence, C contains a circuit C'. Note that C' is itself a cycle and hence $C - C'$ is a cycle. Now the result follows inductively. ∎

DEFINITION **133.** Let $M = M(A)$ be a binary matroid. We call $C^* \subseteq E(M)$ a *cocycle* of $M(A)$ if C^* is a cycle of $M(A)^*$. Equivalently, $C^* \subseteq E(M)$ is a *cocycle* of $M(A)$ if the characteristic vector of C^* is in $\mathcal{C}^*(M)$.

2.2 Graphs

Definition **134.** We say a graph G is *even* if each vertex has even degree.

Remark **135.** Let G be a graph. Then $C^* \subseteq E(G)$ is a cocycle of $M(G)$ if and only if C^* is a cut of G, and $C \subseteq M(G)$ is a cycle of $M(G)$ if and only if $G[V, C]$ is even.

Remark **136.** By Lemma 2.3, if C is a cycle and C^* is a cocycle in a binary matroid, then $|C \cap C^*|$ is even. In particular, this holds when C is a circuit and C^* is a cocircuit.

Exercise **137.** Let C be a circuit and C^* be a cocircuit of a matroid M. Prove that $|C \cap C^*| \neq 1$.

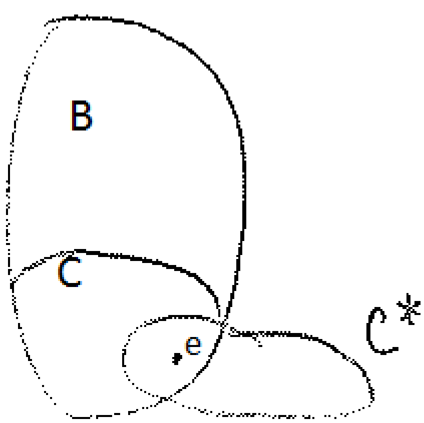
Lemma **138.** [2.5] *Let M, N be matroids on a common ground set E. If*

(1) each circuit of M is a circuit of N, and

(2) each cocircuit of M is a cocircuit of N, then

$M = N$.

Proof. Suppose that $M \neq N$. Then there is a circuit C of N that is independent in M. Let B be a basis of M containing C. Let $e \in C$, and let C^* be the fundamental cocircuit of e in (M, B).



By (2), C^* is a cocircuit of N. However $|C \cap C^*| = 1$ Contradiction. ∎

Theorem **139.** [2.6] *A matroid M is binary if and only if for each circuit C and cocircuit C^*, $|C \cap C^*|$ is even.*

Proof. If M is binary, C is a circuit of M, and C^* is a cocircuit of M, then by Lemma 2.3, $|C \cap C^*|$ is even.

Conversely, suppose that $|C \cap C^*|$ is even for each circuit C and cocircuit C^* of M.

$$A_1 = \begin{array}{cc} E & \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ & & & & & & & \\ & & & & & & & \end{bmatrix} & \mathcal{C}^* \end{array} \qquad A_2 = \begin{array}{cc} E & \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ & & & & & & & \\ & & & & & & & \end{bmatrix} & \mathcal{C} \end{array}$$

Let $\mathcal{C}$ denote the set of circuits of M and $\mathcal{C}^*$ denote the set of cocircuits of M. Let A_1 be the "$(\mathcal{C}^*, E)$-incidence matrix" and A_2 be the "$(\mathcal{C}, E)$-incidence matrix."

Claim. We have $M(A_1^*) = M(A_2)$.

Proof of Claim. The rows of A_1 are orthogonal to the rows of A_2; it suffices to prove that

$$\operatorname{rank}(A_1) + \operatorname{rank}(A_2) \geq |E|.$$

Let B be a basis of M. Now $\begin{bmatrix} I & F(M,B) \end{bmatrix}$ is a submatrix of A_1, so $\operatorname{rank}(A_1) \geq r(M)$. Similarly $\begin{bmatrix} F(M,B)^t & I \end{bmatrix}$ is a submatrix of A_2, so $\operatorname{rank}(A_2) \geq r(M^*) = |E| - r(M)$. Thus, $\operatorname{rank}(A_1) + \operatorname{rank}(A_2) \geq |E|$ as required. ∎

Now each circuit of M is a circuit of $M(A_1)$ and, by the claim, each circuit of M is a circuit of $M(A_1)$. By Lemma 2.5, $M = M(A_1)$. ∎

Theorem **140.** [2.7 - Tutte] *A matroid M is binary if and only if it has no $U_{2,4}$-minor.*

Proof. Let M be an excluded minor for the class of binary matroids. By Assignment 1, Problem 6, M is simple. By Lemma 1.2, if $r(M) \leq 2$, then $M \cong U_{2,4}$. Hence we may assume that $r(M) \geq 3$. By Theorem 2.6, there is a circuit C and cocircuit C^* with $|C \cap C^*|$ odd.

Claim (1). We have $C = C^*$.

Proof of Claim 1. Suppose not, then, up to duality, there exists $e \in C - C^*$. Now $C - \{e\}$ is a circuit in M/e and C^* is a cocircuit in M/e. Since $|C^* \cap (C - \{e\})|$ is odd, M/e is not binary — contradiction. ∎

Note that, since $r(M) \geq 3$,

$$|E - C^*| \geq r(M \setminus C^*) = r(M) - 1 \geq 2.$$

Let $e \in E - C^*$.

Claim (2). There is a cocircuit $\tilde{C}$ of $M \setminus e$ with $\tilde{C} \subseteq C^*$ and $|\tilde{C}|$ odd.

Proof of Claim 2. Let $f \in E - (C^* \cup \{e\})$. Now C^* is a cocircuit in M/f, and M/f is binary. So C^* is a cocycle in $M/f \setminus e$. By Lemma 2.4 and duality, we can partition C^* into cocircuits of $M/f \setminus e$; one of these, say $\tilde{C}$, is odd. Now $\tilde{C}$ is a cocircuit of $M/f \setminus e = (M \setminus e)/f$ and, hence, also of $M \setminus e$. ∎

Now C is a circuit in $M \setminus e$, $\tilde{C}$ is a cocircuit in $M \setminus e$, and $|C \cap \tilde{C}| = |\tilde{C}|$ is odd. Hence $M \setminus e$ is non-binary — contradiction. ∎

2.3 Affine Matroids

DEFINITION **141.** A binary matroid $M = (E, \mathcal{I})$ is *affine* if E is a cocycle of M.

REMARK **142.** Note that, for a graph G,

(1) $M(G)$ is affine if and only if G is bipartite, and

(2) $M(G)^*$ is affine if and only if G is even.

REMARK **143.** Given $A \in \mathrm{GF}(2)^{r\times E}$, $M(A)$ is affine if and only if there exists $x \in \mathrm{GF}(2)^r$ such that

$$x^t A = \mathbf{1}^t;$$

this can be checked efficiently.

THEOREM **144.** [2.8] *A binary matroid M is affine if and only if it has no odd-circuits.*

Proof. Suppose that M has an odd circuit C. Note that $M|C = M(G)$ where G is an odd circuit. Since G is not bipartite, $M(G) = M|C$ is not affine. Hence M is not affine.

Conversely suppose that M is not affine. Then there is no solution to

$$x^t A = \mathbf{1}^t.$$

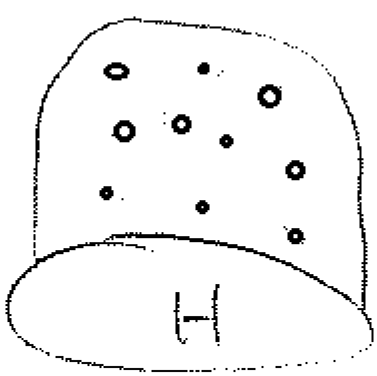
By the Fundamental Theorem of Linear Algebra, there exists $y \in \mathrm{GF}(2)^E$ such that

$$Ay = 0 \quad \text{and} \quad \mathbf{1}^t y = 1.$$

Let $C = \mathrm{support}(y)$. Since $Ay = 0$, C is a cycle of $M(A)$ and, since $\mathbf{1}^t y = 1$, $|C|$ is odd. By Lemma 2.4, C is a disjoint union of circuits, one of which must be odd. ∎

2.4 Affine Geometries

DEFINITION **145.** Let H be a hyperplane of $\mathrm{PG}(n-1,2)$. We denote $\mathrm{PG}(n-1,2) \setminus H$ by $\mathrm{AG}(n-2,2)$.

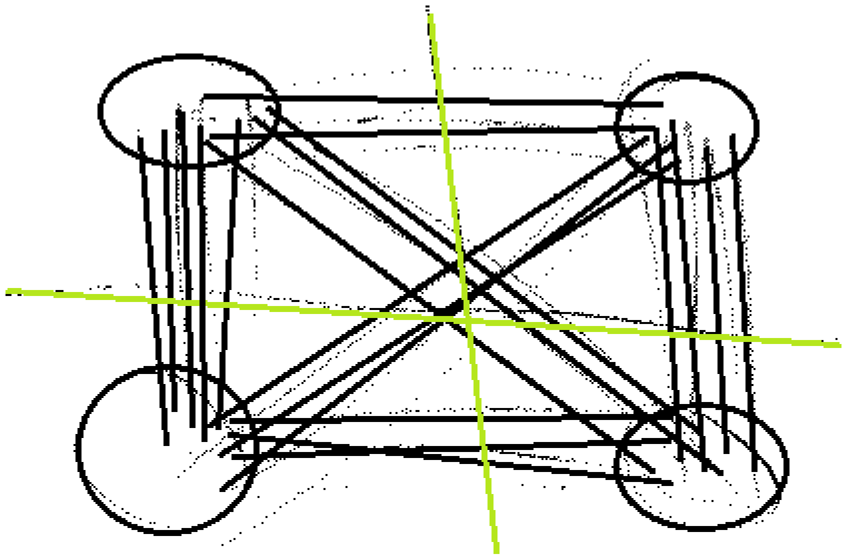


AG(r-1, 2)

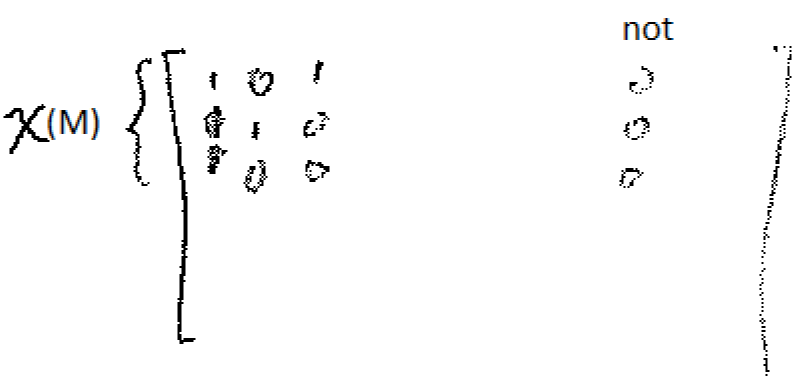
Exercise **146.** Let M be a simple rank-r binary matroid. Prove that M is affine if and only if M is isomorphic to a restriction of $AG(r-1,2)$.

2.5 Critical Number

Remark **147.** Note that G is 4-colourable if and only if there exist cocycles C_1, C_2 of $M(G)$ such that $E(G) = C_1 \cup C_2$.



Definition **148.** The *critical number*, $\chi(M)$, of a matroid M, is the minimum number of cocycles required to cover $E(M)$.



Remark **149.** This is well-defined if M has no loops, as in this case every element of M is in a cocycle and hence we may cover M with cocycles. Any loop of a matroid, on the other hand, is not contained in a cocycle, and hence we would not be able to cover M with cocycles.

Recall that $\chi(G)$ denotes the chromatic number of a graph G.

REMARK **150.** Note that:

(1) $\chi(G) \leq 2^k$ if and only if $\chi(M(G)) \leq k$;

(2) $\chi(M(G)) = \lceil \log_2(\chi(G)) \rceil$.

2.6 Bose-Burton Geometries

DEFINITION **151.** Let F be a rank-$(r-c)$ flat of $\mathrm{PG}(r-1,2)$. We denote $\mathrm{PG}(r-1,2) \setminus F$ by $\mathrm{BB}(r-1,2,c)$.

REMARK **152.** It follows immediately from the definitions that $\mathrm{BB}(r-1,2,1) = \mathrm{AG}(r-1,2)$.

EXERCISE **153.** Let M be a simple rank-r binary matroid. Prove that $\chi(M) \leq c$ if and only if M is isomorphic to a restriction of $BB(r-1,2,c)$.

2.7 Reformulations of the Four Colour Theorem

THEOREM **154.** *(1) If $G = (V,E)$ is a loopless planar graph, then E is the union of two cocycles of $M(G)$.*

(2) If $G = (V,E)$ is a bridgeless planar graph, then G is the union of two even subgraphs.

THEOREM **155.** [2.9] *If $M = (E, \mathcal{I})$ is a binary matroid, then there is a partition (C, C^*) of E into a cycle C and a cocycle C^*.*

Proof. Exercise. Hint: Suppose $M = M(A_1)$ and $M^* = M(A_2)$. Is $\mathbf{1}^t \in \mathrm{rowspace}\left(\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\right)$? Apply the Fundamental Theorem of Linear Algebra. ∎

COROLLARY **156.** *Every graph has a cut whose removal leaves an even subgraph.*

3 Extremal Matroid Theory

Much of what we do in this section will be motivated by, and a generalization of, graph theory.

3.1 Turan

THEOREM **157.** [TURAN'S THEOREM] *If $G = (V, E)$ is a simple n-vertex graph with no subgraph isomorphic to K_t, then*

$$|E| \leq \frac{t-2}{t-1}\binom{n}{2}.$$

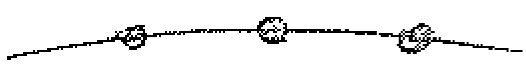
REMARK **158.** Note that, when $n = k(t-1)$, equality is attained by $K_{\underbrace{k,\ldots,k}_{t-1 \text{ times}}}$.

DEFINITION **159.** Let N be a simple binary matroid. A binary matroid M is *N-free* if no restriction of M is isomorphic to N. We let $\mathrm{ex}(N, r)$ denote the maximum number of elements in a simple N-free rank-r binary matroid. We denote $\mathrm{ex}(\mathrm{PG}(t-1,2), r)$ by $\mathrm{ex}(t, r)$.

PROBLEM **160.** Determine $\mathrm{ex}(t, r)$.

3.2 Triangle-free binary matroids

REMARK **161.** Note that $\mathrm{PG}(1,2) = U_{2,3} = M(K_3)$.



DEFINITION **162.** A 3-element circuit in a matroid is called a *triangle*.

REMARK **163.** Note that $\mathrm{AG}(r-1,2)$ has no odd-circuit, and so is triangle-free. Now

$$\begin{aligned}|\mathrm{AG}(r-1,2)| &= |\mathrm{PG}(r-1,2)| - |\mathrm{PG}(r-2,2)| \\ &= (2^r - 1) - \left(2^{r-1} - 1\right) \\ &= 2^{r-1}.\end{aligned}$$

Hence $\mathrm{ex}(2, r) \geq 2^{r-1}$.

THEOREM **164.** [3.1] *For any r, $\mathrm{ex}(2, r) = 2^{r-1}$.*

Proof. Let M be a simple rank-r triangle-free binary matroid considered as a restriction of $\mathrm{PG}(r-1,2)$. Let $e \in E(M)$. The number of lines of $\mathrm{PG}(r-1,2)$ that contain e is

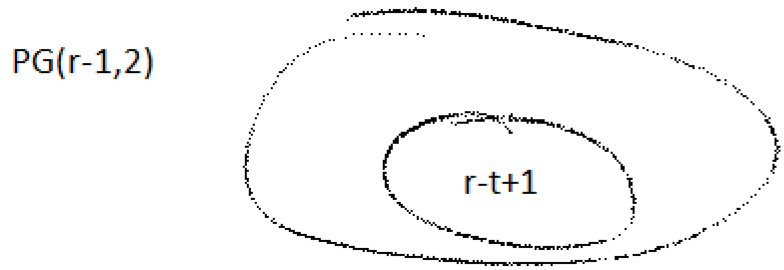
$$\tfrac{1}{2}(|\mathrm{PG}(r-1,2) - 1|) = 2^{r-1} - 1.$$

Each of these lines contains at most one other element of M, so

$$|M| \leq 1 + (2^{r-1} - 1) = 2^{r-1}.$$

∎

3.3 $PG(t-1,2)$-free binary matroids



Recall that $\mathrm{BB}(r-1,2,c)$ is obtained from $\mathrm{PG}(r-1,2)$ by deleting a rank-$(r-c)$ flat.

Lemma **165.** [3.2] *We have that* $\mathrm{BB}(r-1,2,t-1)$ *is* $\mathrm{PG}(t-1,2)$*-free.*

Proof. If F_1 is a rank-t flat in $\mathrm{PG}(r-1,2)$ and F_2 is a rank-$(r-t+1)$ flat in $\mathrm{PG}(r-1,2)$, then

$$\sqcap(F_1,F_2) = r(F_1) + r(F_2) - r(F_1 \cup F_2) \geq t + r - (t-1) - r = 1.$$

Recall $\sqcap(F_1,F_2) = \dim(F_1 \cap F_2)$. Hence $\mathrm{PG}(r-1,2) \setminus F_2$ is $\mathrm{PG}(t-1,2)$-free. ∎

Remark **166.** Now

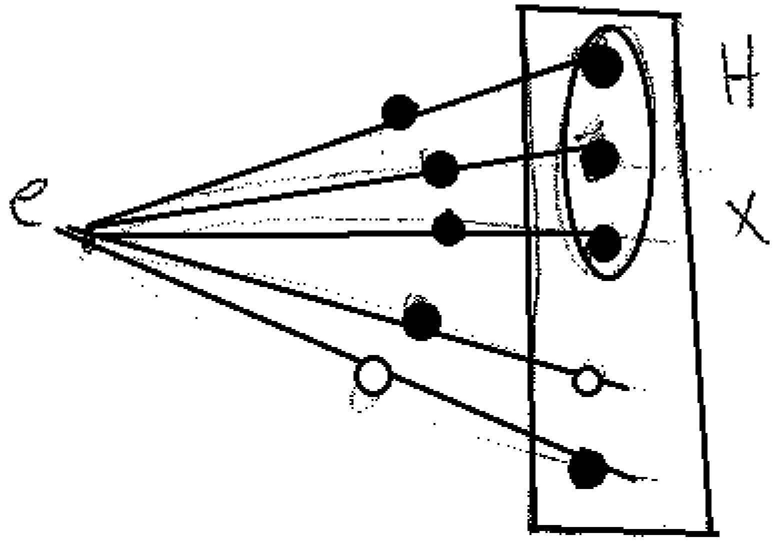
$$|\mathrm{BB}(r-1,2,t-1)| = |\mathrm{PG}(r-1,2)| - |\mathrm{PG}(r-t,2)| = (2^r-1) - \left(2^{r-t+1}-1\right) = \left(1 - \frac{1}{2^{t-1}}\right)2^r.$$

Hence $\mathrm{ex}(t,r) \geq \left(1-\frac{1}{2^{t-1}}\right)2^r$.

Theorem **167.** [Bose-Burton Theorem] *For* $r \geq t \geq 1$,

$$\mathrm{ex}(t,r) = \left(1-\frac{1}{2^{t-1}}\right)2^r.$$

Proof. We proceed by induction on t. When $t=1$, $\mathrm{ex}(1,r)=0$ so the result holds. Suppose that $t>1$ and that the result holds for $k<t$. Let M be a simple $\mathrm{PG}(t-1,2)$-free binary matroid considered as a restriction of $\mathrm{PG}(r-1,2)$.



Let $e \in E(M)$ and let H be a hyperplane of $\mathrm{PG}(r-1,2)$ that avoids e. Let X be the set of points $f \in H \cap E(M)$ such that e, f spans a triangle in M. Note that, since M is $\mathrm{PG}(t-1,2)$-free, $M|X$ is $\mathrm{PG}(t-2,2)$-free. By the induction hypothesis,

$$|X| \le \mathrm{ex}(t-1, r-1) = \left(1 - 1\frac{1}{2^{t-2}}\right)2^{r-1}.$$

The number of lines of $\mathrm{PG}(r-1,2)$ that contain e is $2^{r-1}-1$. So

$$|M| \le 1 + \left(2^{r-1}-1\right) + |X| \le 2^{r-1} + \left(1 - \frac{1}{2^{t-2}}\right)2^{r-1} = \left(1 - \frac{1}{2^{t-1}}\right)2^{r}.$$

∎

THEOREM **168.** [3.3 - BOSE-BURTON] *If M is a simple rank-r $\mathrm{PG}(t-1,2)$-free binary matroid with $|M| = \mathrm{ex}(t,r)$, then $M \cong \mathrm{BB}(r-1,2,t-1)$.*

Proof. Exercise. ∎

EXERCISE **169.** Let X be a set of points in $\mathrm{PG}(r-1,2)$ such that no line of $\mathrm{PG}(r-1,2)$ contains exactly two points of X. Prove that X is a flat of $\mathrm{PG}(r-1,2)$.

3.4 Ramsey

THEOREM **170.** [RAMSEY'S THEOREM] *For each t there exists an n such that in any colouring of the edges of K_n with 2 colours, we get a monochromatic copy of K_t.*

THEOREM **171.** [BIPARTITE RAMSEY THEOREM] *For each t there exists an n such that in any colouring of the edges of $K_{n,n}$ with 2 colours, we get a monochromatic copy of $K_{t,t}$.*

Theorem **172.** [Density Bipartite Ramsey Theorem] *For each integer t and $\epsilon > 0$ there exists an N such that for each $n \geq N$, if $G = (V, E)$ is an n-vertex simple bipartite graph with $|E| \geq \epsilon n^2$, then G has a subgraph isomorphic to $K_{t,t}$.*

Remark **173.** (1) The Density Bipartite Ramsey Theorem implies the Bipartite Ramsey Theorem.

(2) The Density Bipartite Ramsey Theorem is a "bipartite analogue" of Turan's Theorem.

(3) Turan's Theorem does not imply Ramsey's Theorem.

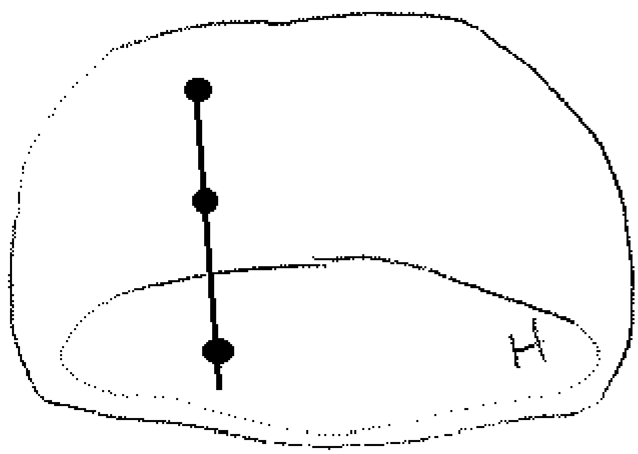
3.5 Hales-Jewett

Theorem **174.** [Geometric Density Hales-Jewett Theorem - Binary Case] *For each $\epsilon > 0$ and integer $t \geq 2$, there exists $R = \mathrm{DHJ}(t, \epsilon)$ such that for each integer $r \geq R$, if M is a simple rank-r $\mathrm{AG}(t-1, 2)$-free affine binary matroid, then $|M| \leq \epsilon 2^r$.*

Proof. We proceed by induction on t. The result is trivial when $t = 2$.

Now assume that for each $\epsilon' > 0$ there exists $\mathrm{DHJ}(t-1, \epsilon')$ such that, for each $r' \geq \mathrm{DHJ}(t-1, \epsilon')$ if N is a simple rank-r' $\mathrm{AG}(t-2, 2)$-free affine binary matroid, then $|N| \leq 2^{r'}$.

