"it's like rain, it just happens"

(loud screams outside)

"~~If it's~~ the zombie apocalypse I'm sorry, I quit"

"I'm functioning at 10% of my ~~normal~~ 10% level"

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## $\mathfrak{sl}(2,\mathbb{C})$ and Combinatorics

Let

$$V = \bigoplus_{j\in\mathbb{Z}} V_j$$

be a vector space (over $\mathbb{C}$). Assume $V$ is finite-dimensional.
Let $X: V_j \to V_{j+2}$ and $Y: V_j \to V_{j-2}$ be endomorphisms.
Let $H = XY - YX$. Assume that $V_j$ (if not zero) is an eigenspace for $H$ with eigenvalue $j$.
Then:

- for $j \geq -1$: $X: V_j \to V_{j+2}$ is surjective
- for $j \leq -1$: $X: V_j \to V_{j+2}$ is injective
- for $k \in \mathbb{N}$: $X^k: V_{-k} \to V_k$ is a bijection

Let $d_j = \dim_{\mathbb{C}} V_j$. Then

$$\cdots \leq d_{-5} \leq d_{-3} \leq d_{-1} = d_1 \geq d_3 \geq d_5 \geq \cdots$$
$$\cdots \leq d_{-4} \leq d_{-2} \leq d_0 \geq d_2 \geq d_4 \geq \cdots$$

Why is it $\mathfrak{sl}(2,\mathbb{C})$?
$SL(2,\mathbb{C})$ is the group of $2\times 2$ matrices over $\mathbb{C}$ of determinant 1.
$V$ has a basis $\underline{u}, \underline{v}$.

$$X(\underline{u}) = \underline{v} \qquad X(\underline{v}) = 0$$
$$Y(\underline{v}) = \underline{u} \qquad Y(\underline{u}) = 0$$

| $X$ | $\underline{u}$ | $\underline{v}$ |
|---|---|---|
| $\underline{u}$ | 0 | 0 |
| $\underline{v}$ | 1 | 0 |

| $Y$ | $\underline{u}$ | $\underline{v}$ |
|---|---|---|
| $\underline{u}$ | 0 | 1 |
| $\underline{v}$ | 0 | 0 |

$\underline{u}$ spans $V_{-1}$
$\underline{v}$ spans $V_1$

Proof: Induction on $d = \dim(V)$. Since $d$ is finite, only finitely many $V_j$ are not zero. Let $k = \max\{j ; V_j \neq 0\}$. Let $\underline{v}_0 \in V_k$ be a nonzero vector $\underline{v}_0 \neq 0$. For each $m \in \mathbb{N}$, let $\underline{v}_m = Y^m(\underline{v}_0)$. Then $\underline{v}_m \in V_{k-2m}$. Let $U = \operatorname{span}\{\underline{v}_0, \underline{v}_1, \dots\}$. $U \subseteq V$. Only finitely many $\underline{v}_m$'s are not zero.

Lemma: $X$, $Y$, and $H$ map $U$ to $U$. ($U$ is a subrepresentation of $V$.)

Proof: Clearly $Y$ maps $U$ to $U$ since $Y(\underline{v}_m) = \underline{v}_{m+1}$ for all $m \in \mathbb{N}$.
Since $H = XY - YX$ it suffices to show that $X: U \to U$.
We'll show that $X: U_j \to U_{j+2}$ for each $j$. $X(\underline{v}_m) = c(k,m)\underline{v}_{m-1}$ for some constants $c(k,m)$. Go by induction on $m \in \mathbb{N}$.

$X(\underline{v}_0) = 0$

$X(\underline{v}_1) = XY\underline{v}_0 = (XY - YX)\underline{v}_0 + YX\underline{v}_0 = H\underline{v}_0 = k\underline{v}_0$

So $c(k,1) = k$ exists.

$X(\underline{v}_2) = XY\underline{v}_1 = (XY-YX)\underline{v}_1 + YX\underline{v}_1 = H\underline{v}_1 + Yk\underline{v}_0 = (k-2)\underline{v}_1 + k\underline{v}_1 = (2k-2)\underline{v}_1$

So $c(k,2) = 2k-2$ exists.

