# Cartier-Foata partial commutation monoids (Heap Inversion)

Example: $\underline{x} = \{x_1, x_2, \dots, x_n\}$ indeterminates.
Let $d \in \mathbb{N}$. How many monomials of degree $d$ are there? Answer depends on whether or not the $x_i$ commute or not.
Don't commute: $x_{i_1} x_{i_2} \cdots x_{i_d}$, $n^d$ choices
Do commute: $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ with $a_1 + a_2 + \cdots + a_n = d$.
multiset of size $d$ with $n$ types of element, $\binom{d+n-1}{n-1}$.

## Generating Functions

Don't commute:

$$\sum_{d=0}^{\infty} (x_1 + x_2 + \cdots + x_n)^d = \frac{1}{1-(x_1+x_2+\cdots+x_n)}$$

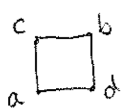
Do commute:

$$\sum_{a_1=0}^{\infty} \cdots \sum_{a_n=0}^{\infty} x_1^{a_1} \cdots x_n^{a_n} = \left(\sum_{a_1=0}^{\infty} x_1^{a_1}\right) \cdots \left(\sum_{a_n=0}^{\infty} x_n^{a_n}\right)$$

$$= \frac{1}{(1-x_1) \cdots (1-x_n)}$$

## Partial Commutation monoid

$\underline{x} = \{x_1, \dots, x_n\}$
Let $G$ be a graph with vertex-set $\{1, 2, \dots, n\}$, $\{i,j\} \in E$ if and only if $x_i$ and $x_j$ don't commute. None commute: $K_n$. All commute: $\overline{K_n}$.

eg. 

, $C_4$ a,b commute
c,d commute
nothing else does

$$\varepsilon \cup (a^* b^*)(c^* d^*)(a^* b^*)$$

$$\frac{1}{(1-c)(1-d)} \; \frac{1}{1-\left(\frac{a+b-ab}{(1-a)(1-b)}\right)\left(\frac{c+d-cd}{(1-c)(1-d)}\right)} \; \frac{1}{(1-a)(1-b)}$$

take the commutative projection:

$$\frac{1}{(1-c)(1-d)(1-a)(1-b) - (a+b-ab)(c+d-cd)}$$

$$= \frac{1}{(1-a)(1-b)(1-c)(1-d) - ((ac+ad+bc+bd) - (abc+abd+acd+bcd) + abcd)}$$

2015 12 02

$$= \frac{1}{1-(a+b+c+d)+(ab+cd)}$$

Stable set polynomial of a graph

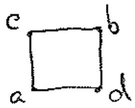
$G=(V,E)$ a simple graph

$\underline{x}=\{x_v ; v\in V\}$ indeterminates indexed by $V$.

$S\subseteq V$ is stable if no two vertices in $S$ are joined by an edge

$$\underline{x}^S = \prod_{v\in S} x_v$$

$$S(G;\underline{x}) = \sum_{S\subseteq V \text{ stable}} \underline{x}^S$$

eg $G$  $\frac{1}{S(G;-\underline{x})}$

generating function for all words in $\{a,b,c,d\}^*$ modulo partial commutation ($ab=ba$ and $cd=dc$)

Cartier-Foata monoid of $G$

$G=(V,E)$ simple graph

$V^*$ set of all strings of vertices

Define an equivalence relation $\approx$ on $V^*$: For any $\alpha,\beta\in V^*$ and $v,w\in V$, $\alpha vw\beta \approx \alpha wv\beta$ if and only if $\{v,w\}\notin E$. Extend this by transitivity.

$$M(G) = V^*/\approx$$

The equivalence class of a word $\alpha = v_1v_2\cdots v_n$ is $[\alpha]$ is a woid.

For a woid $[\alpha]=[v_1v_2\cdots v_n]$, let

$$\underline{x}^{[\alpha]} = x_{v_1}\cdots x_{v_n}$$

$$M(G;\underline{x}) = \sum_{[\alpha]\in M(G)} \underline{x}^{[\alpha]}.$$

Heap Inversion Formula: Let $G=(V,E)$ be a graph. Then

$$M(G;\underline{x}) = \frac{1}{S(G;-\underline{x})}.$$

Note the similarity to $H(t)=E(-t)^{-1}$.

The <u>head</u> of a woid. Let $\alpha = v_1 v_2 \cdots v_n$ be a word in $V^*$. It represents a woid $[\alpha]$ in $M(G)$. Let $Z$ be the set of vertices $v_i$ in $\alpha$ that can be commuted into the first position of some word $\beta$ equivalent to $\alpha$ ($\beta \approx \alpha$). Note: $Z$ is a stable set in $G$. Note: if $\alpha \approx \beta$ then $\mathrm{head}(\alpha) = \mathrm{head}(\beta)$. Head of a woid $\mathrm{head}([\alpha])$ is well-defined.