Let $R \geq \mathrm{DHJ}\left(t-1, \left(\frac{\epsilon}{2}\right)^2\right) + 1$ and be such that the claim below may be proven, and let M be a simple rank-r $\mathrm{AG}(t-1, 2)$-free affine binary matroid with $r \geq R$. Assume that $|M| > \epsilon 2^r$. Consider M as a restriction of $\mathrm{PG}(r-1, 2)$ and let H be a hyperplane which is disjoint from M.

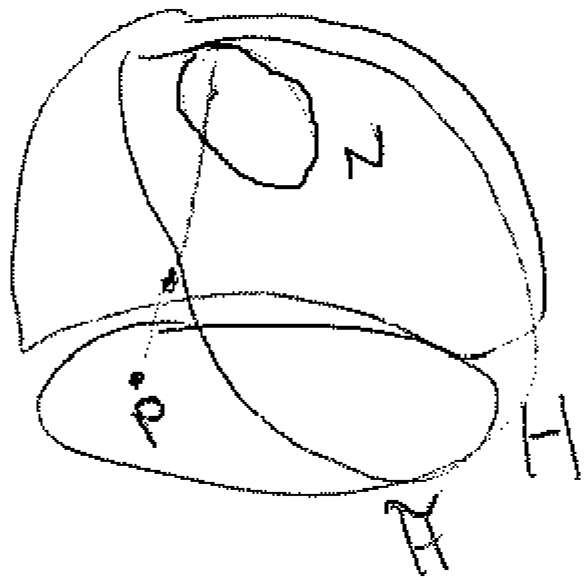


Let $\mathcal{L}$ be the set of all line of $\mathrm{PG}(r-1, 2)$ that contain two elements of M. Note that each line in $\mathcal{L}$ meets H.

Claim. There is a point $p \in H$ that is contained in more than $\left(\frac{\epsilon}{2}\right)^2 2^{r-1}$ lines in $\mathcal{L}$.

Proof of Claim. Exercise. ∎

Let $\tilde{H}$ be a hyperplane of $\mathrm{PG}(r-1, 2)$ that does not contain p, and let N be the restriction of $\tilde{H}$ to the points q such that $\{p, q\}$ spans a line in $\mathcal{L}$.



Now N is a simple affine binary matroid with rank at most $(r-1)$ such that $|N| > \left(\frac{\epsilon}{2}\right)^2 2^{r-1}$ and

$$r - 1 \geq \mathrm{DHJ}\Big(t-1, \left(\tfrac{\epsilon}{2}\right)^2\Big).$$

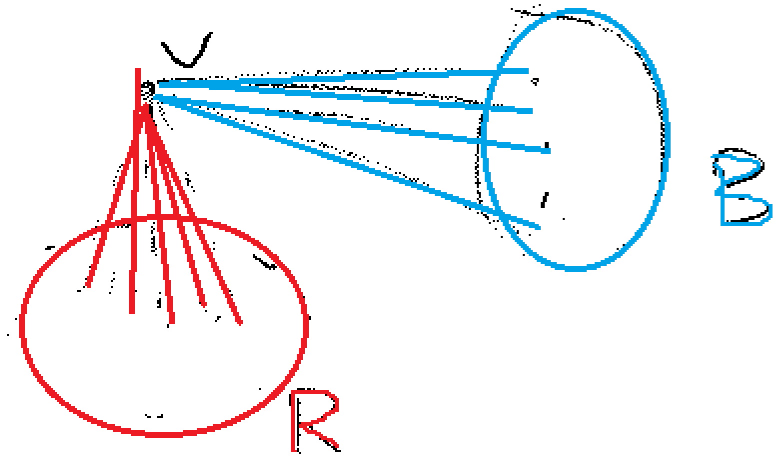
This means that N has an $\mathrm{AG}(t-2,2)$-restriction. Then M has an $\mathrm{AG}(t-1,2)$-restriction. ∎

THEOREM **175.** [GEOMETRIC HALES-JEWETT THEOREM] *For each $t \in \mathbb{Z}_+$ there exists $R = \mathrm{HJ}(t)$ such that, for each integer $r \geq R$ and for each 2-colouring of the elements of $\mathrm{AG}(r-1,2)$, there is a monochromatic copy of $\mathrm{AG}(t-1,2)$.*

Proof. Let $R = \mathrm{DHJ}(t, 1/4)$ and let $r \geq R$. Consider a partition (C_1, C_2) of $E(\mathrm{AG}(r-1,2))$ with $|C_1| \geq |C_2|$. Now $|C_1| \geq \frac{1}{4}r^2 = \frac{1}{2}|\mathrm{AG}(r-1,2)|$. Then, by the Geometric Density Hales-Jewett Theorem $\mathrm{AG}(r-1,2)|C_1$ contains a restriction isomorphic to $\mathrm{AG}(t-1,2)$. ∎

3.6 More Ramsey

THEOREM **176.** [RAMSEY'S THEOREM] *For each $r, b \in \mathbb{Z}_+$ there exists $R(r,b)$ such that for each $n \geq R(r,b)$ if we red/blue colour the edges of K_n then we will get a red K_r or a blue K_b.*



Proof. We proceed by induction on $r + b$. The result is trivial when $r = 1$ or $b = 1$.

Assume that if $r', b' \in \mathbb{Z}^+$ with $r' + b' < r + b$ then there exists $R(r', b')$ such that, for each $n \geq R(r', b')$ if we red/blue colour the edges of K_n then we get either a red $K_{r'}$ or a blue $K_{b'}$. Let $R(r, b) = R(r-1, b) + R(r, b-1) + 1$. Consider a red-blue colouring of the edges of K_n where $n \geq R(r, b)$. Let $v \in V(K_n)$. Let $R = \{w \in V(K_n) - \{v\}\,;\, vw \text{ is red}\}$ and $B = \{w \in V(K_n) - \{v\}\,;\, vw \text{ is blue}\}$. Assume that G has no red K_r or blue K_b. Then

(1) $G[R]$ has no red K_{r-1} or blue K_b, and

(2) $G[B]$ has no red K_r or blue K_{b-1}.

Thus inductively we have

$$n \geq (r-1, b) + R(r, b-1) + 1 > |R| + |B| + 1 = n,$$

which is a contradiction. ∎

Theorem **177.** [Geometric Ramsey Theorem] *For each* $b, r \in \mathbb{Z}_+$ *there exists* $\mathrm{GR}(b, r)$ *such that for each* $n \geq \mathrm{GR}(b, r)$*, in any blue/red colouring of the elements of* $\mathrm{PG}(n-1, 2)$ *we will find a blue* $\mathrm{PG}(b-1, 2)$ *or a red* $\mathrm{PG}(r-1, 2)$.

Proof. Exercise. Hint: Consider a cocircuit C of $\mathrm{PG}(n-1, 2)$ and apply the Geometric Hales-Jewett Theorem to $\mathrm{PG}(n-1, 2)|C$. ∎

3.7 Erdös-Stone

Definition **178.** For a simple graph H, we let $\mathrm{ex}(H, n)$ be the maximum number of edges in a simple n-vertex H-free graph.

Remark **179.** Let H be a simple graph with chromatic number at least two. If G has chromatic number less than $\chi(H)$, then G is H-free. Hence, if n is divisible by $\chi(H)-1$, then

$$\mathrm{ex}(H,n) > \left(\frac{\chi(H)-2}{\chi(H)-1}\right)\binom{n}{2}.$$

Theorem **180.** [Erdös-Stone Theorem - 1946] *If H is a simple graph with chromatic number at least two, then*

$$\lim_{n\to\infty}\frac{\mathrm{ex}(H,n)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}.$$

Remark **181.** Let N be a simple binary matroid. If M is a binary matroid with $\chi(M) < \chi(N)$, then M is N-free. So

$$\mathrm{ex}(N,r) \geq |\mathrm{BB}(r-1,2,\chi(N)-1)| = \left(1-2^{1-\chi(N)}\right)2^r.$$

Theorem **182.** [Geometric Erdös-Stone Theorem - 2014] *If N is a simple binary matroid then*

$$\lim_{r\to\infty}\frac{\mathrm{ex}(N,r)}{2^r} = 1-2^{1-\chi(N)}.$$

Remark **183.** Note that:

(1) When $\chi(N)=1$ (that is, when N is affine), $1-2^{1-\chi(N)}=0$.

(2) When $\chi(N)\geq 2$, the theorem gives the asymptotic behaviour of $\mathrm{ex}(N,r)$.

Lemma **184.** [3.4] *If M is a loopless binary matroid then M has an affine restriction N with $|N| \geq \frac{1}{2}|M|$.*

Proof. Let $M = M(A)$ where $A \in \mathrm{GF}(2)^{r\times E}$. Choose $x \in \mathrm{GF}(2)^r$ uniformly at random and consider the cocycle C^* whose characteristic vector is x^tA. Note that for each $e \in E(M)$,

$$\mathrm{Prob}(e \in C^*) = \tfrac{1}{2}.$$

By linearity of expectation, the expected size of C^* is $\frac{1}{2}|E|$.

So there exists a cocycle C with size $\geq \frac{1}{2}|E|$. By definition, $M|C$ is affine. ∎

Remark **185.** For a loopless graph G, Lemma 3.4 shows that G has a cut of size at least $\frac{1}{2}|E(G)|$.

Problem **186.** Given a loopless binary matroid $M = M(A)$, can we find a cocycle C with $|C| \geq \frac{1}{2}|M|$ in polynomial-time?

Theorem 187. [3.5] *For each $m \geq 0$ and $\epsilon > 0$ there exists* GES$(m, 1, \epsilon)$ *such that*

$$\mathrm{ex}(\mathrm{AG}(m-1,2), r) < \epsilon 2^r$$

for all $r \geq \mathrm{GES}(m, 1, \epsilon)$.

Proof. Let $\mathrm{GES}(m,1,\epsilon) = \mathrm{DHJ}(m, \epsilon/2)$. Let M be a simple rank-r matroid with $|M| \geq \epsilon 2^r$ and $r \geq \mathrm{GES}(m1, 1\epsilon)$. By Lemma 3.4, M has an affine restriction N with $|N| \geq \frac{\epsilon}{2} 2^r$. By the Geometric Density Hales-Jewett Theorem, N has a restriction isomorphic to $\mathrm{AG}(m-1, 2)$. ∎

Remark 188. Note that Theorem 3.5 implies the Geometric Erdös-Stone Theorem in the case that N is affine (that is, $\chi(N) = 1$).

Theorem 189. [3.6] *For each $m \geq 0$ and $\epsilon > 0$ there exists* GES$(m, 2, \epsilon)$ *such that*

$$\mathrm{ex}(\mathrm{BB}(m-1,2,2), r) \leq \left(\tfrac{1}{2} + \epsilon\right) 2^r$$

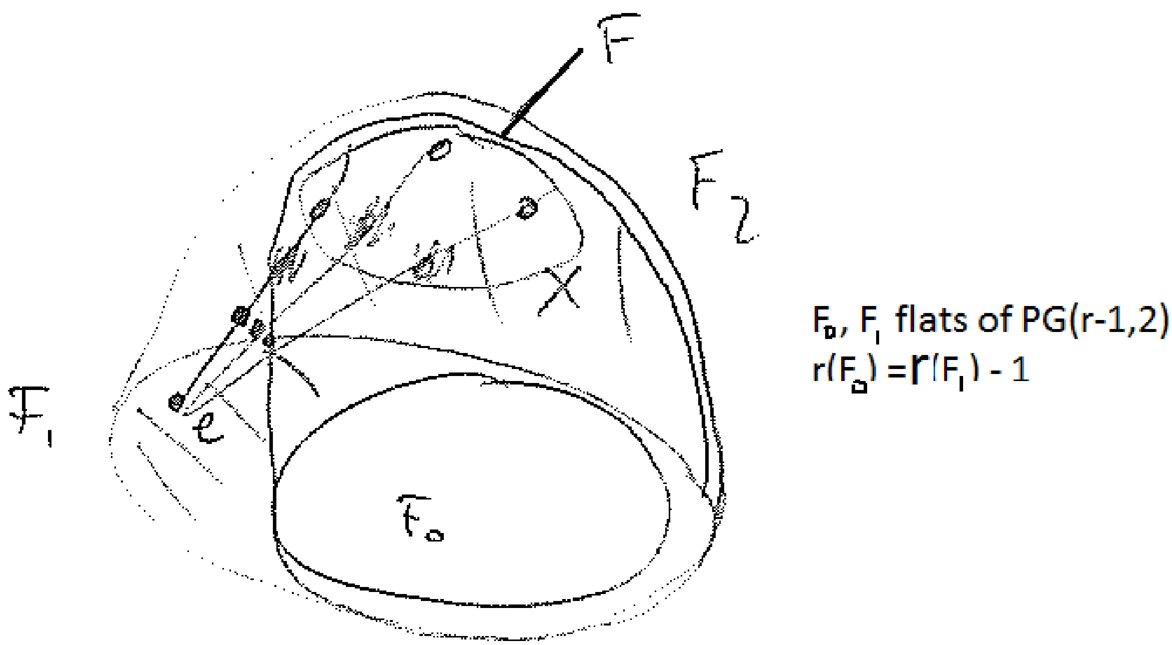
for all $r \geq \mathrm{GES}(m, 2, \epsilon)$.

Proof. Let $\epsilon > 0$ and $m \in \mathbb{Z}_+$.

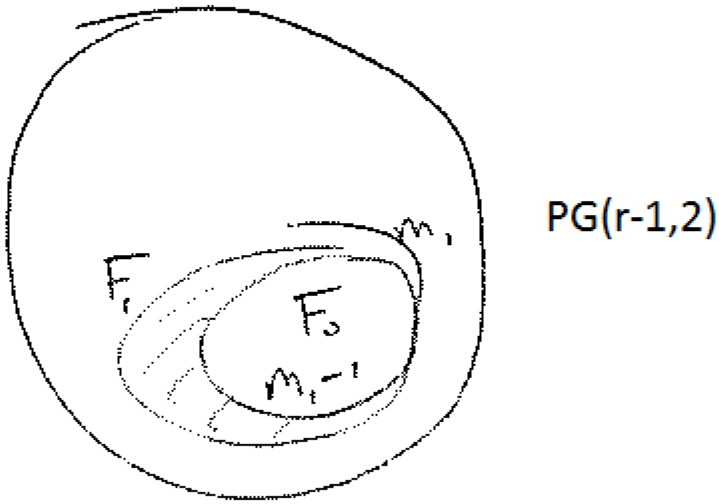
(1) Let $m_1 = \mathrm{DHJ}\left(m-1, \frac{\epsilon}{2}\right)$

(2) Choose m_2 minimum such that $2^{m_1} \leq \frac{\epsilon}{2} 2^{m_2}$.

Now let $\mathrm{GES}(m, 2, \epsilon) = \max\left(m_2, \mathrm{GES}\left(m_1, 1, \frac{1}{2}\right)\right)$. Let M be a simple rank-r binary matroid with $|M| \geq \left(\frac{1}{2} + \epsilon\right) 2^r$ and $r \geq \mathrm{GES}(m, 2, \epsilon)$. Consider M as a restriction of $\mathrm{PG}(r-1, 2)$. Note that $r \geq \mathrm{GES}\left(m_1, 1, \frac{1}{2}\right)$, so, by Theorem 3.5, M has a restriction isomorphic to $\mathrm{AG}(m, -1, 2)$.



So there exist flats $F_0 \subseteq F_1$ in $\mathrm{PG}(r-1,2)$ such that $r(F_1) = m_1$, $r(F_0) = m_1 - 1$, and $F_1 - F_0 \subseteq E(M)$. Let $M' = M \setminus (F_1 \cap E(M))$. By (2), $|F_1 \cap E(M)| \le \frac{\epsilon}{2} 2^r$, so $|M'| \ge \left(\frac{1}{2} + \frac{\epsilon}{2}\right) 2^r$.



Claim. There is a rank-(m_1+1) flat F_2 with $F_1 \subseteq F_2$, such that $|F_2 \cap M'| \ge \left(\frac{1}{2} + \frac{\epsilon}{2}\right) 2^{m_1}$.

Proof of Claim. Exercise. ∎

Note that $|F_2 - F_1| = 2^{m_1}$. Let $e \in F_1 - F_0$, let F be a rank-m_1 flat with $F_0 \subseteq F \subseteq F_2 - \{e\}$. Let $\mathcal{L}$ denote the lines containing and a point of $F - F_0$. Note that $|\mathcal{L}| = 2^{m_1 - 1}$. Let $Z \subseteq F - F_0$ be the set of points f that the line spanned by $\{e, f\}$ is contained in $E(M)$.

Now

$$\left(\frac{1}{2} + \frac{\epsilon}{2}\right) 2^{m_1} \le |F_2 \cap M'| \le |Z| + |\mathcal{L}| = |Z| + \frac{1}{2} 2^{m_1}.$$

Hence $|Z| \ge \frac{\epsilon}{2} 2^{m_1}$.

Let $N = M|Z$. Now N is a rank-$(\le m_1)$ affine matroid with $|N| \ge \frac{\epsilon}{2} 2^{m_1}$. Moreover, $m_1 = \mathrm{DHJ}\left(m-1, \frac{\epsilon}{2}\right)$. Hence N has a restriction N' isomorphic to $\mathrm{AG}(m-2,2)$. Let F' be the flat of M' spanned by N' and let F'' be the rank-m flat spanned by $F' \cup \{e\}$. Now $F'' - F_0 \subseteq E(M)$ and $r(F' \cap F_0) = m-2$, so $M|(F'' - F_0) \cong \mathrm{BB}(m-1,2,2)$. ∎

Remark **190.** Note that, if N is a simple rank-m binary matroid with $\chi(N) = 2$, then N is isomorphic to a restriction of $\mathrm{BB}(m-1,2,2)$. So Theorem 3.6 implies the Geometric Erdös-Stone Theorem in the case that $\chi(N) = 2$.

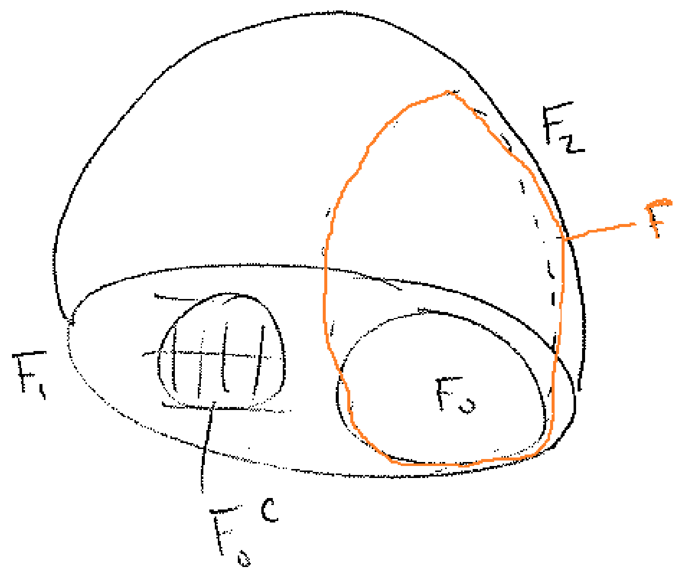
Theorem **191.** [3.7] *For each $1 \le t < m$ and $\epsilon > 0$, there exists $\mathrm{GES}(m,t,\epsilon)$ such that*

$$\mathrm{ex}(\mathrm{BB}(m-1,2,t), r) \le \left(1 - 2^{1-t} + \epsilon\right) 2^r$$

for all $r \ge \mathrm{GES}(m,t,\epsilon)$.

Proof. Exercise. ∎

$$r(F_1) = m_1$$
$$r(F_0) = m_1 - t + 1$$
$$r(F_2) = m_1 + 1$$
$$r(F_0^c) = t - 1$$
$$r(F) = m_1 - t + 2$$



4 Matroid Union and Partition

PROBLEM **192.** [MATROID PARTITION PROBLEM] Let $M_1 = (E, \mathcal{I}_1)$, $M_2 = (E, \mathcal{I}_2)$ be matroids. Determine

$$\max\{|I_1 \cup I_2|;\, I_1 \in \mathcal{I}_1,\, I_2 \in \mathcal{I}_2\}.$$

EXAMPLE **193.** If we can answer the Matroid Partition Problem, then we can answer the following question: Given a graph $G = (V, E)$ does G have two disjoint spanning trees? Indeed, the answer is yes if and only if the maximum above is $2r(M(G))$, where M_1 and M_2 are both taken to be $M(G)$.

DEFINITION **194.** Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be matroids. Define $M_1 \cup M_2 := (E, \mathcal{I})$ where

$$\mathcal{I} := \{I_1 \cup I_2;\, I_1 \in \mathcal{I}_1,\, I_2 \in \mathcal{I}_2\}.$$

An element of $\mathcal{I}$ is called *partitionable.*

THEOREM **195.** [4.1] *If $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ are matroids then $M_1 \cup M_2$ is a matroid.*

We need to prove some preliminary results before we can prove Theorem 4.1.

4.1 Direct Sum

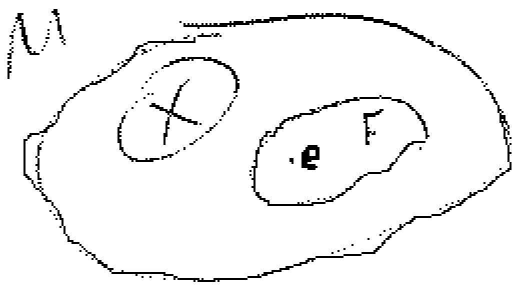
DEFINITION **196.** Let $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ be matroids with $E_1 \cap E_2 = \varnothing$. Define $M_1 \oplus M_2 := (E_1 \cup E_2, \mathcal{I}')$ where

$$\mathcal{I}' := \{I_1 \cup I_2;\, I_1 \in \mathcal{I}_1,\, I_2 \in \mathcal{I}_2\}.$$

Lemma **197.** [4.2] *If $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ are matroids with $E_1 \cap E_2 = \varnothing$ then $M_1 \oplus M_2$ is a matroid.*

Proof. Trivial. ∎

4.2 Extending freely into a flat.



Definition **198.** Let F be a flat of a matroid $M = (E, \mathcal{I})$ and let $e \notin E$. Now define $M' := (E \cup \{e\}, \mathcal{I}')$ such that

(1) $M' \setminus e = M$ and

(2) for $X \subseteq E$,

$$r_{M'}(X \cup \{e\}) := \begin{cases} r_M(X), & F \subseteq \mathrm{cl}_M(X) \\ r_M(X) + 1, & \text{otherwise.} \end{cases}$$

We say that M' is obtained from M by *freely extending* into the flat F.

Lemma **199.** [4.3] *If F is a flat of a matroid M and $e \notin E(M)$ then M' is a matroid.*

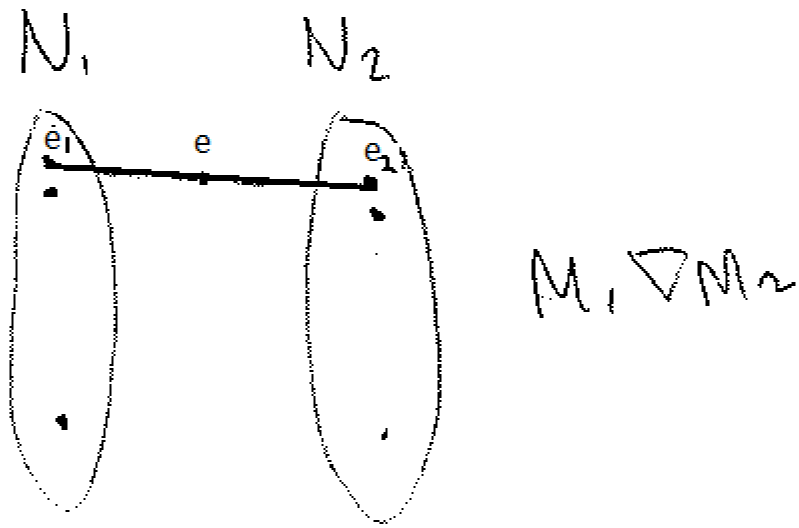
Proof. Exercise. ∎

Definition **200.** Let M be a matroid and let $e, f \in E$ be distinct. We say that e is *freer* than f if for each circuit C of M containing e we have $f \in \mathrm{cl}_M(C)$.

Remark **201.** Note that in the above construction, for each $f \in F$, e is *freer* then f.