Induction step: Assume that $c(k,m)$ exists.

$$X(\underline{v}_m) = c(k,m)\underline{v}_{m-1}$$

Show that $c(k,m+1)$ exists.

$$X(\underline{v}_{m+1}) = (XY-YX)\underline{v}_m + YX\underline{v}_m = H\underline{v}_m + Yc(k,m)\underline{v}_{m-1}$$
$$= (k-2m)\underline{v}_m + c(k,m)\underline{v}_m$$

So $c(k,m+1) = (k-2m) + c(k,m)$ exists. So $U$ is invariant under $X$, $Y$, and $H$. $\square$

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By induction we can show

$$c(k,m) = km - 2\binom{m}{2} = m(k-m+1).$$

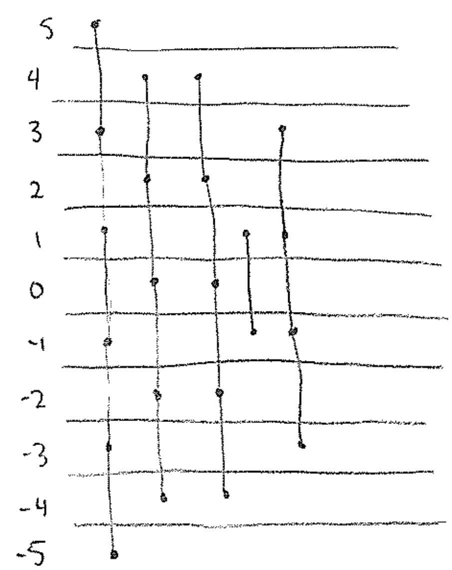
Each $\underline{v}_m$ is in $V_{k-2m}$. Since $V$ is finite dimensional, eventually $\underline{v}_m = \underline{0}$ for $m$ sufficiently large. $X(\underline{v}_m) = m(k-m+1)\underline{v}_{m-1}$. We know that $\underline{v}_0 \neq \underline{0}$. Assume that $\underline{v}_{m-1} \neq \underline{0}$. Either $\underline{v}_m = 0$ or $m(k-m+1) = 0$. If $\underline{v}_m = \underline{0}$ then $m(k-m+1) = 0$. So either $m=0$ or $m = k+1$. So $-k$ is also the smallest index such that $V_j \neq 0$.

Let $d_j = \dim V_j$. Then

$$\cdots \le d_{-5} \le d_{-3} \le d_{-1} = d_1 \ge d_3 \ge d_5 \ge \cdots$$

and

$$\cdots \le d_{-4} \le d_{-2} \le d_0 \ge d_2 \ge d_4 \ge \cdots$$



Eg Boolean Representations

Let $A$ be a finite set. $\mathcal{P}(A)$ is the set of all subsets of $A$.
$V = \mathbb{C}\mathcal{P}(A)$, vector space of all formal linear combinations of $S \subseteq A$.
Basis: $\{[S] ; S \subseteq A\}$. Define $X, Y : V \to V$ on basis vectors. Extend linearly. Then $H = XY - YX$.

$$X([S]) = \sum_{a \in A \setminus S} [S \cup \{a\}]$$

$$Y([S]) = \sum_{b \in S} [S \setminus \{b\}]$$

What is $H = XY - YX$? Apply $H$ to $[S]$ where $\#S = i$.

$$\circledast \quad XY([S]) = (i)[S] + \sum_{T; |T \Delta S| = 2, |T| = |S|} [T]$$

$$\circledast \quad YX([S]) = (n-i)[S] + \sum_{T; |T \Delta S| = 2, |T| = |S|} [T]$$

$$H([S]) = (i)[S] - (n-i)[S] = (2i-n)[S].$$

For every $i$-element subset $S$, $[S]$ is an eigenvector of $H$ with eigenvalue $2i-n$. So this is a representation of $\mathfrak{sl}(2, \mathbb{C})$.
The $i$-element subsets are a basis for the $(2i-n)$-eigenspace of $H$.

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$A$ a finite set, $|A| = n$. Let $\mathcal{P}_k(A)$ be the set of all $k$-element subsets of $A$, and let

$$\mathcal{P}(A) = \bigcup_{k=0}^{n} \mathcal{P}_k(A)$$

be the set of all subsets of $A$. Set $V_k = \mathbb{C}\mathcal{P}_k(A)$ and

$$X : V_k \to V_{k+1} \quad \text{by} \quad X([S]) = \sum_{a \in A \setminus S} [S \cup \{a\}]$$

$$Y : V_k \to V_{k-1} \quad \text{by} \quad Y([S]) = \sum_{b \in S} [S \setminus \{b\}]$$

Then $H = XY - YX$ has $V_k$ as eigenspace with eigenvalue $2k-n$.