Proof (of Heap Inversion): Show $S(G;-\underline{x})M(G;\underline{x}) = 1$. Sign reversing involution on $S(G)\times M(G)$ (where $S(G)$ is the set of stable sets of $G$) that cancels everything except $(\emptyset, [\varepsilon])$.
Given $(Z,[\alpha])$ with either $Z \neq \emptyset$ or $\alpha \neq \varepsilon$, $[Z]$ is a well-defined woid. Consider the set $A = \mathrm{head}([Z][\alpha])$. This is not empty.
Fix a total order on $V$, $V = \{v_1, v_2, \ldots, v_n\}$.
Let $b$ be the first vertex in $A$. Then either $b \in Z$ or $b \in \mathrm{head}([\alpha])$ (not both).

$$(Z,[\alpha]) \mapsto \begin{cases} (Z\setminus b, [b\alpha]) & \text{if } b \in Z \\ (Z\cup b, [\hat{\alpha}]) & \text{if } b \in \mathrm{head}([\alpha]) \end{cases}$$

where $\hat{\alpha}$ move $b$ to the front of $\alpha$ and delete it.
Note that

$$\mathrm{head}(f([Z][\alpha])) = \mathrm{head}([Z][\alpha]).$$

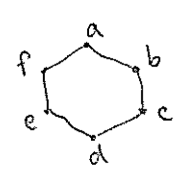
It follows that $f: S(G)\times M(G) \to S(G)\times M(G)$ is an involution that fixes only $(\emptyset, [\varepsilon])$.

$$f: (Z,[\alpha]) \mapsto (Z',[\alpha'])$$

sizes of $Z$ and $Z'$ have opposite parity, that gives the sign change in $S(G;-\underline{x})$. ∎

2015 12 04

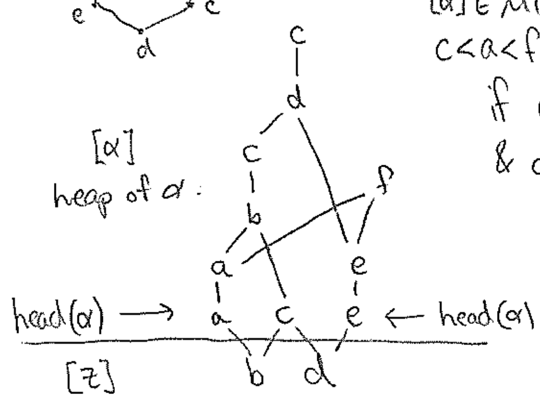
eg.




$\alpha = acabceedfce \in V^*$
$\downarrow$
$[\alpha] \in M(G) = V^*/\approx$
$c < a < f < b < e < d$

if $\alpha \approx \beta$ then $\mathrm{heap}(\alpha) = \mathrm{heap}(\beta)$
& conversely

[α]
heap of α:



2015 12 04(2)

$$S(G)\times\mathcal{M}(G)\xrightarrow{\varphi}S(G)\times\mathcal{M}(G)$$

- the only fixed element $(\emptyset,[\varepsilon])$
- all other elements cancel in pairs

<u>Stable set polynomials</u>

$$S(G;\underline{x})=\sum_{Z\subseteq V\text{ stable}}\underline{x}^Z \qquad \underline{x}^Z=\prod_{v\in Z}x_v$$

Set all $x_v=t$.

$$S(G;t)=\sum_{Z\subseteq V\text{ stable}}t^{|Z|}=\sum_{k=0}^{n}c_k t^k$$

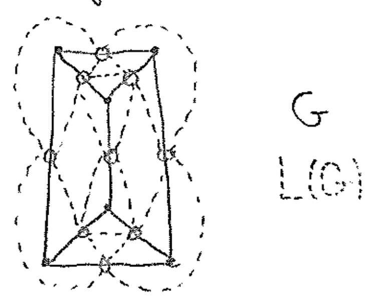
where $c_k$ is the number of $k$-element stable sets.

<u>Line-graphs</u>

$L(G)$ of $G$

vertices of $L(G)$ are edges of $G$.

$e,f$ are adjacent in $L(G)$ if they have a common vertex.




Stable sets in $L(G)$ are matchings in $G$.

Heilmann-Lieb (1972): For any graph $G$, $S(L(G);t)$ has only real zeros.

Newton's Inequalities

$$\frac{c_k^2}{\binom{n}{k}^2}\geq\frac{c_{k-1}}{\binom{n}{k-1}}\frac{c_{k+1}}{\binom{n}{k+1}}\Rightarrow c_k^2>c_{k-1}c_{k+1}\quad(\text{if } c_k\neq 0)$$

Line graphs are claw-free (no induced $K_{1,3}$)

Hamidowne (90s): for a claw-free graph $H$, $S(H,t)$ has log-concave coefficients $c_k\geq c_{k-1}c_{k+1}$.

Chudnovsky-Seymour (early 2000s): If $H$ is claw-free then $S(H;t)$ has only real zeros.