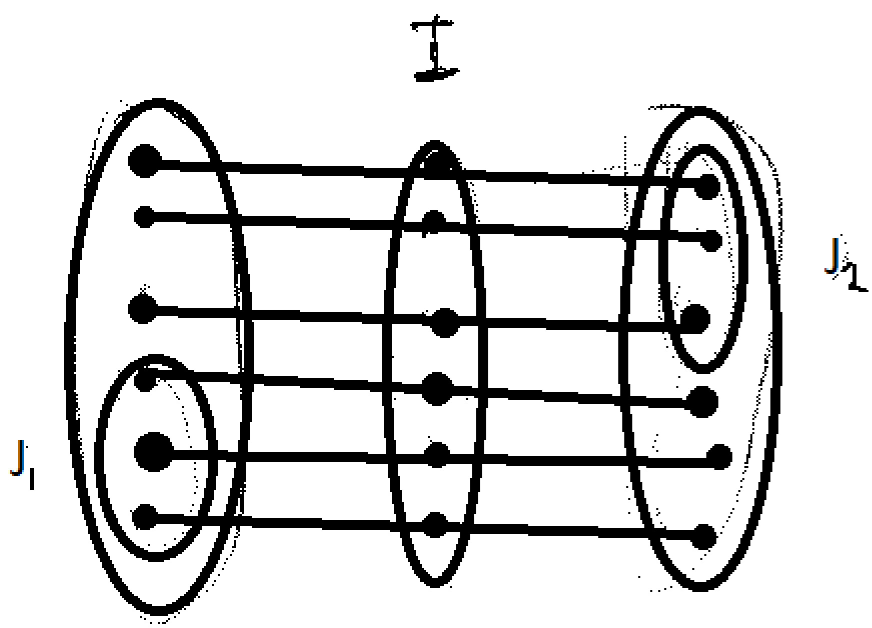
4.3 Constructing $M_1 \cup M_2$

Definition **202.** Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be matroids. Take copies $N_1 = (E_1, \mathcal{I}_1)$ and $N_2 = (E_2, \mathcal{I}_2)$ of M_1 and M_2, respectively, where E, E_1, E_2 are pairwise disjoint. For each $e \in E$, let e_1 and e_2 denote the copies of e in N_1 and N_2 respectively. Now construct $M_1 \nabla M_2$ from $N_2 \oplus N_2$ by, for each element $e \in E$, freely extending into the line $\{e_1, e_2\}$.



Lemma **203.** [4.4] *If* $M_1 = (E, \mathcal{I}_1)$ *and* $M_2 = (E, \mathcal{I}_2)$ *are matroids then* $(M_1 \nabla M_2)|E = M_1 \cup M_2$.

Proof. Let I be an independent set of $M_1 \cup M_2$; let (I_1, I_2) be a partition of I into an M_1-independent set I_1 and an M_2-independent set I_2; let J_1, J_2 be the copies of I_1, I_2 in N_1, N_2 respectively.



Hence $J_1 \cup J_2$ is independent in $M_1 \nabla M_2$. For each $e \in I$ we will (one element at a time) replace the copy (e_1 or e_2) of e in $J_1 \cup J_2$ with e; the resulting set will remain independent since e is freer than e_1 and e_2. Therefore I is $(M_1 \nabla M_2)$-independent.

Conversely, consider an $(M_1 \nabla M_2)$-independent set $I \subseteq E$. It suffices to show that there is a partition (I_1, I_2) of I such that $\{e_1;\, e \in \mathcal{I}_1\} \cup \{e_2;\, e \in \mathcal{I}_2\}$ is $(M_1 \nabla M_2)$-independent. We will find the partition by replacing each element $e \in E$, one at a time, with e_1 or e_2.

Since e is spanned by $\{e_1, e_2\}$, $I - \{e\}$ does not span both e_1 and e_2. Hence either $(I - \{e\}) \cup \{e_1\}$ or $(I - \{e\}) \cup \{e_2\}$ is $(M_1 \nabla M_2)$-independent. ∎

Proof of Theorem 4.1. Lemma 4.4 implies Theorem 4.1, because we know $(M_1 \nabla M_2)|E$ is a matroid. ∎

Lemma **204.** [4.5] *Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be matroids. For $A \subseteq E$,*

$$r(M_1 \cup M_2) \le |E - A| + r_{M_1}(A) + r_{M_2}(A).$$

Proof. Note that $(M_1 \cup M_2)|A = (M_1|A) \cup (M_2|A)$. So

$$\begin{aligned} r(M_1 \cup M_2) &\le |E - A| + r((M_1 \cup M_2)|A) \\ &= |E - A| + r((M_1|A) \cup (M_2|A)) \\ &\le |E - A| + r(M_1|A) + r(M_2|A) \\ &= |E - A| + r_{M_1}(A) + r_{M_2}(A). \end{aligned}$$

∎

Theorem **205.** [Matroid Partition Theorem* - Edmonds] *Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be matroids. then*

$$r(M_1 \cup M_2) = \min_{A \subseteq E} \left(|E - A| + r_{M_1}(A) + r_{M_2}(A) \right).$$

(*Here we stated the version for two matroids, while the general theorem applies to any finite number of matroids.)

Lemma **206.** [4.6] *Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be matroids. If $M_1 \cup M_2$ has no coloops then*

$$r(M_1 \cup M_2) = r(M_1) + r(M_2).$$

Proof. Consider the construction of $M_1 \nabla M_2$. For $S \subseteq E$, let $\tilde{S} = (E - S) \cup \{e_1;\, e \in S\} \cup \{e_2;\, e \in S\}$. Then $\tilde{E} = E_1 \cup E_2$, which has rank $r(M_1) + r(M_2)$ in $M_1 \nabla M_2$, and $\tilde{\varnothing} = E$.

For a contradiction, suppose that $r_{M_1 \nabla M_2}(\tilde{\varnothing}) < r(M_1) + r(M_2)$. Choose $S \subseteq E$ minimal such that $r(\tilde{S}) = r(M_1) + r(M_2)$ and let $e \in S$. Let $T = S - \{e\}$.

First suppose that e is a coloop of $(M_1 \nabla M_2)|\tilde{T}$. Then e is a coloop of $(M_1 \nabla M_2)|E$. But $(M_1 \nabla M_2)|E = M_1 \cup M_2$ — contradiction.

So it must be the case that e is not a coloop of $(M_1 \nabla M_2)|\tilde{T}$. Then $\tilde{S} - \{e_1, e_2\}$ spans e. However, e is freer than $\{e_1, e_2\}$, so $\tilde{S} - \{e_1, e_2\}$ spans e_1 and e_2. So

$$r_{M_1 \nabla M_2}(\tilde{S} - \{e_1, e_2\}) = r_{M_1 \nabla M_2}(\tilde{S}) = r(M_1) + r(M_2).$$

But then

$$r_{M_1 \nabla M_2}(\tilde{T}) = r_{M_1 \nabla M_2}(\tilde{T} - \{e\}) = r_{M_1 \nabla M_2}(\tilde{S} - \{e_1, e_2\}) = r(M_1) + r(M_2)$$

— contradiction. ∎

THEOREM **207.** [4.7] *If X is the set of coloops of $M_1 \cup M_2$, then*

$$r(M_1 \cup M_2) = |X| + r_{M_1}(E - X) + r_{M_2}(E - X)\,.$$

Proof. Note that $(M_1 \cup M_2) \setminus X$ has no coloops and $r(M_1 \cup M_2) = |X| + r((M_1 \cup M_2) \setminus X)$. Moreover $(M_1 \cup M_2) \setminus X = (M_1 \setminus X) \cup (M_2 \setminus X)$. By Lemma 4.6,

$$r(M_1 \setminus X \cup (M_2 \setminus X)) = r(M_1 \setminus X) + r(M_2 \setminus X) = r_{M_1}(E_X) + r_{M_2}(E - X)\,.$$

The result follows. ∎

REMARK **208.** Note that Theorem 4.7 implies the Matroid Partition Theorem, which we now state in full generality.

THEOREM **209.** [MATROID PARTITION THEOREM - EDMONDS] *If $M_1, \ldots, M_k$ are matroids on a common ground set E then*

$$r(M_1 \cup \cdots \cup M_k) = \min_{A \subseteq E}(|E - A| + r_{M_1}(A) + \cdots + r_{M_k}(A))\,.$$

Proof. Exercise. ∎

4.4 Matroid Algorithms (Oracle Model)

DEFINITION **210.** A matroid $M = (E, \mathcal{I})$ is *given* by E and access to an oracle for computing $r_M(X)$ for any given set $X \subseteq E$. Calls to the rank oracle take one time unit.

THEOREM **211.** [4.8 - EDMONDS, 1965] *There is an algorithm that, given matroids $M_1, \ldots, M_k$ on a common ground set E, computes $r(M_1 \cup \cdots \cup M_k)$ in $O\big(k^3 |E|^3\big)$-time.*

Proof. Do CO 450/650. ∎

4.5 Packing Bases

PROBLEM **212.** Given a matroid M and an integer k, does M have k disjoint bases?

REMARK **213.** (1) A matroid M has k disjoint bases if and only if $r(\underbrace{M \cup \cdots \cup M}_{k \text{ copies}}) = kr(M)$.

(2) If B is a basis of M and $e \in E(M)$, then B contains a basis of M/e.

(3) For $X \subseteq E(M)$, if M has k disjoint bases then so does M/X.

(4) If M has k disjoint bases, then $|M| \geq kr(M)$.

THEOREM **214.** [4.9 - TUTTE, NASH-WILLIAMS] *A matroid $M = (E, \mathcal{I})$ has k disjoint bases if and only if $|M/X| \geq kr(M/X)$ for each $X \subseteq E$.*

Proof. If M has k disjoint bases, then by (2) and (3) above, $|M/X| \geq kr(M/X)$ for each $X \subseteq E(M)$.

Conversely, suppose that M does not have k disjoint bases. Then, by (1) above,

$$r(\underbrace{M \cup \cdots \cup M}_{k \text{ copies}}) < kr(M).$$

By the Matroid Partition Theorem, there exists $X \subseteq E$ such that

$$kr(M) > |E - X| + kr_M(X).$$

Thus

$$|M/X| = |E - X| < k(r(M) - r_M(X)) = kr(M/X).$$

∎

Corollary **215.** [4.10] *If G is $(2k)$-edge-connected, then G has k edge-disjoint spanning trees.*

Proof. If G is $(2k)$-edge-connected, then G/X is $(2k)$-edge-connected for each $X \subseteq E(G)$. So, by Theorem 4.9, it suffices to prove that, if H is a $(2k)$-edge-connected graph, then

$$|E(H)| \geq k(|V(H)| - 1)\,.$$

Now

$$|E(H)| = \tfrac{1}{2} \sum_{v \in V(H)} \deg_H(v) \geq \tfrac{1}{2} \sum_{v \in V(H)} 2k = k\,|V(H)|\,,$$

as required. ∎

4.6 Covering with independent sets

Problem **216.** Given a matroid $M = (E, \mathcal{I})$ and $k \in \mathbb{Z}$, can we cover E with k independent sets?

Example **217.** Can we cover the edges of a graph G with k forests? This question can be resolved by applying the answer to the above problem to $M(G)$.

Remark **218.** (1) We can cover E with k independent sets if and only if

$$r(\underbrace{M \cup \cdots \cup M}_{k \text{ copies}}) = |E|\,.$$

(2) If we can cover M with k independent sets then we can cover $M|A$ with k independent sets for each $A \subseteq E$.

(3) If we can cover M with k independent sets, then $|M| \le kr(M)$.

Theorem **219.** [4.11] *A matroid M can be covered with k independent sets if and only if $|A| \le kr(A)$ for each $A \subseteq E$.*

Proof. If M can be covered by k independent sets, then, by (2) and (3) above, $|A| \le kr(A)$, for each $A \subseteq E(M)$.

Conversely, suppose that M cannot be covered by k independent sets. Then, by (1),

$$r(\underbrace{M \cup \cdots \cup M}_{k \text{ copies}}) < |E|.$$

By the Matroid Partition Theorem, there exists $A \subseteq E$ such that

$$|E| > r(\underbrace{M \cup \cdots \cup M}_{k \text{ copies}}) = |E - A| + kr_M(A).$$

Thus $kr_M(A) < |A|$. ∎

Corollary **220.** [4.12] *Each simple planar graph can be covered by three forests.*

Proof. By Theorem 4.11, it suffices to prove that, if H is a simple planar graph, then $|E(H)| \le 3r(M(H))$. By possibly identifying vertices, we may assume that H is connected. We may also assume that $|V(H)| \ge 3$. Then

$$|E(H)| \le 3(|V(H)| - 6) = 3r(M(H)) - 3 < 3r(M(H)),$$

as required. ∎

Remark **221.** Since forests are bipartite, Corollary 4.12 implies that, if G is a loopless planar graph, then

$$\chi(G) \le 8.$$

Corollary **222.** [4.13] *Let $\mathcal{M}$ be a class of binary matroids that is closed under taking restrictions and let $k \in \mathbb{Z}_+$ such that, for each simple matroid $N \in \mathcal{M}$,*

$$|N| \le kr(N).$$

Then, for each loopless $M \in \mathcal{M}$, $\chi(M) \le k$.

Proof. Let $M \in \mathcal{M}$ be loopless and let $\bar{M}$ be the simplification of M. Then $\chi(M) = \chi(\bar{M})$. By Theorem 4.11, there is a partition $(I_1, \ldots, I_k)$ of $E(\bar{M})$ into independent sets. Since $M|I_j$ has no odd-circuit, $M|I_j$ is affine. Hence there are cocycles $C_1, \ldots, C_k$ with $I_1 \subseteq C_1, \ldots, I_k \subseteq C_k$. Now $C_1 \cup \cdots \cup C_k = E(\bar{M})$, so $\chi(\bar{M}) \le k$. ∎

Exercise **223.** [Jaeger] If M is a cographic matroid without coloops, then $\chi(M) \le 3$.

REMARK **224.** The above exercise be restated as: Each bridgeless graph is the union of three even subgraphs.

EXERCISE **225.** Prove that if G is a 4-edge-connected graph then G is the union of two even subgraphs.

REMARK **226.** The above exercise implies that: Planar graphs with girth four are 4-colourable.

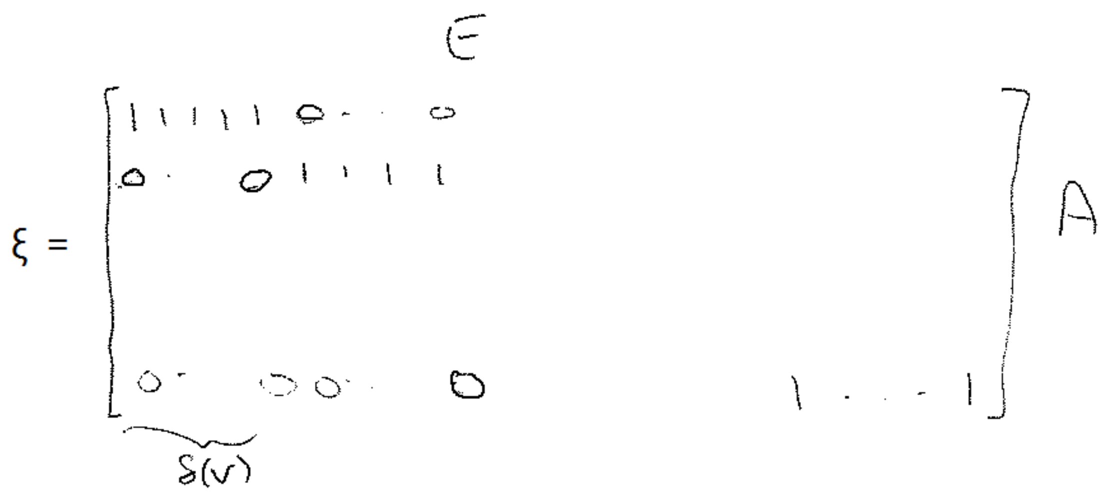
4.7 Matroid Intersection

DEFINITION **227.** Let M_1 and M_2 be the matroids on a common ground set. We let $\nu(M_1, M_2)$ denote the maximum size of a common independent set.

DEFINITION **228.** Let $G = (V, E)$ be a bipartite graph with bipartition (A, B). Let $M_A := (E, \mathcal{I}_A)$ where $I \in \mathcal{I}_A$ if each vertex in A is incident with at most one edge in I. Let $M_B := (E, \mathcal{I}_B)$ where $I \in \mathcal{I}_B$ if each vertex in B is incident with at most one edge in I.

PROPOSITION **229.** *If G is a bipartite graph with bipartition (A, B) then M_A and M_B are matroids.*

Proof. Simply note that $M_A = M(\xi)$, where ξ is given as follows:



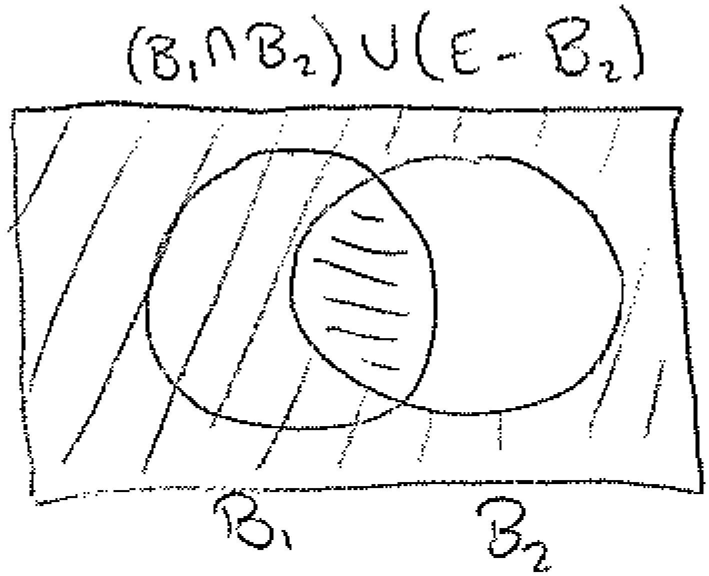
■

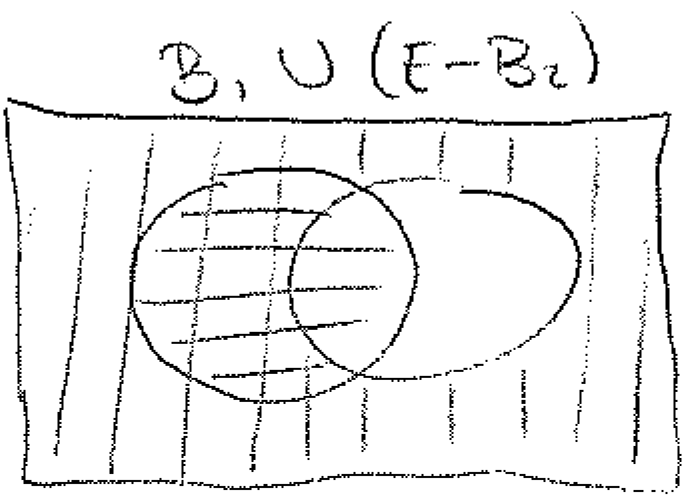
REMARK **230.** Now $M \subseteq E$ is a common independent set of M_A and M_B if and only if M is a matching. Hence $\nu(M_A, M_B) = \nu(G)$.

LEMMA **231.** [4.14] *Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be matroids. Then*

$$\nu(M_1, M_2) = r(M_1 \cup M_2^*) - r(M_2^*) .$$

Proof. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ denote the set of bases of M_1 and M_2 respectively. For $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$,





$$|B_1 \cap B_2| + r(M_2^*) = |B_1 \cap B_2| + |E - B_2| = |(B_1 \cap B_2) \cup (E - B_2)| = |B_1 \cup (E - B_2)|.$$

Hence

$$\begin{aligned}\nu(M_1, M_2) + r(M_2^*) &= \left(\max_{(B_1,B_2)\in\mathcal{B}_1\times\mathcal{B}_2} |B_1 \cap B_2|\right) + r(M_2^*)\\ &= \max_{(B_1,B_2)\in\mathcal{B}_1\times\mathcal{B}_2} |B_1 \cup (E - B_2)|\\ &= r(M_1 \cup M_2^*).\end{aligned}$$

∎

THEOREM **232.** [MATROID INTERSECTION THEOREM - EDMONDS 1965] *For matroids* $M_1 = (E, \mathcal{I}_1)$ *and* $M_2 = (E, \mathcal{I}_2)$,

$$\nu(M_1, M_2) = \min_{A \subseteq E} (r_{M_1}(A) + r_{M_2}(E - A)) .$$

Proof. By Lemma 4.1 and the Matroid Partition Theorem we have

$$\begin{aligned}
\nu(M_1, M_2) &= r(M_1 \cup M_2^*) - r(M_2^*) \\
&= \min_{A \subseteq E} \big(|E - A| + r_{M_1}(A) + r_{M_2^*}(A) \big) - r(M_2^*) \\
&= \min_{A \subseteq E} (|E| - |A| + r_{M_1}(A) + |A| - (r(M_2) - r_{M_2}(E - A)) - (|E| - r(M_2))) \\
&= \min_{A \subseteq E} (r_M(A) + r_{M_2}(E - A)) .
\end{aligned}$$

∎

4.8 Application to bipartite matching

DEFINITION **233.** Recall that for a graph $G = (V, E)$, $C \subseteq V$ is a *cover* of G if $G - C$ has no edges.

THEOREM **234.** [KÖNIG'S THEOREM] *In a bipartite graph, the maximum size of a matching is equal to the minimum size of a cover.*

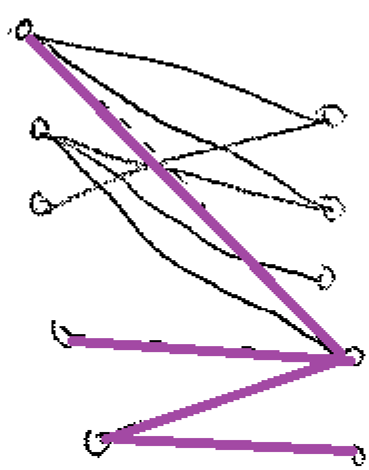
Proof. Let $G = (V, E)$ be bipartite with bipartition (A, B), and let M_A, M_B be the associated matroids. If C is a cover, then

$$\nu(G) \le \nu(G - C) + |C| = |C|.$$

It remains to prove that there is a cover attaining equality.

Recall that $\nu(G) = \nu(M_A, M_B)$. By the Matroid Intersection Theorem, there is a partition (X_A, X_B) of E such that

$$\nu(M_A, M_B) = r_{M_A}(X_A) + r_{M_B}(X_B) .$$



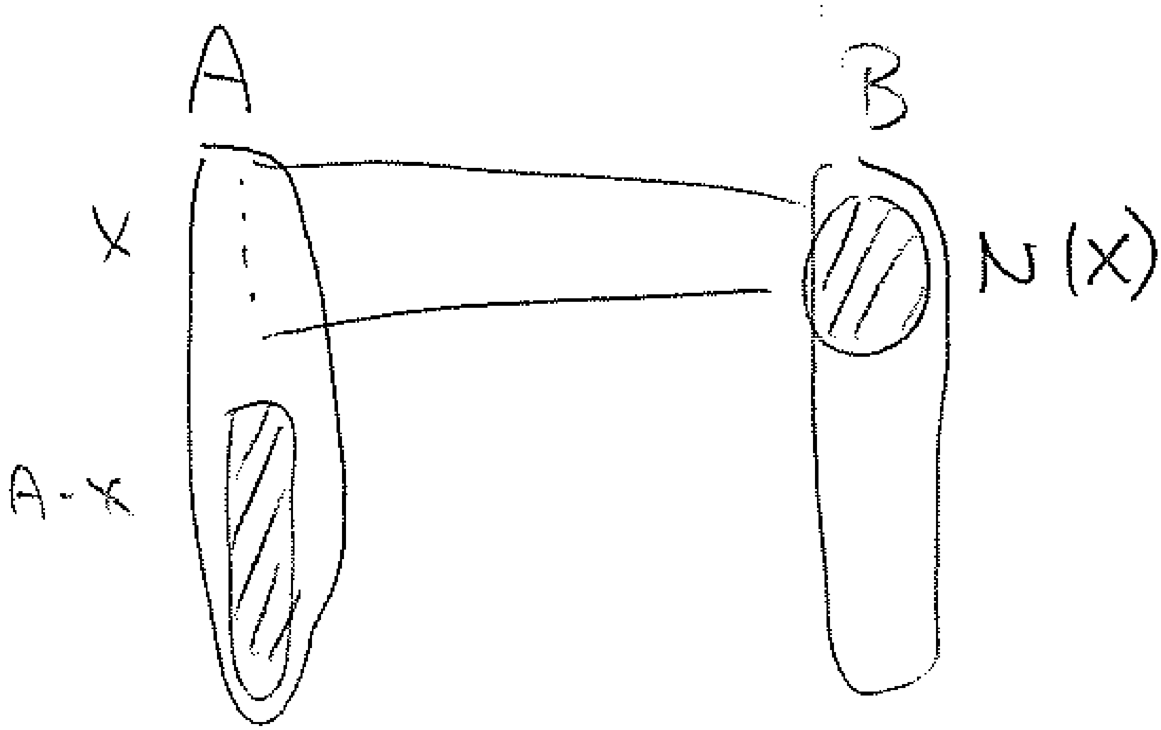
Let C_A be the vertices in A incident with edges in X_A and let C_B be the vertices in B incident with edges in X_B

Now let $C = C_A \cup C_B$. Thus C is a cover and

$$|C| = |C_A| + |C_B| = r_{M_A}(X_A) + r_{M_B}(X_B) = \nu(M_A, M_B) = \nu(G),$$

as required. ∎

THEOREM **235.** [MENDELSON'S THEOREM] *Let G be a bipartite graph with bipartition (A,B). Let X be the set of avoidable vertices in A, and let $C = (A - X) \cup N(X)$. Then C is a cover and $|C| = \nu(G)$.*



Proof. Exercise. ∎

5 Matroid Connectivity

LEMMA **236.** [5.0] *Let (X,Y) be a partition of the ground set of a matroid $M = (E,\mathcal{I})$. Then $M = (M|X) \oplus (M|Y)$ if and only if $r_M(X) + r_Y(Y) = r(M)$.*

Proof. If $M = (M|X) \oplus (M|Y)$, then

$$r_M(X) + r_M(Y) = r_M(X \cup Y) = r(M)\,.$$

Conversely, suppose that $r_M(X) + r_M(Y) = r(M)$. Thus $\sqcap_M(X,Y) = 0$. By Assignment 1 Problem 8, $\sqcap(X',Y') = 0$ for each $X' \subseteq X$ and $Y' \subseteq Y$. That is $r_M(X' \cup Y') = r_M(X') + r_M(Y')$. Hence $M = (M|X) \oplus (M|Y)$. ∎

Definition **237.** A matroid $M = (E, \mathcal{I})$ is *connected* if for each proper partition (X, Y) of E, $r_M(X) + r_M(Y) > r(M)$. Equivalently, $\sqcap_M(X, Y) \geq 1$.

Definition **238.** Let $M = (E, \mathcal{I})$ be a matroid. For $X \subseteq E$, the *connectivity* of X in M is

$$\lambda_M(X) := \sqcap_M(X, E - X).$$

Remark **239.** Recalling the definition of $\sqcap$, we have

$$\lambda_M(X) = r_M(X) + r_M(E - X) - r(M).$$

Lemma **240.** [5.1] *If M is a matroid then*

$$\lambda_M(X) = r_M(X) + r_{M^*}(X) - |X|.$$

Proof. We have

$$r_M(X) + r_{M^*}(X) = r_M(X) + |X| - (r(M) - r_M(E - X)) = \lambda_M(X) + |X|.$$

∎

Lemma **241.** [5.2] *(1) $\lambda(X) = \lambda(E - X)$, for each $X \subseteq E$.*

(2) $\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y)$, for each $X, Y \subseteq E$.

(3) $\lambda_M(X) = \lambda_{M^*}(X)$ for each $X \subseteq E$.

Proof. Immediate by Lemma 5.1 and the definition. ∎

Lemma **242.** [5.3] *For disjoint $X, C \subseteq E$ in a matroid $M = (E, \mathcal{I})$ we have*

$$\lambda_{M/C}(X) = \lambda_M(X) - \sqcap_M(X, C).$$

Proof. Note

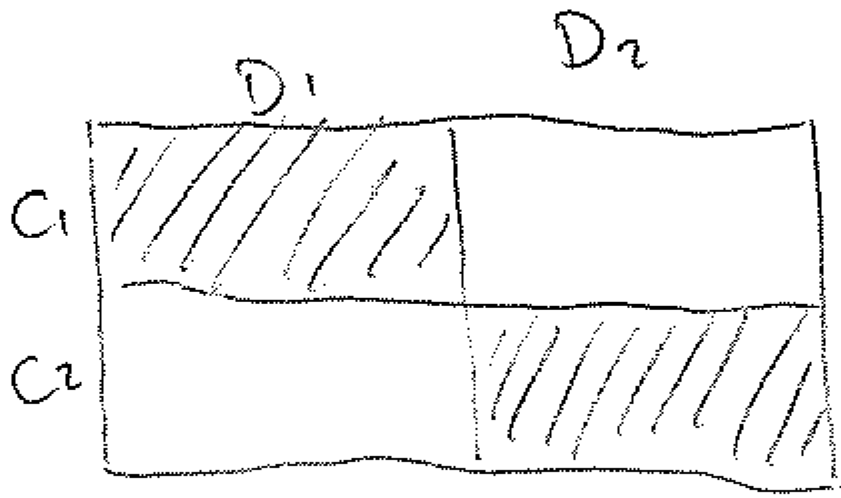
$$\begin{aligned}\lambda_{M/C}(X) &= r_{M/C}(X) + r_{(M/C)^*}(X) - |X| \\ &= r_M(X) - \sqcap_M(X, C) + r_{M^*}(X) - |X| \\ &= \lambda_M(X) - \sqcap_M(X, C).\end{aligned}$$

∎

5.1 Bixby-Coullard Inequality

Theorem **243.** [Bixby-Coullard Inequality] *If $e \in E$ in a matroid $M = (E, \mathcal{I})$, and (C_1, C_2) and (D_1, D_2) are partitions of $E - \{e\}$, then*

$$\lambda_{M \setminus e}(C_1) + \lambda_{M/e}(D_1) \geq \lambda_M(C_1 \cap D_1) + \lambda_M(C_2 \cap D_2) - 1.$$



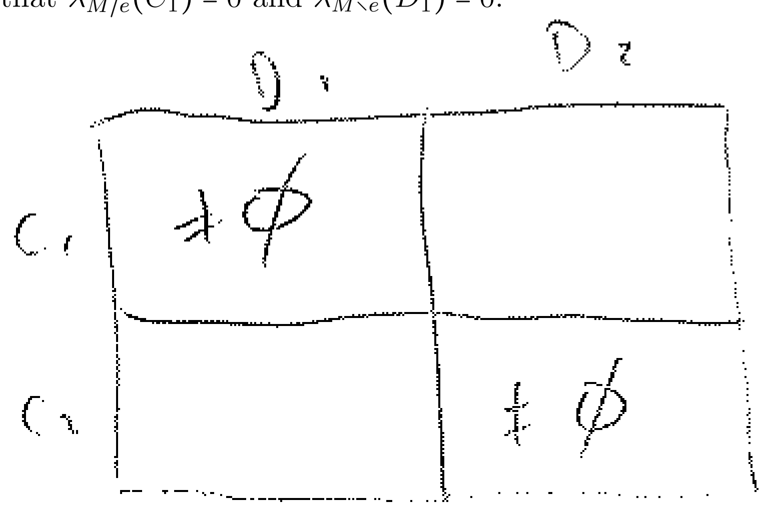
Proof. We see

$$
\begin{aligned}
&\lambda_{M/e}(C_1) + r_{M\setminus e}(D_1)\\
&\quad = r_{M/e}(C_1) + r_{M/e}(C_2) - r(M/e) + r_{M\setminus e}(D_1) + r_{M\setminus e}(D_2) - r(M\setminus e)\\
&\quad = r_M(C_1\cup\{e\}) + r_M(C_2\cup\{e\}) - r(M) - R_M(\{e\}) + r_M(D_1) + r_M(D_2) - r(M\setminus e)\\
&\quad \geq r_M(C_1\cap D_1) + r_M(C_1\cup D_1\cup\{e\}) + r_M(C_2\cap D_2) + r_M(C_2\cup D_2\cup\{e\}) - \sqcap_M(\{e\}, E-\{e\}) - 2r(M)\\
&\quad = \lambda_{M_1}(C_1\cap D_1) + \lambda_M(C_2\cap D_2) - \sqcap_M(\{e\}, E-\{e\})\\
&\quad \geq \lambda_M(C_1\cap D_1) + \lambda_M(C_2\cap D_2) - 1.
\end{aligned}
$$

■

Theorem **244.** [5.4 - Bixby's Lemma] *If e is an element in a connected matroid M, then either $M\setminus e$ or M/e is connected.*

Proof. Suppose not. Then there exist proper partitions (C_1, C_2) and (D_1, D_2) of $E-\{e\}$ such that $\lambda_{M/e}(C_1) = 0$ and $\lambda_{M\setminus e}(D_1) = 0$.



By possibly swapping D_1, D_2 we may assume that $C_1 \cap D_1$ and $C_2 \cap D_2$ are both non-empty. By the Bixby-Coullard Inequality,

$$0 = \lambda_{M/e}(C_1) + \lambda_{M \setminus e}(D_1) \geq \lambda_M(C_1 \cap D_1) + \lambda_M(C_2 \cap D_2) - 1.$$

Hence either $\lambda_M(C_1 \cap D_1) = 0$ or $\lambda_M(C_2 \cap D_2) = 0$, contradicting that M is connected. ∎

THEOREM **245.** [5.5] *A matroid $M = (E, \mathcal{I})$ is connected if and only if, for each distinct $e, f \in E$, there exists a circuit C of M containing both e and f.*

Proof. Exercise. Hint: Use Theorem 5.4 and induction. ∎

COROLLARY **246.** [5.6] *Let G be a graph. Then $M(G)$ is connected if and only if G is loopless and 2-connected.*

Proof. By Menger's Theorem, G is loopless and 2-connected if and only if each pair of distinct edges is contained in a circuit. Now the result follows from Theorem 5.5. ∎

5.2 Fundamental graphs

DEFINITION **247.** Let B be a basis of a matroid $M = (V, \mathcal{I})$. The *fundamental graph* of (M, B) is

$$FG(M, B) \coloneqq (V, E),$$

where

$$E \coloneqq \{uv;\, u \in B,\, v \in V - B, (B - \{u\}) \cup \{v\} \text{ is a basis}\}.$$

THEOREM **248.** [5.7] *A matroid M, with basis B, is connected if and only if $FG(M, B)$ is connected.*

Proof. Exercise. ∎

5.3 Connectivity between two sets

DEFINITION **249.** Let $M = (E, \mathcal{I})$ be a matroid. Let $S, T \subseteq E$ be disjoint. We define the *connectivity* between S and T to be

$$\kappa_M(S, T) \coloneqq \min\{\lambda_M(X)\,;\, S \subseteq X \subseteq E - T\}.$$

REMARK **250.** Note that:

(1) $\kappa_M(S, T) = \kappa_M(T, S)$ since $\lambda_M(X) = \lambda_M(E - X)$.

(2) $\kappa_{M^*}(S, T) = \kappa_M(S, T)$ since $\lambda_{M^*}(X) = \lambda_M(X)$.

LEMMA **251.** [5.8] *If N is a minor of a matroid M with $S, T \subseteq E(N)$, then*

$$\kappa_N(S,T) \le \kappa_M(S,T)\,.$$

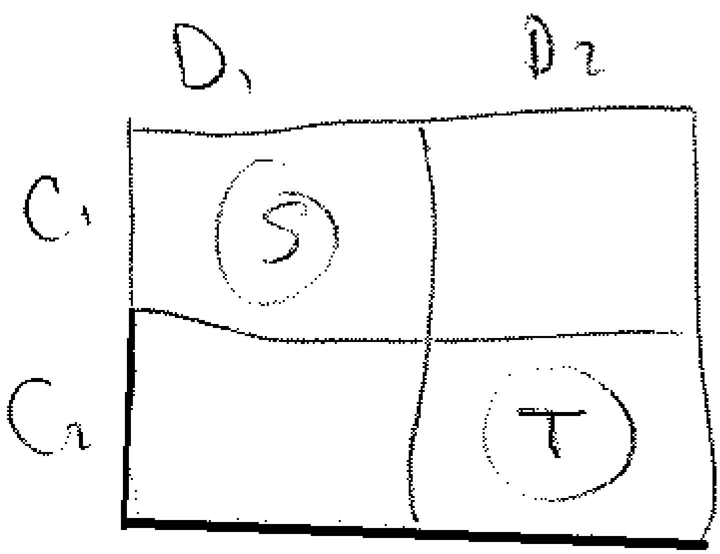
Proof. By (2) above, it suffices to consider the case that $N = M \setminus e$. Let $S \subseteq X \subseteq E - T$ with $\lambda_M(X) = \kappa_M(S,T)$. By (1) we may assume that $e \notin X$. Now $\kappa_M(S,T) = \lambda_M(X) \ge \lambda_N(X) \ge \kappa_N(S,T)$. ∎

THEOREM **252.** [TUTTE'S LINKING THEOREM] *Let $M = (E, \mathcal{I})$ be a matroid and let $S, T \subseteq E$ be disjoint. Then there exists a minor N of M with $E(N) = S \cup T$ such that $\lambda_N(S) = \kappa_M(S,T)$*

Proof. The result is trivial when $S \cup T = E$, so assume that this is not the case. Let $e \in E - (S \cup T)$. The results follows inductively if either $\kappa_{M \setminus e}(S,T) = \kappa_M(S,T)$ or $\kappa_{M/e}(S,T) = \kappa_M(S,T)$. So assume that

$$\kappa_{M\setminus e}(S,T) = \kappa_M(S,T) - 1 \quad \text{and} \quad \kappa_{M/e}(S,T) = \kappa_M(S,T) - 1.$$

Let $(D_1, D_2), (C_1, C_2)$ be partitions of $E - \{e\}$ with $S \subseteq D_1 \cap C_1$ and $T \subseteq D_2 \cap C_2$ such that $\lambda_{M \setminus e}(D_1) \le \kappa_M(S,T) - 1$ and $\lambda_{M/e}(C_1) \le \kappa_M(S,T) - 1$.



By the Bixby-Coullard Inequality,

$$2(\kappa_M(S,T) - 1) \ge \lambda_{M\setminus e}(D_1) + \lambda_{M/e}(C_1) \ge \lambda_M(D_1 \cap C_1) + \lambda_M(D_2 \cap C_2) - 1.$$

Hence either $\lambda_M(D_1 \cap C_1) \le \kappa_M(S,T) - 1$ or $\lambda_M(D_2 \cap C_2) \le \kappa_M(S,T) - 1$. So $\kappa_M(S,T) \le \kappa_M(S,T) - 1$ — contradiction. ∎

EXERCISE **253.** (1) Prove that

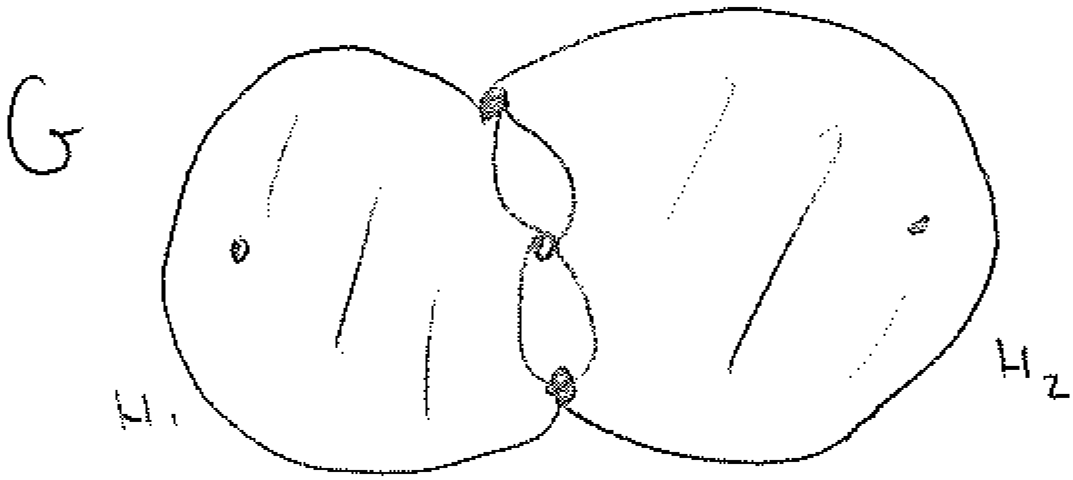
$$\kappa_M(S,T) = \nu(M \setminus S/T, M/S \setminus T) - r(M) + r_M(S) + r_M(T)\,.$$

(2) Derive Tutte's Linking Theorem from the Matroid Intersection Theorem.

REMARK **254.** By (1) we can compute $\kappa_M(S,T)$ efficiently.

5.4 Application to graphs

Definition **255.** Let $G = (V, E)$ be a graph. A *separation* of G is a pair (H_1, H_2) of edge-disjoint subgraphs of G with $H_1 \cup H_2 = G$. The *order* of the separation, denoted $\operatorname{ord}(H_1, H_2)$, is $|V(H_1) \cap V(H_2)|$.



Lemma **256.** [5.9] *If (H_1, H_2) is a separation in a graph G then*

$$\lambda_{M(G)}(E(H_1)) = \operatorname{ord}(H_1, H_2) - \operatorname{comps}(H_1) - \operatorname{comps}(H_2) + \operatorname{comps}(G)\,.$$

Proof. We have

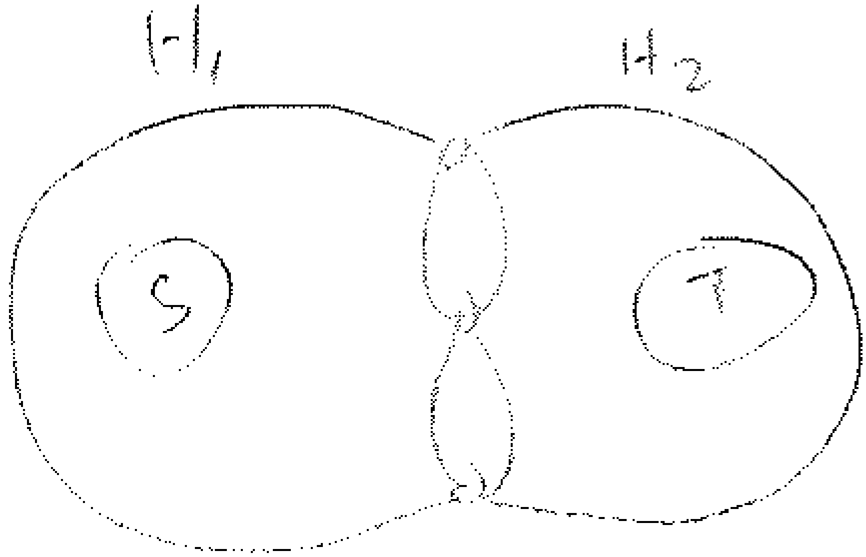
$$\begin{aligned}
\lambda_{M(G)}(E(H_1)) &= r(E(H_1)) + r(E(H_2)) - r(E(G)) \\
&= |V(H_1)| - \operatorname{comps}(H_1) + |V(H_2)| - \operatorname{comps}(H_2) - (|V| - \operatorname{comps}(G)) \\
&= \operatorname{ord}(H_1, H_2) - \operatorname{comps}(H_1) - \operatorname{comps}(H_2) + \operatorname{comps}(G)\,.
\end{aligned}$$

∎

Remark **257.** Note that if H_1, H_2, and G are all connected, then we get

$$\lambda_{M(G)}(E(H_1)) = \operatorname{ord}(H_1, H_2) - 1.$$

Definition **258.** Let $G =$ be a graph and let $S, T \subseteq V(G)$. We define $\kappa_G(S, T)$ to be the minimum order of a separation (H_1, H_2) of G with $S \subseteq V(H_1)$ and $T \subseteq V(H_2)$.



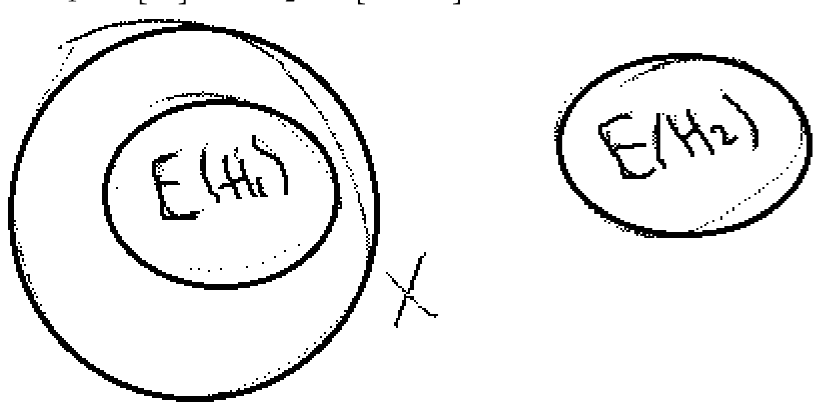
Lemma **259.** [5.10] *Let H_S, H_T be edge-disjoint subgraphs of a graph G. If H_S, H_T and G are all connected, then*

$$\kappa_G(V(H_S), V(H_T)) = \kappa_{M(G)}(E(H_S), E(H_T)) + 1.$$

Proof. For each separation (H_1, H_2) of G we have $\lambda_{M(G)}(E(H_1)) \le \mathrm{ord}(H_1, H_2) - 1$. Hence

$$\kappa_G(V(H_S), V(H_T)) \le \kappa_{M(G)}(E(H_S), E(H_T)) + 1.$$

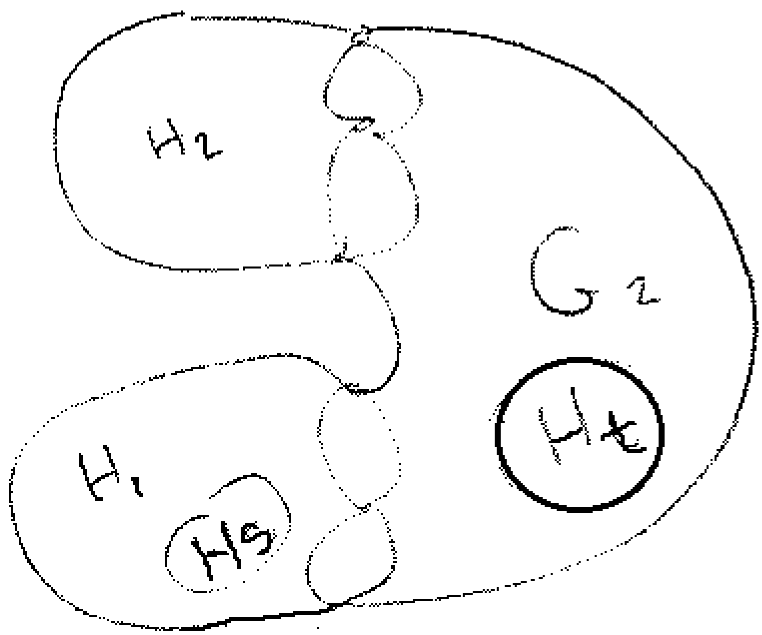
Let $X \subseteq E$ such that $E(H_S) \subseteq X \subseteq E - E(H_T)$ and $\lambda_{M(G)}(X) = \kappa_{M(G)}(E(H_S), E(H_T))$. Let $G_1 = G[X]$ and $G_2 = G[E - X]$.



We assume that X has been chosen to minimize

$$\mathrm{comps}(G_1) + \mathrm{comps}(G_2).$$

If G_1 and G_2 are both connected, then we are done. So, up to symmetry, we may assume that G_1 is not connected.



Let H_2 be a component of G_1 that does not contain H_S and let $H_1 = G_1 - V(H_2)$. Note that

$$\text{comps}(G_2 \cup H_2) \geq \text{comps}(G_2) - (\text{ord}(H_2, G_2) - 1)$$

Now

$$\begin{aligned}\lambda_{M(G)}(E(H_1)) &= \text{ord}(H_1, G_2 \cup H_2) - \text{comps}(H_1) - \text{comps}(G_2 \cup H_2) + 1 \\ &\leq \text{ord}(H_1, G_2 \cup H_2) + \text{ord}(H_2, G_2) - (\text{comps}(G_1) - 1) - (\text{comps}(G_2) + 1) + 1 \\ &= \lambda_{M(G)}(E(G)).\end{aligned}$$

However

$$\text{comps}(G_1) + \text{comps}(G_2) > \text{comps}(H_1) + \text{comps}(H_2 \cup G_2),$$

contradicting our choice of X. ∎

Theorem 260. [Menger's Theorem] *Let G be a graph and let $S, T \subseteq V(G)$. The maximum number of vertex-disjoint (S,T)-paths is equal to $\kappa_G(S,T)$.*

Proof. Let k be the maximum number of vertex disjoint (S,T)-paths. Clearly $k \leq \kappa_G(S,T)$.

Note that adding edges into S or into T does not change either $\kappa_G(S,T)$ or the maximum number of vertex disjoint (S,T)-paths. Hence we may assume that G has edge-disjoint connected subgraphs H_S and H_T with $V(H_S) = S$ and $V(H_T) = T$. By lemma 5.10, $\kappa_{M(G)}(E(H_S), E(H_T)) = \kappa_G(S,T) - 1$. By Tutte's Linking Theorem, there exists a minor N of $M(G)$ with $E(N) = E(H_S) \cup E(H_T)$ such that

$$\lambda_N(E(H_S)) = \kappa_{M(G)}(E(H_S), E(H_T)) = \kappa_G(S,T) - 1.$$

Suppose that $N = M/C \setminus D$. Then

$$\sqcap_{M(G)}(E(H_S), E(E_T)) = \kappa_G(S,T) - 1. \tag{3}$$

Let $H = G[V,C]$. By (3), there are $\kappa_G(S,T)$ components of H containing a vertex of both S and T. Hence $k \geq \kappa_G(S,T)$. ∎

5.5 Tutte Connectivity

Definition **261.** Let $M = (E,\mathcal{I})$ be a matroid. A partition (X,Y) of E is a *k-separation* of M if

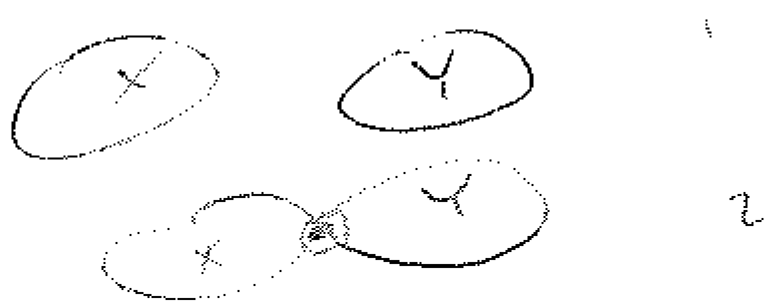
(i) $\lambda_M(X) \leq k-1$, and

(ii) $|X|,|Y| \geq k$.

Remark **262.** Note that $\lambda_M(X) \leq \min(|X|,|Y|)$, so (ii) is a nondegeneracy condition.

Example **263.** A 1-separation is a pair (X,Y) such that X and Y are non-empty and skew.

Example **264.** A 2-separation is a pair (X,Y) such that $|X|,|Y| \geq 2$ and $\sqcap(X,Y) \leq 1$.



Definition **265.** A matroid is *k-connected* if it has no ℓ-separation for any $\ell < k$.

Remark **266.** (1) A matroid M is 2-connected if and only if it is connected.

(2) A matroid M is k-connected if and only if M^* is k-connected.

(3) If T is a triangle in a matroid M and $|M| \geq 6$ then $\lambda_M(T) = r_M(T) + r_{M^*}(T) - |T| \leq r_M(T) = 2$. So $(T, E(M) - T)$ is a 3-separation and, hence, M is not 4-connected. This means that neither $M(K_n)$ nor $\mathrm{PG}(n-1, 2)$ is 4-connected for any $n \geq 4$.

EXERCISE **267.** [CHALLENGE] Show that, for each k there is a graph G such that $M(G)$ is k-connected and $|E(G)| \geq 2k$.

5.6 Vertical connectivity

DEFINITION **268.** Let $M = (E, \mathcal{I})$ be a matroid. A partition (X, Y) of E is a *vertical k-separation* of M if

(i) $\lambda_M(X) \leq k - 1$, and

(ii) $r_M(X), r_M(E - X) \geq k$.

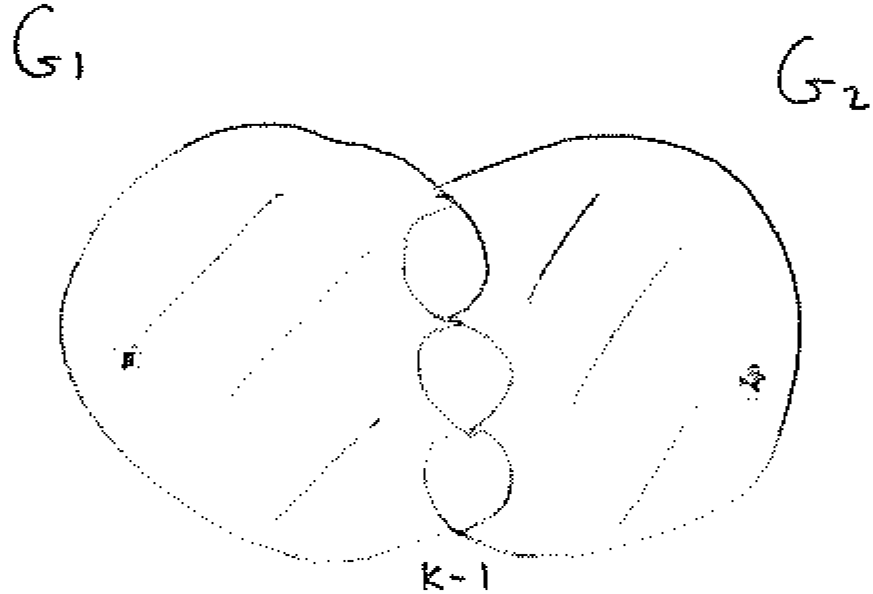
DEFINITION **269.** A matroid is *vertically k-connected* if it has no vertical ℓ-separation for any $\ell < k$.

REMARK **270.** Vertical k-connectivity need not be preserved under duality (see, for example, $U_{4,6}$).

THEOREM **271.** [5.11 - CUNNINGHAM] *Let G be a connected graph and let $k \geq 1$. Then G is k-connected if and only if $M(G)$ is vertically k-connected.*

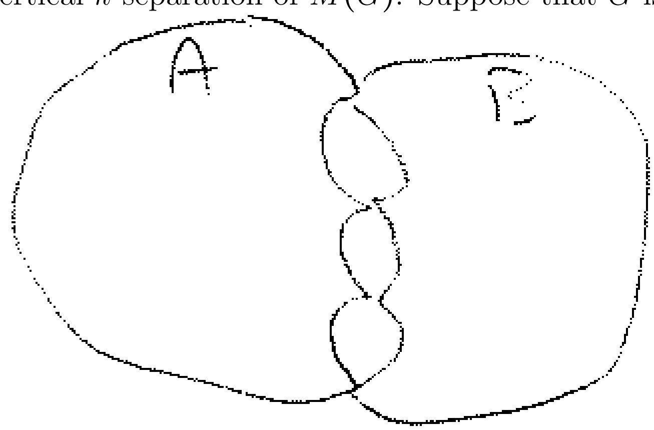
Proof. The result is vacuous for $k = 1$; suppose $k \geq 2$. Inductively we may assume that G is $(k-1)$-connected and $M(G)$ is vertically $(k-1)$-connected.

If G is not k-connected, then G has a separation (G_1, G_2) of order less than k with $V(G_1) - V(G_2) \neq \emptyset$ and $V(G_2) - V(G_1) \neq \emptyset$.



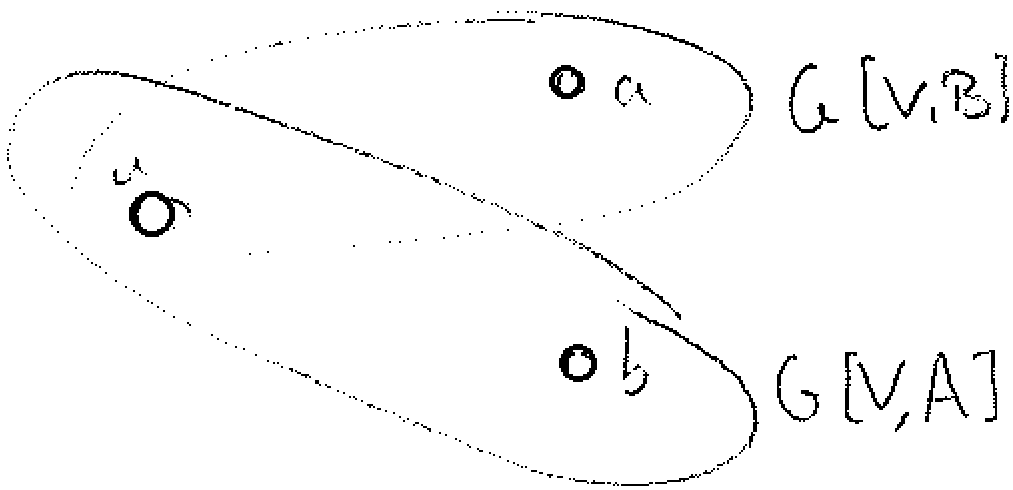
Since G is $(k-1)$-connected, G_1 and G_2 are connected. So $\lambda_{M(G)}(E(G_1)) = (k-1) - 1 = k-2$. Note that $|V(G_1)|, |V(G_2)| \geq k$, so $r(E(G_1)) \geq k-1$ and $r(E(G_2)) \geq k-1$. Hence $(E(G_1), E(G_2))$ is a vertical $(k-1)$-separation in $M(G)$, hence, $M(G)$ is not vertically k-connected

Conversely, suppose that $M(G)$ is not vertically $(k-1)$-connected and let (A, B) be a vertical k-separation of $M(G)$. Suppose that G is k-connected.



Claim. There exist $u, v \in V(G)$ such that u, v are in distinct components of both $G[V, A]$ and $G[V, B]$.

Proof of Claim. Since (A, B) is a vertical k-separation, neither $G[V, A]$ nor $G[V, B]$ is connected.



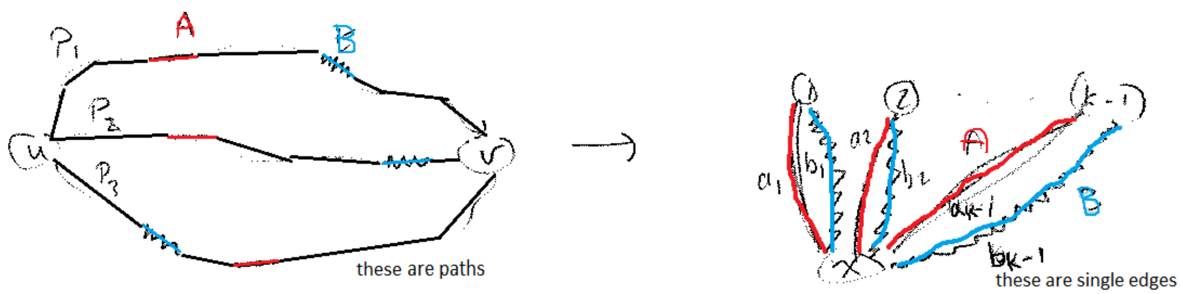
So there exist $u, a, b \in V$ such that

- u, a are in distinct components of $G[V, A]$, and
- u, b are in distinct components of $G[V, B]$.

We may assume that

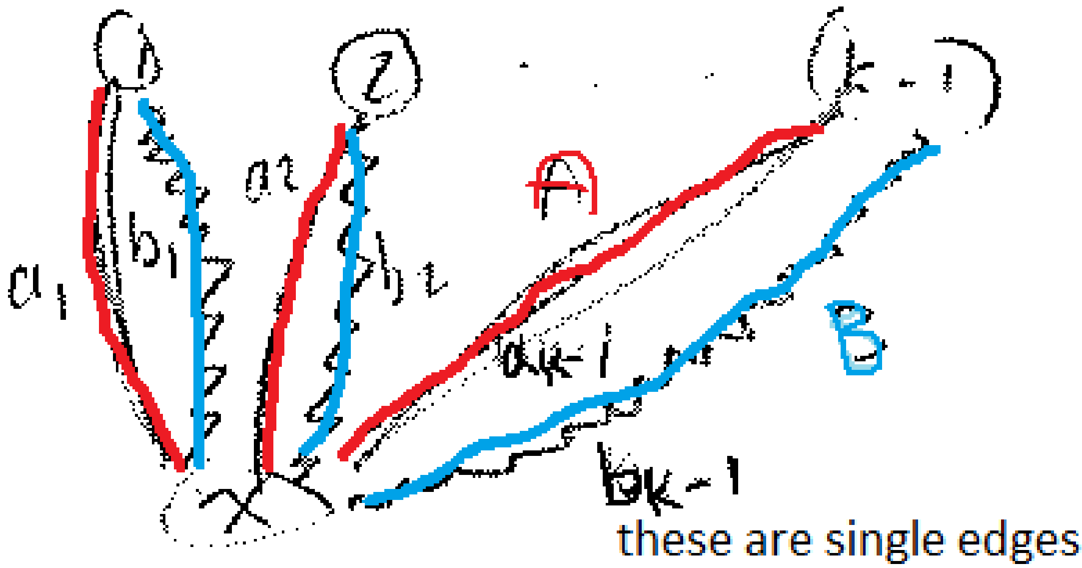
- u, a are in the same component of $G[V, B]$, and
- u, b are in the same component of $G[V, A]$.

Hence a, b are in distinct components of both $G[V, A]$ and $G[V, B]$. ∎



Since G is k-connected, there exist k-internally disjoint (u, v)-paths. Each of these paths contains an edge from both A and B.

So we get the following minor N:



Here, $a_1, \ldots, a_{k-1} \in A$ and $b_1, \ldots, b_{k-1} \in B$. Now

$$\begin{aligned} k-2 &\geq \lambda_{M(G)}(A) \\ &\geq \kappa_{M(G)}(\{a_1, \ldots, a_{k-1}\}, \{b_1, \ldots, b_{k-1}\}) \\ &\geq \sqcap_N(\{a_1, \ldots, a_{k-1}\}, \{b_1, \ldots, b_{k-1}\}) = k-1. \end{aligned}$$

This contradiction completes the proof. ∎

Definition **272.** Let M be a matroid. The *girth* of M is the length of a shortest circuit.

Exercise **273.** Let M be a matroid with $|E(M)| \geq 2k$. Show that M is (Tutte) k-connected if and only if M is vertically k-connected and M has girth at least k.

Theorem **274.** [5.12] *For a connected graph G, $M(G)$ is k-connected if and only if G is k-connected and G has girth at least k.*

Proof. Immediate by the exercise and Theorem 5.11. ∎

5.7 3-connectivity

Remark **275.** By Theorem 5.12, for a connected graph G, $M(G)$ is 3-connected if and only if G is simple and 3-connected.

Definition **276.** A matroid M is *internally* 3*-connected* if M is connected and, for each 2-separation (X,Y) of M either $|X| = 2$ or $|Y| = 2$.

Recall that the simplification of a matroid is uniquely determined up to isomorphism.

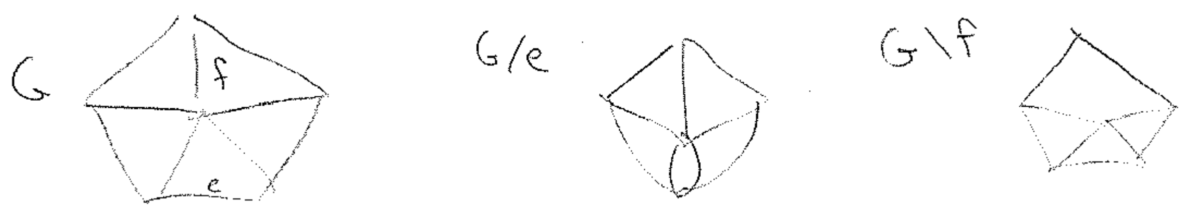
Definition **277.** Let M be a matroid. We let $\mathrm{si}(M)$ denote the *simplification* of M, and we let $\mathrm{co}(M)$ denote the *cosimplification* of M. In particular, $\mathrm{co}(M) = (\mathrm{si}(M^*))^*$.

Remark **278.** (1) If M is internally 3-connected and (X,Y) is a 2-separation with $|X| = 2$, then X is a series-pair or a parallel pair. (Indeed, we see $1 = \lambda_M(X) = r_M(X) + r_{M^*}(X) - 2$.)

(2) If M is 3-connected with $|M| \geq 4$ and $M \setminus e$ is internally 3-connected, then $M \setminus e$ has no parallel pairs, so $\mathrm{co}(M \setminus e)$ is 3-connected.

(3) Dually, if M is 3-connected with $|M| \geq 4$, and M/e is internally 3-connected, then $\mathrm{si}(M/e)$is 3-connected.

Lemma **279.** *Let e be an element of* 3*-connected matroid M. Then $M \setminus e$ or M/e is internally* 3*-connected.*



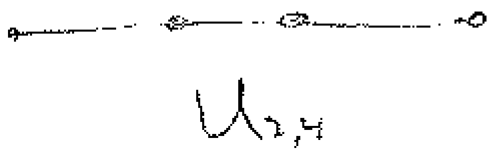
Proof. Suppose otherwise. Then, there exist 2-separations (D_1, D_2) and (C_1, C_2) in $M \setminus e$ and M/e, respectively, with $|D_1|, |D_2|, |C_1|, |C_2| \geq 3$. By possibly swapping C_1 and C_2, we may assume that $|C_1 \cap D_1| \geq 2, |C_2 \cap D_2| \geq 2$.

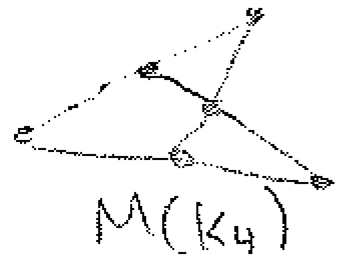
By the Bixby-Coullard Inequality,

$$2 \geq \lambda_{M \setminus e}(D_1) + \lambda_{M/e}(C_1) \geq \lambda_M(D_1 \cap C_1) + \lambda_M(D_2 \cap C_2) - 1.$$

So either $\lambda_M(D_1 \cap C_1) \leq 1$ or $\lambda_M(D_2 \cap C_2) \leq 1$. However, this contradicts the fact that M is 3-connected. ∎

Corollary **280.** [5.13] *If M is a 3-connected matroid with $|M| \geq 4$, then M has a $U_{2,4}$- or $M(K_4)$-minor.*





Proof. Exercise. ∎

THEOREM **281.** *Let e be an element of a matroid M. Then there exists $N \in \{M \setminus e, M/e\}$ such that for each partition (X_1, X_2) of $E(N)$ there is a partition (A, B) of either X_1 or X_2 such that*

$$\lambda_M(A)\,, \lambda_M(B) \leq \lambda_N(X_1)\,.$$

Proof. Exercise. ∎

PROBLEM **282.** Given a 3-connected matroid M, does there exist an element $e \in E(M)$ such that M/e or $M \setminus e$ is 3-connected?

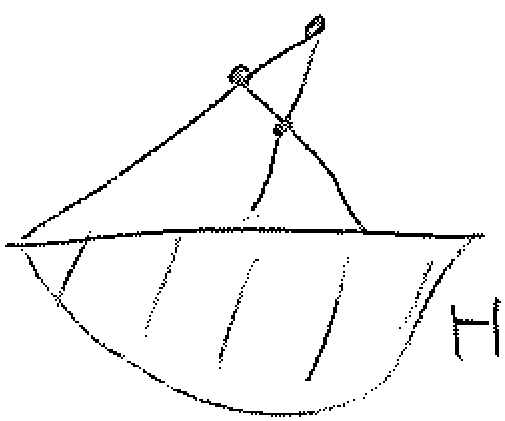
Recall that a 3-element circuit in a matroid is called a triangle.

DEFINITION **283.** A 3-element cocircuit in a matroid is called a *triad*.

REMARK **284.** Note that:

(1) If M/e is internally 3-connected, but not 3-connected, then e is in a triangle.

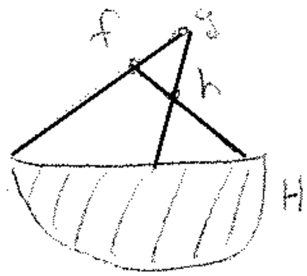
(2) If $M \setminus e$ is internally 3-connected, but not 3-connected, then e is in a triad.

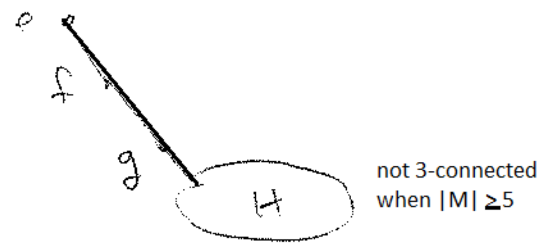


Lemma **285.** [5.14] *Let e be an element in a 3-connected matroid M. If e is in neither a triangle nor a triad, then $M \setminus e$ or M/e is 3-connected.*

Proof. By Bixby's Lemma and the discussion above. ∎

Remark **286.** Up to duality we may now assume, in answering our problem above, that M has a triangle $\{e, f, g\}$.





Lemma **287.** [5.15] *Let e, f, g, h be distinct elements in a 3-connected matroid M. If $\{e, f, g\}$ is a triangle and $\{f, g, h\}$ is a triad, then $M \setminus e$ is internally 3-connected.*

Proof. Exercise. Hint: Consider M/e. ∎

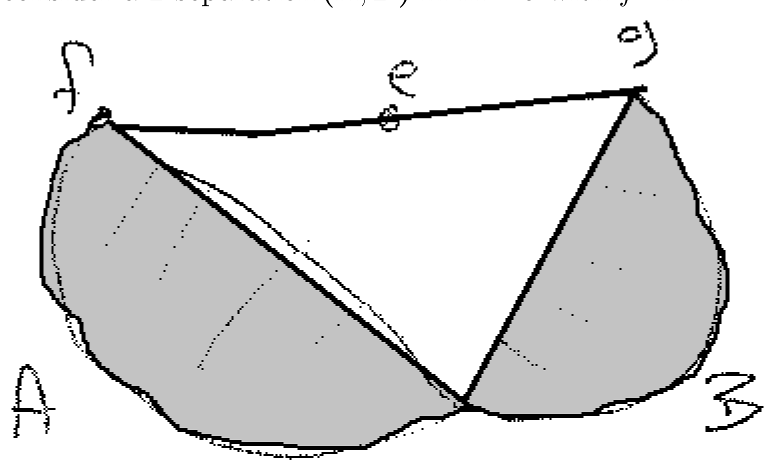
Lemma **288.** [Tutte's Triangle Lemma] *Let $T = \{e, f, g\}$ be a triangle in a 3-connected matroid M with $|M| \geq 4$. If e is not contained in a triad, then either $M \setminus e$, $M \setminus f$, or $M \setminus g$ is 3-connected.*

Proof. Suppose that none of $M \setminus e$, $M \setminus f$, $M \setminus g$ is 3-connected.

Claim (1)**.** We have $|M| \geq 8$.

Proof of Claim 1. Exercise. Hint: If $|M| \leq 7$ then $r(M) \leq 3$ or $r(M^*) \leq 3$. ∎

Now consider a 2-separation (A, B) in $M \setminus e$ with $f \in A$.



Claim (2). $f \in A - \mathrm{cl}_M(B)$ and $g \in B - \mathrm{cl}_M(A)$.

Proof of Claim 2. If $g \in A$, then since $\{f, g\}$ spans e

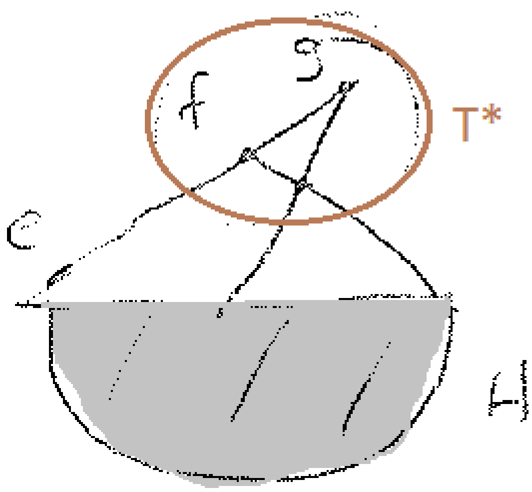
$$\lambda_M(B) = r_M(A \cup \{e\}) + r_M(B) - r(M) = r_M(A) + r_M(B) - r(M \setminus e) = \lambda_{M\setminus e}(A) = 1.$$

Contradicting that M is 3-connected; thus $g \in B$. Similarly, if $f \in \mathrm{cl}_M(B)$, then $\lambda_M(B \cup \{e, f\}) = \lambda_{M\setminus e}(B) = 1$, which is again a contradiction, unless $|A| = 2$ and $f \in \mathrm{cl}_M(B)$ then the element in $A - \{f\}$ is a coloop in $M \setminus e$. However since M is 3-connected, $M \setminus e$ is connected, and we again have a contradiction. Thus $f \notin \mathrm{cl}_M(B)$ and, similarly, $g \notin \mathrm{cl}_M(A)$. ∎

Since $|M| \geq 8$, $\max(|A|, |B|) \geq 4$. Up to symmetry we may assume that $|A| \geq 4$.
Note that

$$\begin{aligned}\lambda_{M/f}(B \cup \{e\}) &= \lambda_M(B \cup \{e\}) - \sqcap_M(\{f\}, B \cup \{e\}) \\ &= \lambda_M(B \cup \{e\}) - 1 \\ &\leq (\lambda_{M\setminus e}(B) + 1) - 1 = 1.\end{aligned}$$

Hence $(A - \{f\}, B \cup \{e\})$ is a 2-separation in M/f, so M/f is not internally 3-connected.
Then, by Bixby's Lemma $M \setminus f$ is internally 3-connected. Since $M \setminus f$ is not 3-connected, f is contained in a triad T^*. Now $|T \cap T^*| \neq 1$. By hypothesis, $e \notin T^*$, so $T \cap T^* = \{f, g\}$.



Then, by Lemma 5.15, $M \setminus e$ is internally 3-connected. But $M \setminus e$ is not 3-connected, so e is in a triad — contradiction. ∎

Remark **289.** In fact, Tutte showed that either $M \setminus e$ or $M \setminus f$ is 3-connected.

Remark **290.** By Lemma 5.14 and Tutte's Triangle Lemma, either

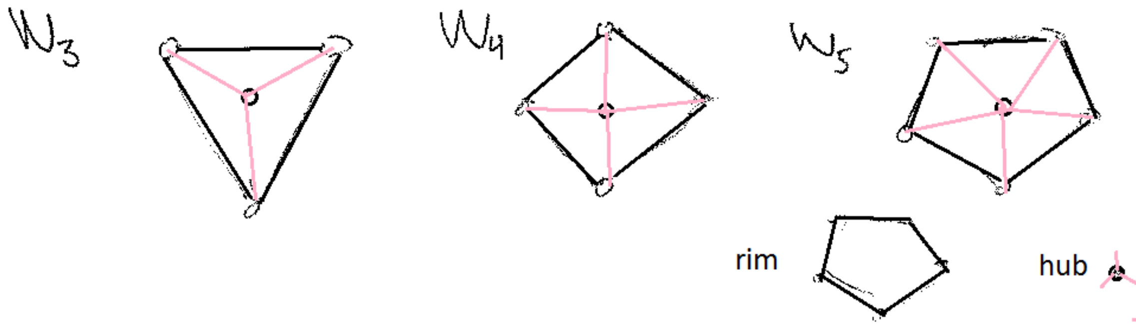
(1) there is an element e such that $M \setminus e$ or M/e is 3-connected, or

(2) each element of M is in both a triangle and a triad.

Problem **291.** Which 3-connected matroids have the property that each element is in both a triangle and a triad?

5.8 Wheels

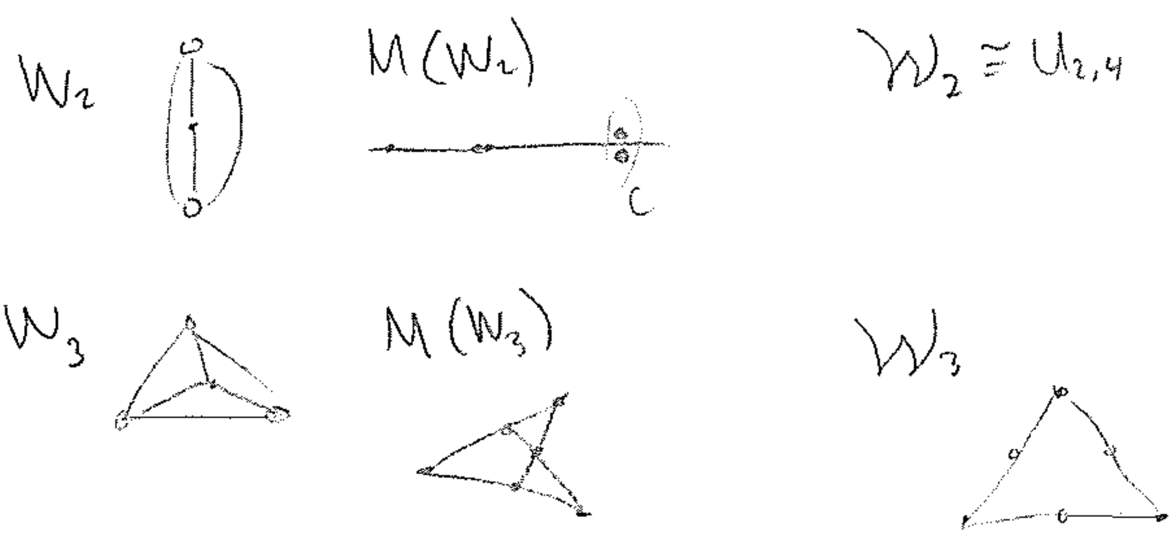
Definition 292.



Remark 293. Note that each edge in a wheel is in both a triangle and a triad.

Definition 294. Let W_n be a wheel with hub h and rim C. Note that C is a circuit-hyperplane of $M(W_n)$. Let $\mathcal{W}_n$ be the matroid obtained from $M(W_n)$ by *relaxing* C. Precisely, $E(\mathcal{W}_n) \coloneqq E(W_n)$ and $\mathcal{I}(\mathcal{W}_n) \coloneqq \mathcal{I}(M(W_n)) \cup \{C\}$. We call $\mathcal{W}_n$ the *rank-n whirl*.

Remark 295. By Assignment 1, question 9, $\mathcal{W}_n$ is a matroid.



Remark 296. Whirls are not binary. Why?

Lemma 297. [5.16] *Let M be a 3-connected matroid with $|M| \geq 4$. IF each element of M is contained in both a triangle and a triad, then M is a wheel or a whirl.*

Proof Sketch. Recall that if X is a circuit and C^* is a cocircuit, then $|C \cap C^*| \neq 1$.

Claim (1). We may assume that $|M| \geq 7$.

Claim (2). Our matroid M has no $U_{2,4}$-restriction.

Let $B \subseteq E(M)$ be the set of elements in at least two triangles, and let B^* be the set of elements in at least two triads.

Claim (3). (a) For each triangle T we have $|t \cap B^*| = 1$.

(b) For each triad T^* we have $|T^* \cap B| = 1$.

Claim (4). The pair (B, B^*) is a partition of $E(M)$.

Claim (5). The set B is a basis of M.

Claim (6). The fundamental graph $FG(M, B)$ is a circuit.

Claim (7). Either M is a wheel or M is a whirl.

∎

Theorem **298.** [Tutte's Wheel and Whirls Theorem] *Let M be a 3-connected matroid with $|M| \geq 4$. If M is not a wheel or a whirl, then there exists $e \in E(M)$ such that $M \setminus e$ or M/e is 3-connected.*

Proof. This follows from Lemma 5.14, duality, Tutte's Triangle Lemma, and Lemma 5.16. ∎

Corollary **299.** [5.17] *If M is a 3-connected matroid with $|M| \geq 4$ then there is a sequence $(N_0, \ldots, N_k)$ of 3-connected matroids such that*

(i) N_0 *is a wheel or a whirl,*

(ii) $N_k = M$, *and*

(iii) $N_i = N_{i+1} \setminus e$ *or* $N_i = N_{i+1}/e$ *for each* $i \in \{0, \ldots, k\}$ *and some* $e \in E(M)$.

Proof. Follows immediately from the above. ∎

Exercise **300.** Let M be a 3-connected matroid with $|M| \neq 1$. Prove that there exists $e \in E(M)$ such that $\mathrm{si}(M/e)$ is 3-connected.

6 Graphic Matroids

Recall the following theorem.

Theorem **301.** [1.19] *If G is a plane graph, then $M(G^*) = M(G)^*$.*

LEMMA **302.** [6.1] *Let $G = (V_G, E)$ and $H = (V_H, E)$ be graphs with no isolated vertices. If $M(G)$ is connected and $M(G)^* = M(H)$, then there are planar embeddings of G and H so that $G^* = H$.*

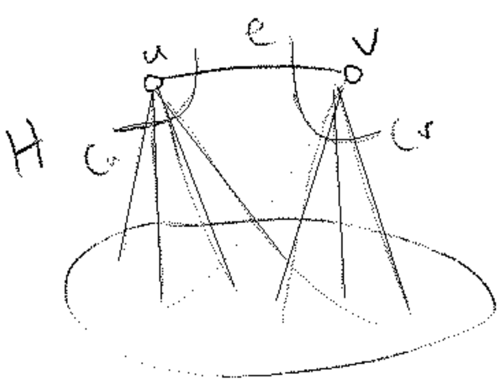
Proof. Since $M(G)$ is connected and $M(G)^* = M(H)$, G and H are both loopless and 2-connected. The result is trivial if $E = \varnothing$; assume otherwise and let $e \in E$. Note that

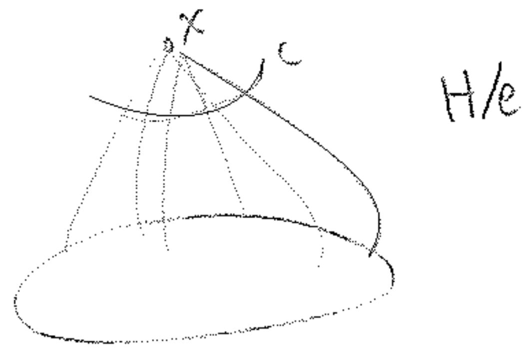
(1) By theorem 5.4 either $M(G)/e$ or $M(G) \setminus e$ is connected.

(2) $M(G \setminus e)^* = M(H/e)$, $M(G/e)^* = M(H \setminus e)$,

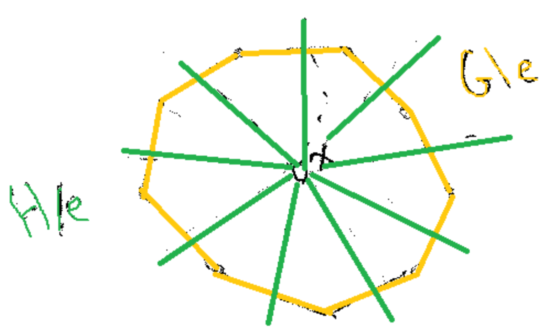
(3) M is connected if and only if M^* is connected.

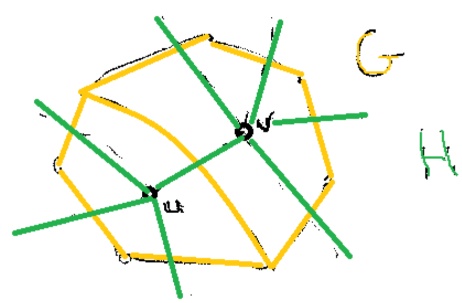
By possibly swapping G and H, we may assume $M(G \setminus e)$ and $M(H/e)$ are connected. Since $M(G \setminus e) = M(H/e)^*$, we inductively have dual planar embeddings of $G \setminus e$, H/e.





Let $C = \delta_{H/e}(x)$ where e has ends $u, v \in V_H$ that are identified to a vertex x of H/e. Since H and H/e are 2-connected, $\delta_H(u)$ and $\delta_H(v)$ are cocircuits of $M(H)$ and C is a cocircuit of $M(H/e)$. In our dual planar embedding of $M(G \setminus e)$, the set C is a facial circuit (because it is a vertex neighbourhood of the dual graph). Moreover, $\delta_H(u)$, $\delta_H(v)$ are circuits of $M(G \setminus e)$ with $C = (\delta_H(u) \cup \delta_H(v)) - \{e\}$.





Since $\delta_H(u)$ and $\delta_H(v)$ are circuits of G and $(\delta_H(u) \cup \delta_H(v)) - \{e\}$ is a facial circuit of $G \setminus e$, the edge e must join two vertices in this face. This gives a consistent way to extend our dual embeddings of H/e, $G \setminus e$ to dual embeddings of G, H. ∎

COROLLARY **303.** [1.24] *A graph G is planar if and only if $M(G)^*$ is graphic.*

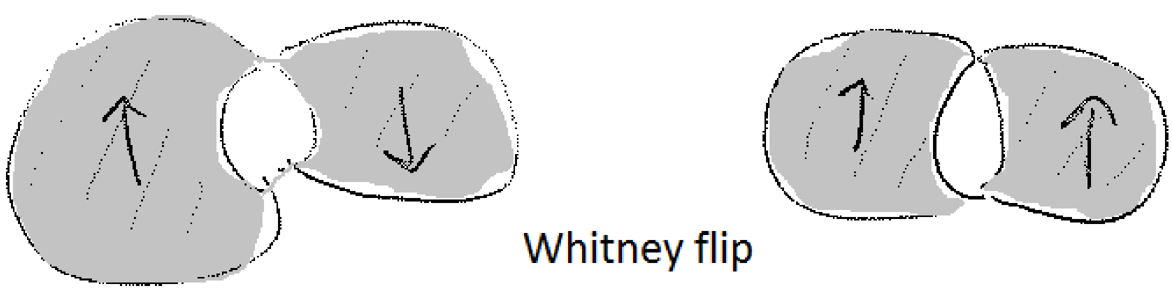
Proof. Exercise. ∎

Remark **304.** Graphicness testing generalizes planarity testing.

Problem **305.** Does a graphic matroid determine the graph?

Solution. No!

Problem **306.** Does a 2-connected graphic matroid determine the graph?



Solution. No! If a graphic matroid is not 3-connected, it does not determine the graph (Whitney flips).

Definition **307.** A cocircuit C of a matroid M is *separating* if $M \setminus C$ is disconnected.

Theorem **308.** *If G and H are loopless 3-connected graphs and $M(G) = M(H)$, then $G = H$ up to vertex labels.*

Proof.

Claim. If G is 3-connected, loopless, then a cocircuit C of $M(G)$ is non-separating if and only if there exists $v \in V(G)$ so that $C = \delta_G(v)$.

Proof of Claim. Since G is 3-connected, $G - v$ is 2-connected so $\delta_G(v)$ is a non-separating cocircuit.

Conversely, consider a non-separating cocircuit C. Now $C = \deg_G(X)$ for some $X \subseteq V(G)$ such that $G[X]$ and $G[V - X]$ are connected. Since C is non-separating, either $G[X]$ or $G[V - X]$ contains no edges. So X or $V - X$ is a single vertex w and $C = \delta_G(w)$. ∎

Now since $M(H) = M(G)$ by the claim, for each $v \in V(G)$ there exists $w \in V(H)$ such that $\delta_H(w) = \delta_G(v)$ and vice versa. So $G = H$ up to vertex labels. ∎

Remark **309.** By this theorem, simple 3-connected planar graphs have "unique" planar embeddings. That is, the facial cycles are independent of the embedding.

6.1 Recognizing graphic matroids

Problem **310.** Given a matroid M, is M graphic?

Problem **311.** [1] Given a binary matroid M, is M graphic?

Algorithm **312.** [Algorithm Overview for Problem 1]

Step (1)**.** Reduce to 3-connected case.

Step (2)**.** Find a sequence of 3-connected matroids $N_0, N_1, \dots, N_k = M$ where N_0 is a wheel and for each i, $N_i = N_{i+1}/e$ or $N_{i+1} \setminus e$ for some $e \in E(M)$.

Step (3)**.** Inductively try to find graphs $G_0, G_1, \dots, G_k$ so that $N_i = M(G_i)$ for each i.

Remark **313.** Testing whether M is 3-connected reduces to the problem of computing $\kappa_M(X, Y)$ for all X, Y with $|X| = |Y| = 2$. This can be solved with Edmonds' matroid intersection algorithm.

Step (1)**.** We may assume M is connected, because M is graphic if and only if each of its components is graphic.

Suppose $M = M(A)$, and (X, Y) is a 2-separation Since $\sqcap_M(X, Y) = 1$, there is a unique non-zero vector $v \in \text{colspace}(A|X) \cap \text{colspace}(A|Y)$. Let $M^+ = M([A|v])$, and $M_X = M^+|(X \cup \{e\})$, $M_Y = M^+|(Y \cup \{e\})$.

Exercise **314.** Show that M_X, M_Y are isomorphic to minors of M.

Exercise **315.** Show that M is graphic if and only if M_X and M_Y are graphic.

6.2 Extensions

PROBLEM **316.** [2A] Let M be a 3-connected binary matroid and let $M \setminus e = M(G)$ be 3-connected, graphic. Is M graphic?

Now

$M = M(A')$

$$A' \quad \begin{array}{cc} E-\{e\} & e \\ \left[\begin{array}{c} A \end{array}\right. & \left.\begin{array}{|c} t \end{array}\right] \end{array}$$

and A is the incidence matrix of G

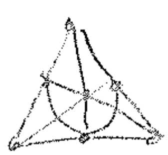
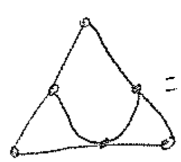
Let $T = \{v \in V;\, t_v = 1\}$.

DEFINITION **317.** A pair (G,T), where G is a graph and $T \subseteq V(G)$, is called a *graft*.

REMARK **318.** A binary extension of a graphic matroid can be described by a graft.

DEFINITION **319.** We write $M(G,T)$ for the associated matroid of a graft (G,T), and call the new element the *graft element*.

EXAMPLE **320.**

$M = F_7$  $M \setminus e =$  $= M(K_4)$

$M = M(K_4, T)$ where $T = V(K_4)$

LEMMA **321.** [6.3] *For a graft (G,T), if $|T|$ is odd then the graft element e is a coloop.*

Proof. Each column of the incidence matrix sums to zero. ∎

LEMMA **322.** [6.4] *Let (G,T) be a graft where G is a 3-connected graph and $|T|$ is even. Then $M(G,T)$ is graphic if and only if $|T| \le 2$.*

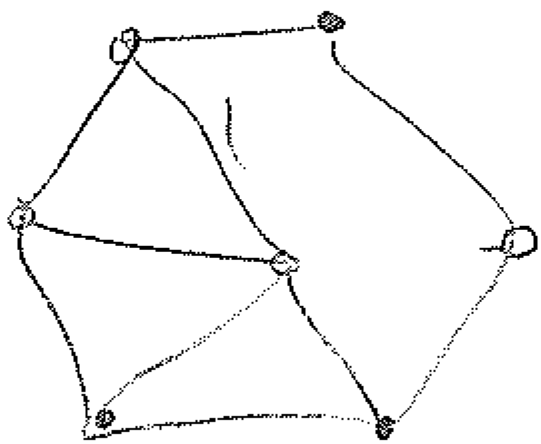
Proof. If $|T| \le 2$ then M is clearly graphic. Conversely, suppose that $M(G,T) = M(H)$. We may assume $M(G,T)$ has no loops.

By Theorem 6.2, we may relabel the vertices of H so that $H \setminus e = G$:

$$M\left(\begin{array}{cc} & e \\ \left[\begin{array}{c} H\backslash e \end{array}\right. & \left.\begin{array}{c} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{array}\right] \end{array}\right) = M\left(\begin{array}{cc} & e \\ \left[\begin{array}{c} G = H\backslash e \end{array}\right. & \left.\begin{array}{c} t \end{array}\right] \end{array}\right)$$

So $|T| \leq 2$. ∎

Remark **323.** Note that this lemma solves problem 2a.



6.3 Coextensions

Problem **324.** [2b] Let M be a 3-connected binary matroid and let $M/e = M(G)$ be 3-connected and graphic. Is M graphic?

Now $M = M(A')$ where

$$A' = \begin{bmatrix} 1 & s \\ 0 & A \end{bmatrix}$$

where A is the incidence matrix of a graph G and $s \in \mathrm{GF}(2)^{E(G)}$. Let

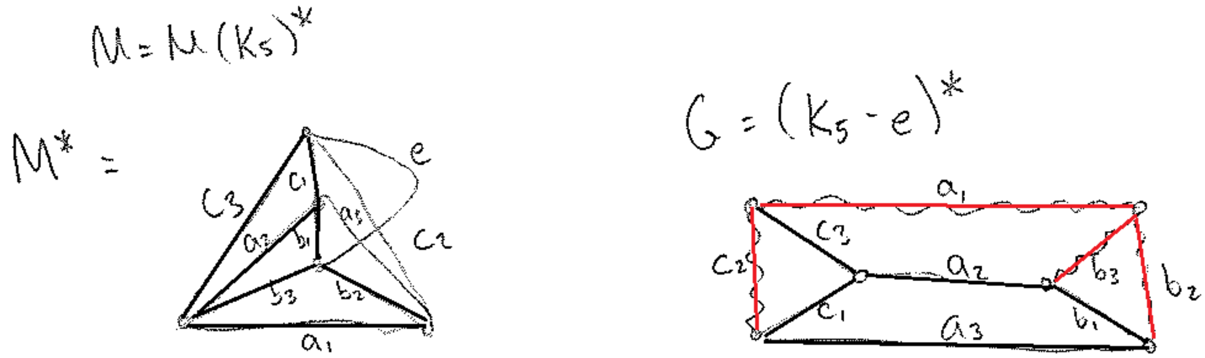
$$\Sigma = \{f \in E(G);\, s_f = 1\}.$$

Definition **325.** A pair (G, Σ) where G is a graph and $\Sigma \subseteq E(G)$ is called a *signed graph*, with *signature* Σ.

Remark **326.** A binary coextension of a graphic matroid can be described by signed graph.

Definition **327.** We write $M(G,\Sigma)$ for the associated matroid of a signed graph (G,Σ). An edge in Σ is called *odd*.

Example **328.**



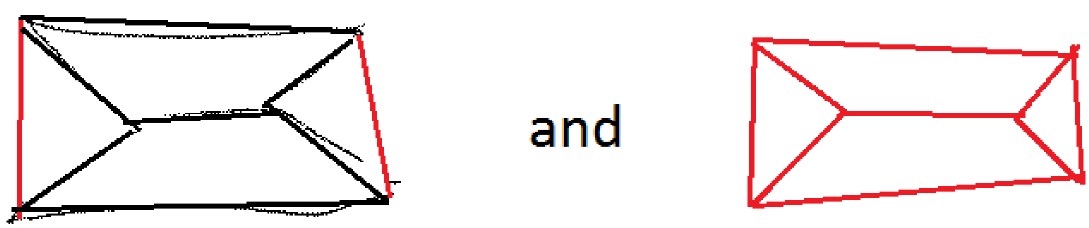
Lemma **329.** [6.5] *Let G be a graph and let, $\Sigma_1,\Sigma_2 \subseteq E(G)$. Then $M(G,\Sigma_1) = M(G,\Sigma_2)$ if and only if $\Sigma_1 \Delta \Sigma_2$ is a cut of G.*

Proof. By lemma 2.1 $M(G,\Sigma_1) = M(G,\Sigma_2)$ if and only if

$$\text{rowspace}\begin{pmatrix} 1 & s_1 \\ 0 & A \end{pmatrix} = \text{rowspace}\begin{pmatrix} 1 & s_2 \\ 0 & A \end{pmatrix},$$

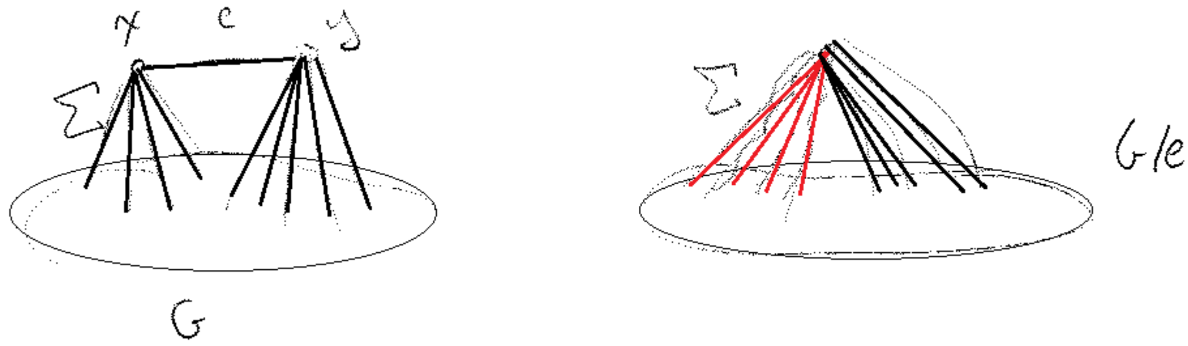
which in turn holds if and only if $s_1 + s_2 \in \text{rowspace}(A)$. ∎

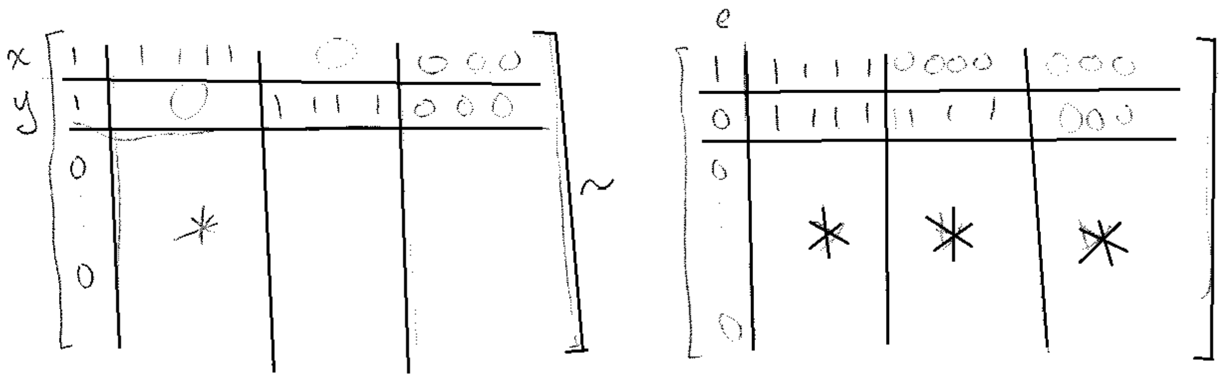
Example **330.**



Lemma **331.** [6.6] *Let G be a graph, let $e = xy \in E(G)$, and let $\Sigma = \delta_G(x) - \{e\}$. Then $M(G) = M(G/e,\Sigma)$.*

Proof.





∎

DEFINITION 332. Let (G, Σ) be a signed graph. A circuit C of G is called Σ-*odd* if $|C \cap \Sigma|$ is odd. Otherwise C is called Σ-*even*.

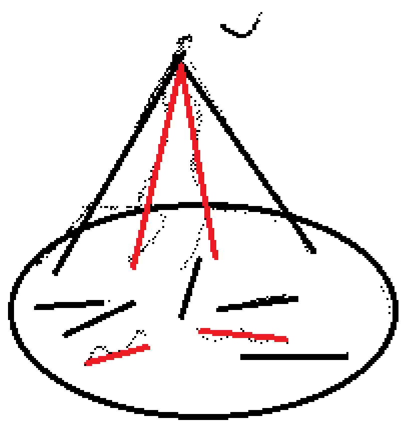
REMARK 333.
- We see Σ-parity is not affected by resigning Σ across a cut.
- If C is Σ-even circuit of G, then C is a circuit of $M(G, \Sigma)$.
- If C is Σ-odd circuit of G, then $C \cup \{e\}$ is a circuit of $M(G, \Sigma)$.

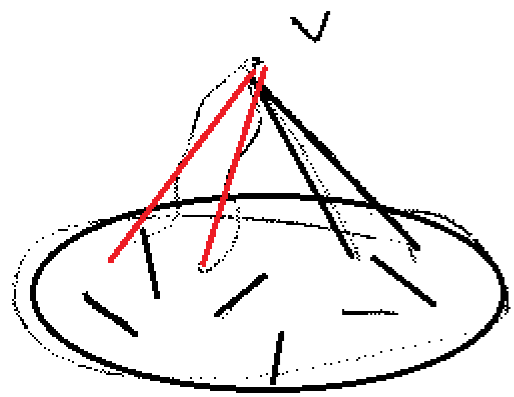
DEFINITION 334. Let (G, Σ) be a signed graph. A vertex $v \in V(G)$ is a *block node* if it is contained in each Σ-odd circuit.

THEOREM 335. [6.7] *Let (G, Σ) be a signed graph where G is loopless and 3-connected. Then $M(G, \Sigma)$ is graphic if and only if (G, Σ) has a block node.*

Proof. Suppose that $M(G, \Sigma) = M(H)$ is graphic. Let x be an end of e in H and let $\Sigma_H = \delta_H(x) - \{e\}$. By Lemma 6.6, $M(G, \Sigma) = M(H/e, \Sigma_H)$, so $M(G) = M(H/e)$, and thus $G = H/e$ up to vertex labels. So (G, Σ) and $(H/e, \Sigma_H)$ are equivalent so we may assume $\Sigma = \Sigma_H$. Therefore (G, Σ) has a block node.

Conversely, suppose that v is a block node of (G, Σ).





Now $(G,\Sigma)-v$ has no odd circuit, so by resigning we may assume that each edge in Σ is incident with v. By Lemma 6.6 it follows that (G,Σ) is graphic. ∎

Remark **336.** We can efficiently check the existence of a block node (check, for each vertex v, if e is a coloop of $M(G,\Sigma)-\delta_G(v)$), so Theorem 6.7 solves problem 2b.

Recall our first problem of this section.

Problem **337.** [1] Given a matroid M by a rank oracle, is M graphic?

Remark **338.** Note:

(1) If M is graphic, M is binary;

(2) Let B be a basis of M and let $f = F(M,B)$. If M is binary, then $M = M\big(\begin{bmatrix} I_B & F \end{bmatrix}\big)$;

(3) There is no efficient algorithm for testing if $M = M([I_B, F])$.

Algorithm **339.** [Algorithm for Problem 1]

Step (1)**.** Test if the binary matroid $M\big(\begin{bmatrix} I & F \end{bmatrix}\big)$ is graphic. If not, then stop because M is not graphic. If so, then find G so that $M\big(\begin{bmatrix} I & F \end{bmatrix}\big) = M(G)$.

Step (2)**.** Test if $M = M(G)$.

Theorem **340.** [6.8 - Seymour] *Let $M = (E,\mathcal{I})$ be a matroid and G be a connected graph. Then $M = M(G)$ if and only if*

(1) $r(M) = |V| - 1$, and

(2) for each $v \in V(G)$ and $e \in \delta_G(v)$, there is a cocircuit C of G with $e \in C \subseteq \delta_G(v)$.

Proof. Exercise. ∎

6.4 Excluded minors for graphic matroids

Theorem **341.** [6.9 - Tutte] *If M is a binary excluded minor for the class of graphic matroids, then* $M \in \{F_7, F_7^*, M(K_{3,3})^*, M(K_5)^*\}$.

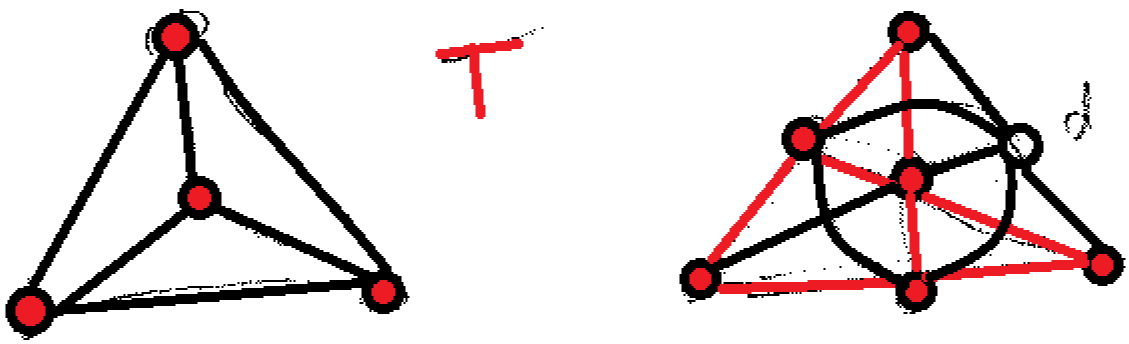
Proof outline. Well M is 3-connected and M is not a wheel (because wheels are graphic) or a whirl (since whirls are not binary). So there exists $e \in E(M)$ such that either $M \setminus e$ is 3-connected or M/e is 3-connected. In the former case, use grafts. In the latter case, use signed graphs. ∎

Definition **342.** Let G be a graph, let $T \subseteq V(G)$, and let $J \subseteq E(G)$. We say that J is a *T-join* if T is the set of odd-degree vertices of $G[V, J]$.

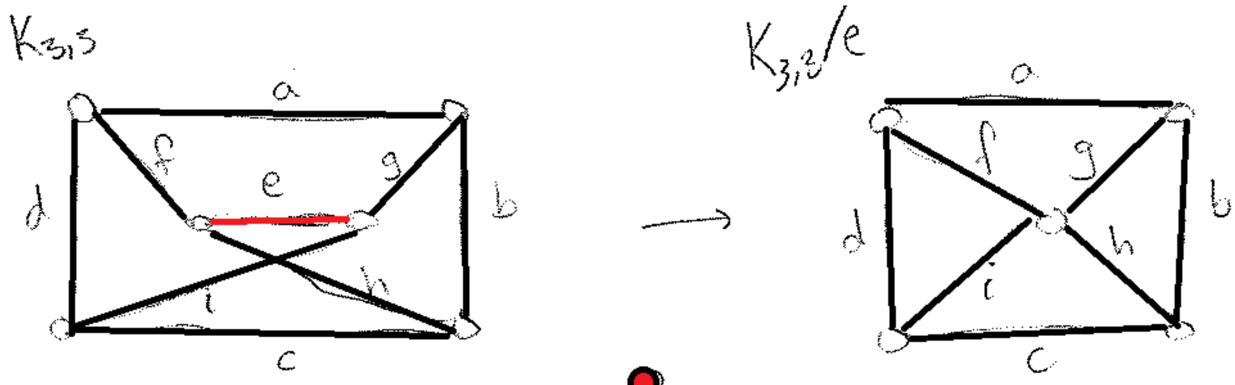
Remark **343.** Let (G,T) be a graft with graft element e and let $J \subseteq E(G)$. Then $J \cup \{e\}$ is a cycle of $M(G,T)$ if and only if J is a T-join.

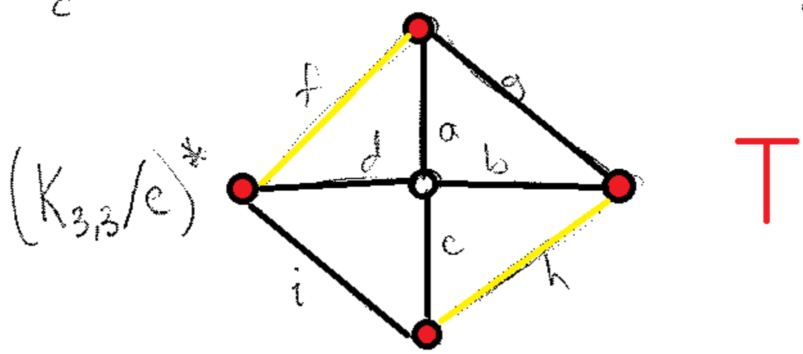
From this observation it follows that if $|T|$ is odd, then e is a coloop of $M(G,T)$, and hence $M(G,T)$ is graphic.

Example **344.** Recall that $F_7 = M(K_4, V(K_4))$.



Example **345.** Also recall that $M(K_{3,3})^*$.





Note that $\{e, f, h\}$ is a cycle of $M(K_{3,3})^*$.

6.5 Graft Minors

DEFINITION **346.** Let (G,T) be a graft. For $f \in E(G)$, we define

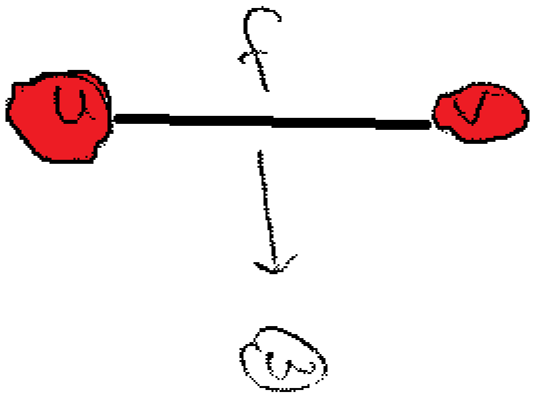
$$M(G,T) \setminus f \coloneqq M(G \setminus f, T)$$
$$M(G,T)/f \coloneqq M(G/f, T')$$

to be *graft deletion* and *graft contraction* respectively, where $T' - \{w\} \coloneqq T - \{u,v\}$ and $w \in T$ if and only if $|T \cap \{u,v\}| = 1$.

THEOREM **347.** [6.10] *If M is a binary matroid excluded minor for the class of graphic matroids and $e \in E(M)$ is such that $M \setminus e$ is 3-connected, then $M \in \{F_7, M(K_{3,3})^*\}$.*

Proof. We can represent M by a graft (G,T) where G is a 3-connected graph and $|T|$ is even and at least four.

Claim (1). Let $f = uv \in E(G)$. If G/f is 3-connected, then $|T| = 4$ and $u, v \in T$.

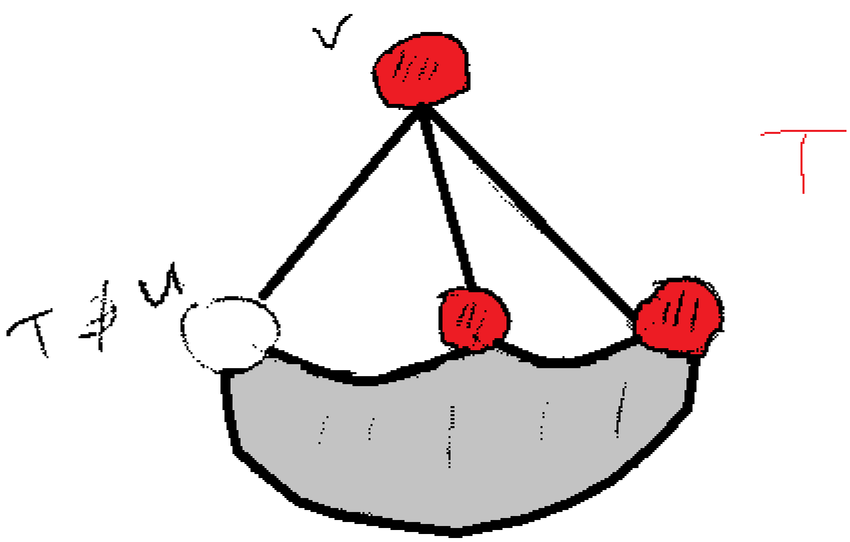


Proof of Claim 1. Well $M/f = M(G/f, T')$ where $|T'|$ is even and $T \cap V(G - u - v) = T \cap V(G - u - v)$. Since M is an excluded minor, M/f is graphic. So, by Lemma 6.4, $|T'| = 2$. Hence $|T| = 4$ and $u, v \in T$. ∎

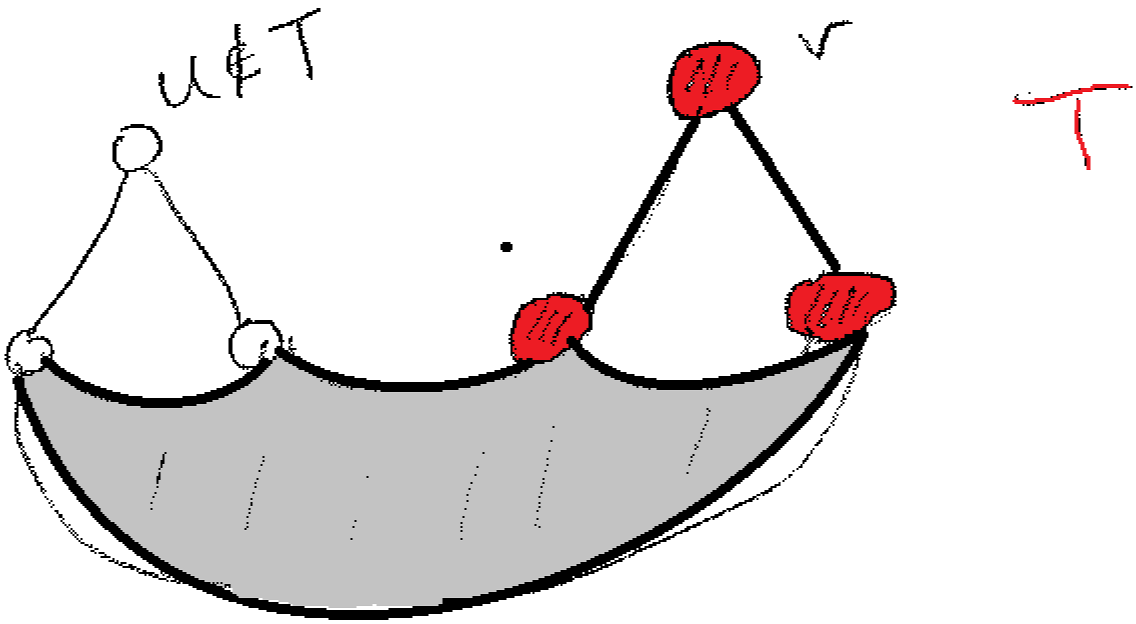
Claim (2). We have $|T| = 4$.

Proof of Claim 2. By an exercise, there exists $f \in E(G)$ such that $si(M(G)/f)$ is 3-connected. Hence G/f is 3-connected. Apply Claim 1 and the desired result follows. ∎

Claim (3). If $f = uv \in E(G)$ and $u \notin T$, then $v \in T$, $\deg(v) = 3$, and $N(v) \subseteq T \cup \{u\}$.



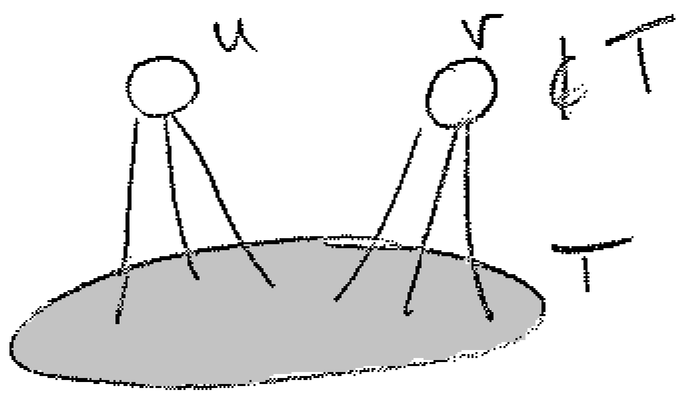
Proof of Claim 3. By Claim 1, G/f is not 3-connected. So $M(G)/f$ is not internally 3-connected. Then, by Bixby's Lemma, $M(G) \setminus f$ is internally 3-connected. The only possible degree two nodes in $G \setminus f$ are u and v.



If $w \in \{u,v\}$ has degree two and $f' \in ww' \in E(G) - \{f\}$, then consider $(G',T') = (G,T) \setminus f/f'$. Now $|T| = |T'|$ unless $w, w' \in T$. So, unless $\deg_{G\setminus f}(v) = 2$, $v \in T$, and $N_{G\setminus f}(v) \subseteq T$, we can get a minor (G'',T'') of $(G,T)\setminus f$ such that G'' is 3-connected and $|T''| = 4$ — contradicting that $M(G,T)\setminus f$ is graphic. ∎

Claim (4). We have $|V(G)| \leq 5$.

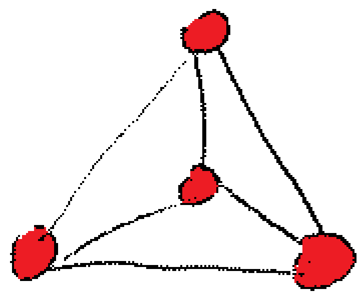
Proof of Claim 4. If not, then there are two vertices $u, v \in V(G) - T$. By Claim 3, $N(u) \subseteq T$, $N(v) \subseteq T$, and $N(u) \cap N(v) = \varnothing$. However G is 3-connected and $|T| = 4$ — contradiction.



∎

Now we are left with two cases. First suppose that $|V(G)| = 4$. This means:

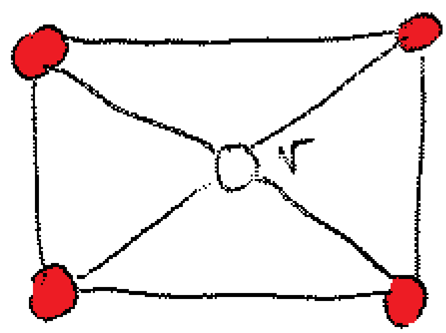
$(G, T) =$



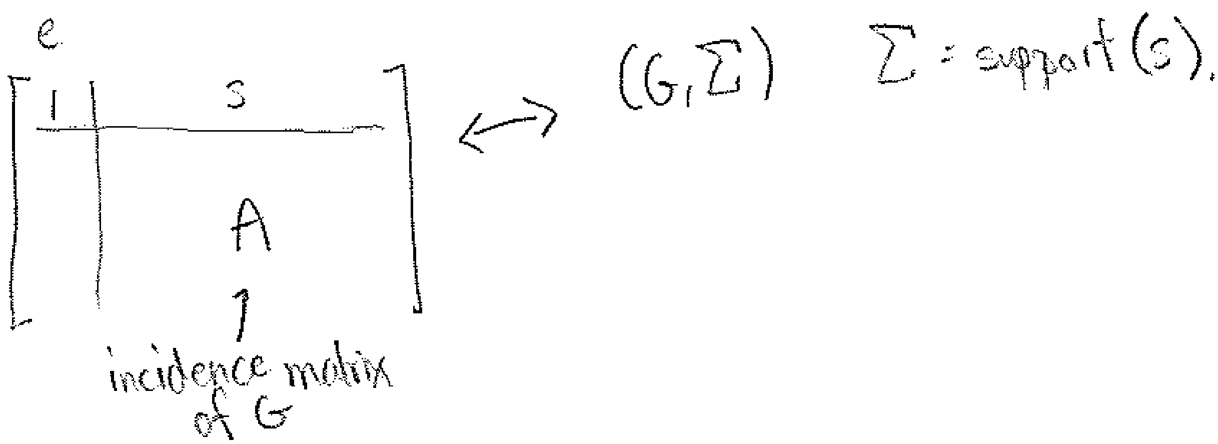
So $M(G,T) = F_7$.

Otherwise we have $|V(G)| = 5$ Let $v \in V(G) - T$. Since $|V(G)|$ is odd, G has a vertex w of even degree. Thus $\deg(w) = 4$. By Claim 3, $v = w$ and each vertex in T has degree 3. Hence:

$(G, T) =$



Thus $M = M(K_{3,3})^*$. ∎



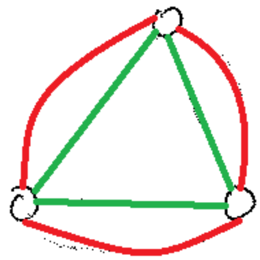
Remark **348.** Let (G, Σ) be a signed graph, with $M/e = M(G, \Sigma)$. Note

(1) $\Sigma \cup \{e\}$ is a cocycle of M, and

(2) if C^* is a cocycle of M that contains e then $M(G, C^* - \{e\}) = M(G, \Sigma) = M$.

F_7:

$$
\begin{array}{c}
\begin{array}{ccccccc} e & & & & & & \end{array} \\
\left[\begin{array}{c|cccccc}
1 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0
\end{array}\right]
\end{array}
$$

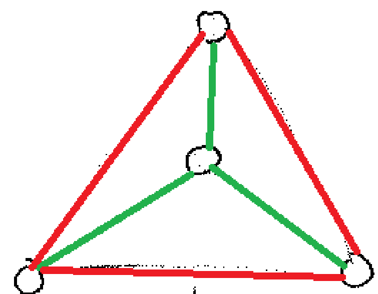


Σ

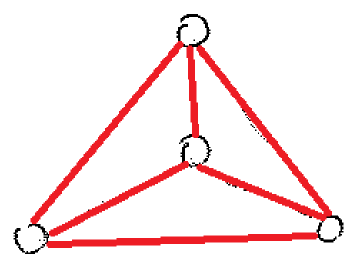
doubled-K_3

F_7^*:

$$
\begin{array}{c}
\begin{array}{ccccccc} & & & e & & & \end{array} \\
\left[\begin{array}{ccc|c|ccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline
1 & & & & 0 & 1 & 1 \\
 & 1 & & & 1 & 0 & 1 \\
 & & 1 & & 1 & 1 & 0 \\
 & & & 1 & 1 & 1 & 1
\end{array}\right]
\end{array}
$$



resign

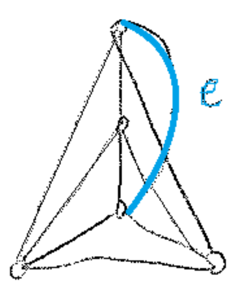


Σ

odd-$K_4 = (K_4, E(K_4))$

$M(K_5)^*$

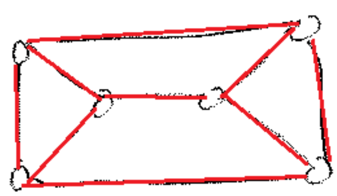
K_5



e

$(K_5 - e)^*$

G



Σ

$M(G, \Sigma) = M(K_5)^*$

REMARK **349.** Note that $E(K_5)$ is a cocycle of $M(K_5)^*$.

6.6 Signed graph Minors

DEFINITION **350.** Let (G, Σ) be a signed graph and let $f = uv \in E(G)$. We define

$$M(G,\Sigma) \setminus f := M(G \setminus f, \Sigma)$$

to be *signed graph deletion.* If $f \notin \Sigma$, then we define

$$M(G,\Sigma)/f := M(G/f, \Sigma)$$

to be *signed graph contraction.* If $f \in \Sigma$ and $u \neq v$, then we define

$$M(G,\Sigma)/f := M(G, \Sigma\Delta\delta(u))/f$$

to be *signed graph contraction.*

REMARK **351.** If $f \in \Sigma$ and $u \neq v$, then

$$M(G,\Sigma)/f = M(G/f, \Sigma\Delta\delta \cup u)\,.$$

Moreover, note that we do not define contraction by f if $f \in \Sigma$ and $u = v$ (in which case e and f are parallel in $M(G,\Sigma)$).

$$\begin{array}{c} \begin{array}{cc} e & f \end{array} \\ \left[\begin{array}{cc} 1 & 1 \\ \hline 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{array}\right] \end{array}$$

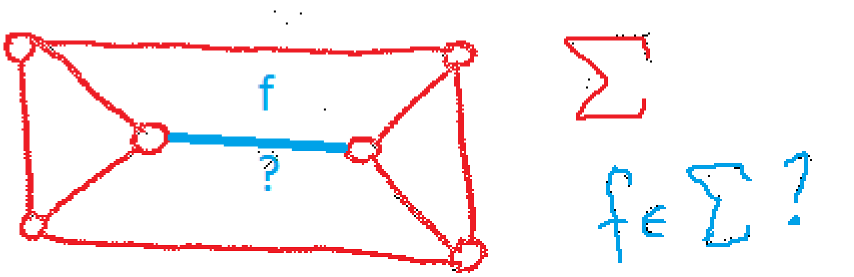
DEFINITION **352.** Let (G,Σ) and (G',Σ') be signed graphs. We say (G',Σ') is a *minor* of (G,Σ) if there exist disjoint edge sets $D, C \subseteq E(G)$ and $X \subseteq V(G)$ such that

(1) $C \cap (\Sigma\Delta\delta(X)) = \varnothing$, and

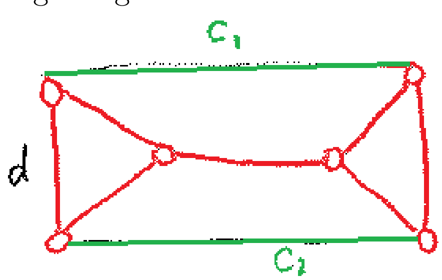
(2) $(G',\Sigma') = (G \setminus D/C, \Sigma'\Delta\delta(X))$.

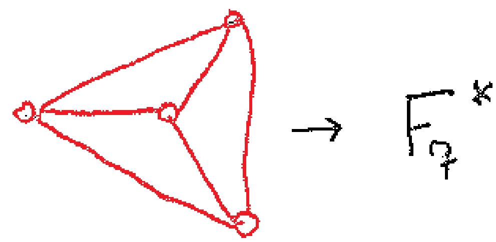
LEMMA **353.** [6.11] *If (G,Σ) has two disjoint odd circuits and G is simple and 3-connected, then $M(G,\Sigma)$ has an F_7^*- or $M(K_5)^*$-minor.*

Proof sketch.



There are two cases. First, if $f \in \Sigma'$ then $M(G', \Sigma') = M(K_5)^*$. Second, if $f \notin \Sigma'$, then resign to get:





Now $M(G', \Sigma') \setminus d / c_1, c_2 \cong F_7^*$. ∎

Recall the following facts and definitions.

Remark 354. (1) A circuit C of G is a circuit of $M(G, \Sigma)$ if and only if $|C \cap \Sigma|$ is even.

(2) A set of edges X of (G, Σ) is odd if $|X \cap \Sigma|$ is odd.

(3) A vertex $v \in V(G)$ is a blocknode of (G, Σ) if each odd circuit of (G, Σ) contains v.

Theorem 355. [6.7] *Let (G, Σ) be a signed graph where G is a loopless and 3-connected graph. Then $M(G, \Sigma)$ is graphic if and only if (G, Σ) has a blocknode.*

Theorem 356. [6.12 - Gerards] *Let (G, Σ) be a signed graph where G is a simple 3-connected graph. Then either*

(1) (G, Σ) has a blocknode, or

(2) (G, Σ) *has two disjoint odd circuits, or*

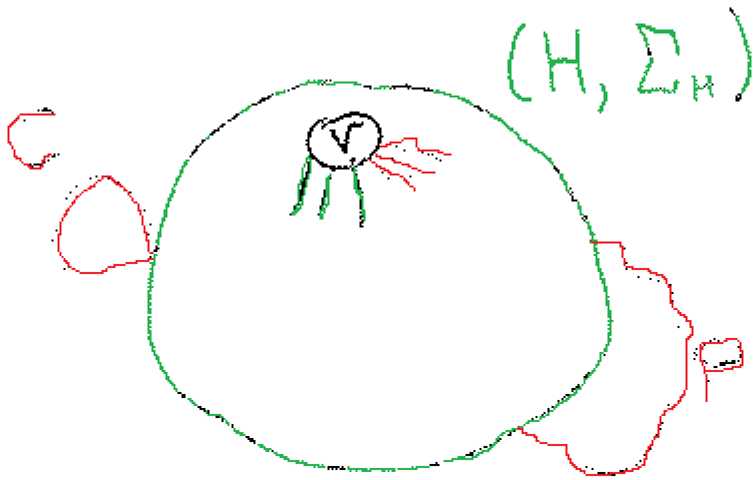
(3) (G, Σ) *has one of the following minors: odd-*K_4*, doubled-*K_3*. (See previous diagrams.)*

Proof. Suppose for a contradiction that the result is false. Consider a counterexample (G, Σ).

Claim (1). Let (H, Σ_H) be a subgraph of (G, Σ) and $v \in V(H)$ such that

(i) (H, Σ_H) has an odd circuit, and

(ii) $\Sigma_H \subseteq \delta_H(v)$.



Then either

(1) there is an odd circuit C of (G, Σ) such that $v \notin V(C)$ and $|V(C) \cap V(H)| = 1$, or

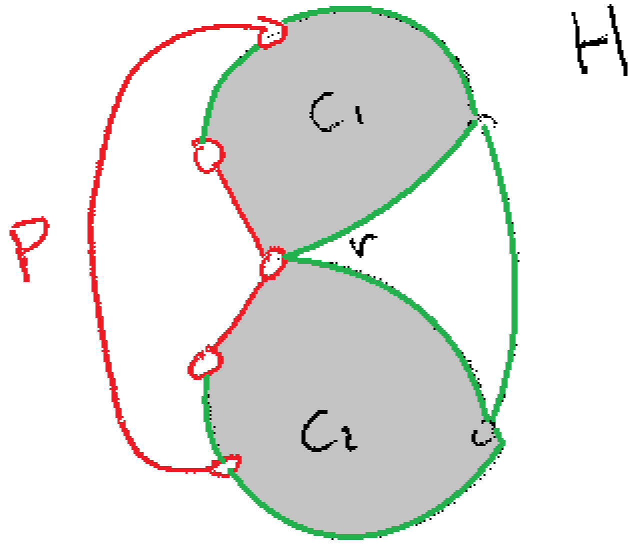
(2) there is an odd path P of (G, Σ) with ends u and w such that $v \notin \{u, w\}$ and $V(P) \cap V(H) = \{u, w\}$. (We call such a P an *odd-ear* of (H, Σ_H) *avoiding* v).

Proof of Claim 1 Sketch. Since (G, Σ) has no blocknode, there is an odd circuit C of (G, Σ) avoiding v. Since (H, Σ_H) has an odd circuit and (G, Σ) has no two disjoint odd circuits, we get $V(C) \cap V(H) \neq \varnothing$.

We may assume that $|V(H) \cap V(C)| \geq 2$ since otherwise the claim holds. Now, since $(H, \Sigma_H) - v$ has no odd edges and C is odd, we easily find an odd ear in C for (H, Σ_H) avoiding v. ∎

Claim (2). If C_1 and C_2 are odd-circuits, then $|V(C_1) \cap V(C_2)| \geq 2$.

Proof of Claim 2 Sketch. Suppose that $V(C_1) \cap V(C_2) = \{v\}$. Let $(H, \Sigma_H) = C_1 \cup C_2 \cup Q$ where Q is a shortest $(V(C_1) - v, V(C_2) - v)$-path in $(G, \Sigma) - v$. By resigning we may assume that $\Sigma_H \subseteq \delta(v)$.

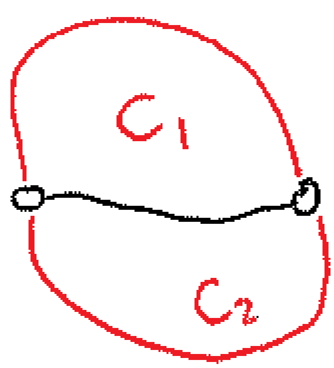


Now we apply Claim 1, and hence have two cases to consider. Firstly, suppose that there is an odd circuit C of (G,Σ) with $v \notin V(C)$ and $|V(C) \cap V(H)| \le 1$. Then C is disjoint from C_1 or C_2 — contradiction.

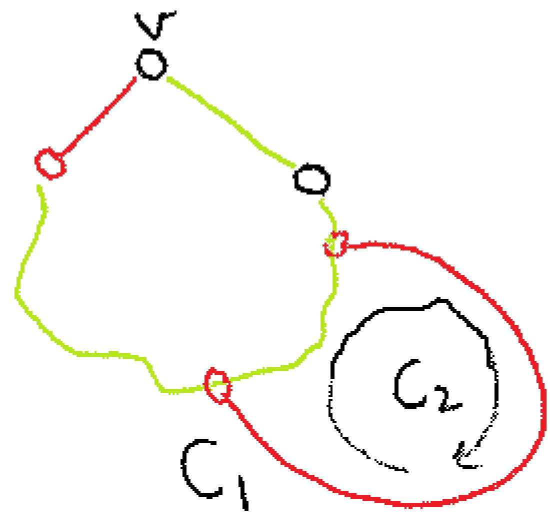
Secondly, suppose that there is an odd ear P' of (H,Σ_H) avoiding v. Then since (G,Σ) has no disjoint odd circuits, P has an end $u \in V(C_1) - v$ and an end $w \in V(C_2) - v$. Then (G,Σ) has a doubled-K_3 minor. ∎

Note that Claim 2 precludes outcome (1) of Claim 1.

Claim (3). There are odd circuits C_1 and C_2 such that $C_1 \cap C_2$ is a path.

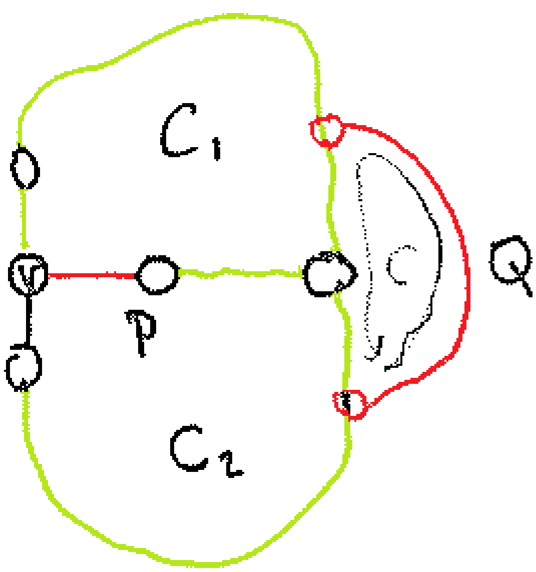


Proof of Claim 3. Let C_1 be an odd circuit and let $v \in C_1$. Resign (G,Σ) so that all odd edges of C_1 are incident with v. By Claims 1 and 2, there is an odd ear P of C_1 avoiding v. Let C_2 be the unique circuit in $(C_1 - v) \cup P$. Then C_1 and C_2 satisfy the claim.

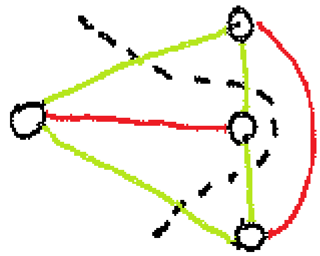


■

Choose C_1 and C_2 satisfying Claim 3 and with $|V(C_1) \cap V(C_2)|$ as small as possible. Let $P = C_1 \cap C_2$ and let v be an end of P. Resign (G, Σ) so that all odd edges of $C_1 \cup C_2$ are incident with v and so that there is only on odd edge (that edge is in P).



By Claims 1 and 2, there is an odd ear Q of $C_1 \cup C_2$ that avoids v. Let C be the unique circuit in $((C_1 \cup C_2) - v) \cup Q$. Note that C is odd and $C \cap C_1$ and $C \cap C_2$ are both paths. So, by the minimality of P, Q cannot have an end on P. Since (G, Σ) has no disjoint odd circuits, Q has an end in $V(C_1) - V(P)$ and an end in $V(C_2) - V(P)$. So (G, Σ) has an odd-K_4 minor.



odd-K_4

∎

Corollary **357.** [6.13] *If M is a binary excluded minor for the class of graphic matroids and there exists $e \in E(M)$ such that M/e is 3-connected, then $M \cong F_7^*$ or $M \cong M(K_5)^*$.*

Proof. There is no element $e \in E(F_7)$ such that F_7/e is 3-connected. It suffices to prove that M has a minor isomorphic to F_7, F_7^*, or $M(K_5)$. Suppose otherwise.

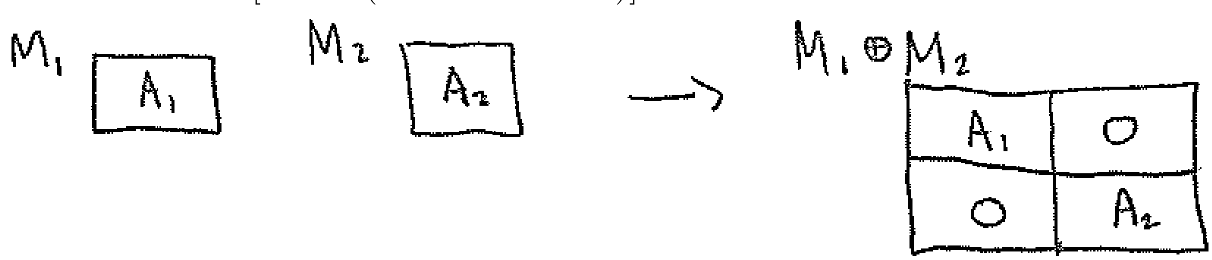
We can represent M by a signed graph (G, Σ) where G is simple and 3-connected. Since M is not graphic (G, Σ) has no block node. By Lemma 6.11, (G, Σ) has no two disjoint odd circuits. Then, by Theorem 6.12, (G, Σ) has a minor isomorphic to odd-K_4 or doubled-K_3. Then M has a minor isomorphic to F_7^* or F_7 — contradiction. ∎

Theorem 6.10 and Corollary 6.13 together prove that the binary excluded minors for the class of graphic matroids are F_7, F_7^*, $M(K_5^*)$, and $M(K_{3,3})^*$. Thus the excluded minors for the class of graphic matroids are $U_{2,4}$, F_7, F_7^*, $M(K_5^*)$, and $M(K_{3,3})^*$.

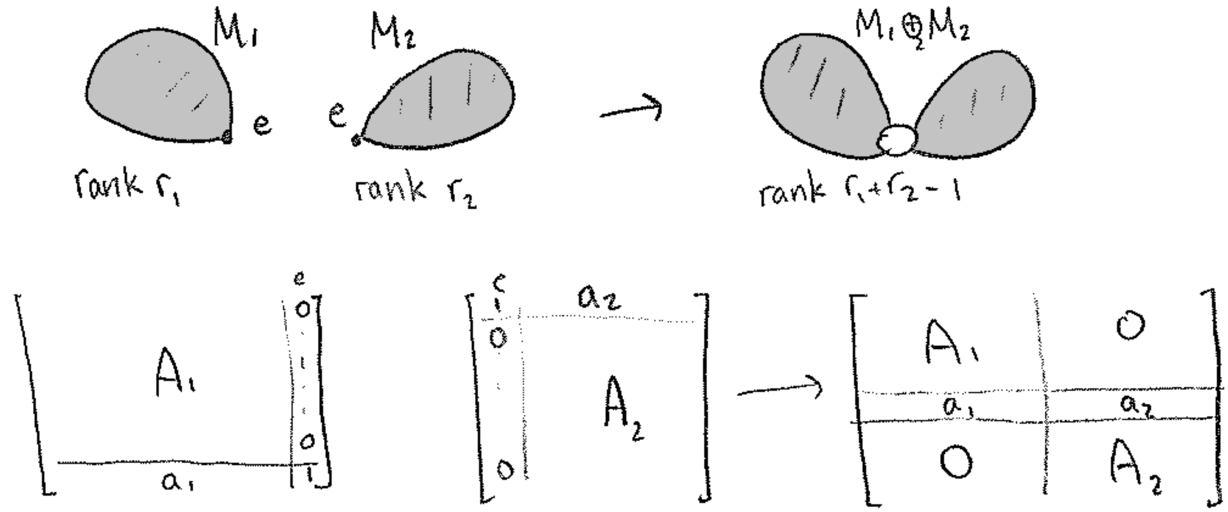
7 Regular Matroid Decomposition

7.1 Sums of binary matroids

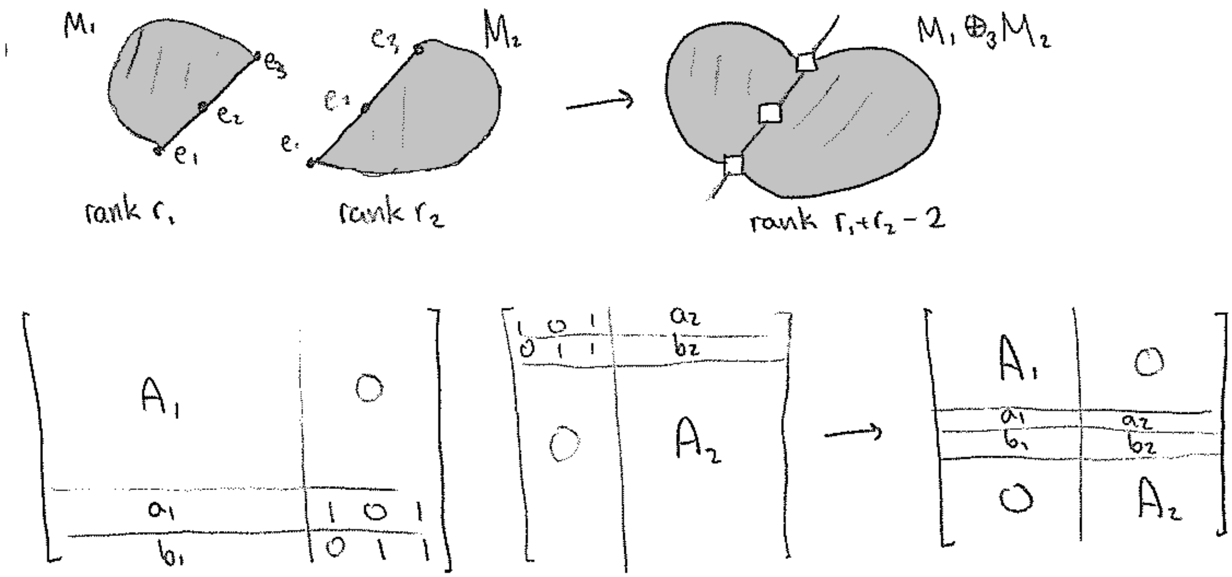
Definition **358.** [1-Sum (or Direct Sum)]



Definition **359.** [2-Sum]



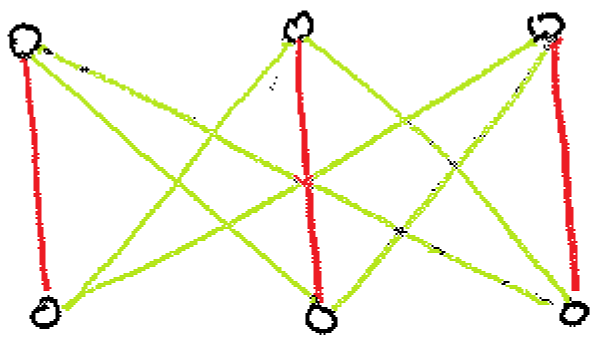
DEFINITION **360.** [3-SUM]



REMARK **361.** The class of regular matroids is closed under 1-, 2-, and 3-sums.

7.2 Seymour's Decomposition Theorem

Definition **362.** We define $R_{10} \coloneqq M(G, \Sigma)$ where



over $GF(3)$

$$\begin{bmatrix} \text{incidence matrix of } K_5 \end{bmatrix}$$

Theorem **363.** [Seymour's Decomposition Theorem] *Every regular matroid can be constructed by* 1-, 2-, *and* 3-*sums from graphic matroids, cographic matroids, and copies of* R_{10}.