Corollary: $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}, \binom{n}{n}$

is palindromic and unimodal.

## G-equivariant representations of $sl_2(\mathbb{C})$

$G$ a permutation group on $A$. $V = \mathbb{C}P(A)$, $\{X, Y, H = XY - YX\}$ a representation of $sl_2$ on $V$. The representation is $G$-equivariant if and only if for every $g \in G$ and every $\underline{v} \in V$,

$$gX(\underline{v}) = Xg(\underline{v}) \text{ and } gY(\underline{v}) = Yg(\underline{v}).$$

Then $gH(\underline{v}) = Hg(\underline{v})$ too.

Proposition: Let $V = \mathbb{C}P(A)$ be a boolean representation of $sl_2$. Let $G \leq S_A$ be a permutation group. Assume $V$ is $G$-equivariant. Let

$$V^G = \{\underline{v} \in V ;\ g(\underline{v}) = \underline{v} \text{ for all } g \in G\}$$

be the $G$-invariant subspace. Then $X : V^G \to V^G$ and $Y : V^G \to V^G$ so if $V^G \neq \{\underline{0}\}$ then it is a representation of $sl_2(\mathbb{C})$.

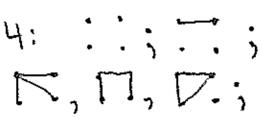
Proof: Assume $\underline{v} \in V^G$ so that $g(\underline{v}) = \underline{v}$ for all $g \in G$. Consider $\underline{w} = X\underline{v}$. For any $g \in G$:

$$g(\underline{w}) = gX\underline{v} = Xg\underline{v} = X\underline{v} = \underline{w}.$$

So $\underline{w} \in V^G$. Same for $Y$. ∎

## Applications

① Unlabelled graphs (Isomorphism classes of graphs)
Let $g(n,k)$ be the number of isomorphism classes of graphs with $n$ vertices and $k$ edges.

$n=4$:  (palindromic)

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $g(4,k)$ | 1 | 1 | 2 | 3 | 2 | 1 | 1 |

Strategy: $G$-equivariant action of $sl_2(\mathbb{C})$ on $V = \mathbb{C}P(A)$. Fix $n \in \mathbb{N}$. Let $N = \{1, 2, \ldots, n\}$ and $A$ be the set of 2-element subsets of $N$. Let $G$ be the permutation group $S_n$ acting on graphs with vertex set $N$. For $g \in G$ and $\underline{\Psi}$ a graph, $g(\underline{\Psi})$ has same vertices $N$, edges $\{\{g(v), g(w)\} ; \{v, w\} \in E(\underline{\Psi})\} \subseteq A$.

## G-equivariant representations

$G \leq S_A$ a permutation group on $A$. For $\sigma \in G$, define

$$\bar{\sigma} : \mathcal{P}(A) \to \mathcal{P}(A) \text{ by } \bar{\sigma}(S) = \{\sigma(a) ; a \in S\}.$$

$\{X, Y, H\}$ is G-equivariant if $X\bar{\sigma} = \bar{\sigma}X$ and $Y\bar{\sigma} = \bar{\sigma}Y$ for all $\sigma \in G$.
Then $H\bar{\sigma} = \bar{\sigma}H$ too.

## G-invariant subspace

$$V^G = \{\underline{v} \in V ; \bar{\sigma}(\underline{v}) = \underline{v} \text{ for all } \sigma \in G\}$$

Fact: If $\{X, Y, H\}$ spans a G-equivariant $sl_2$ representation on $V$, then if $V^G \neq \{0\}$ then $V^G$ is a subrepresentation of $V$. So all those palindromic unimodal inequalities apply to $V^G$.

## Basis for $V^G$

Pick $\underline{v} \in V$. Then

$$\tilde{\underline{v}} = \frac{1}{|G|} \sum_{\sigma \in G} \bar{\sigma}(\underline{v})$$

is G-invariant.
For $G \leq S_A$ acting on $\mathcal{P}(A)$, let the orbits be $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_m$. For each $\mathcal{O}_i$, all sets in $\mathcal{O}_i$ have the same size. Let

$$[\mathcal{O}_i] = \sum_{S \in \mathcal{O}_i} [S].$$

Then each $[\mathcal{O}_i]$ is G-invariant.
Fact: $\{[\mathcal{O}_1], [\mathcal{O}_2], \dots, [\mathcal{O}_m]\}$ is a basis for $V^G$.

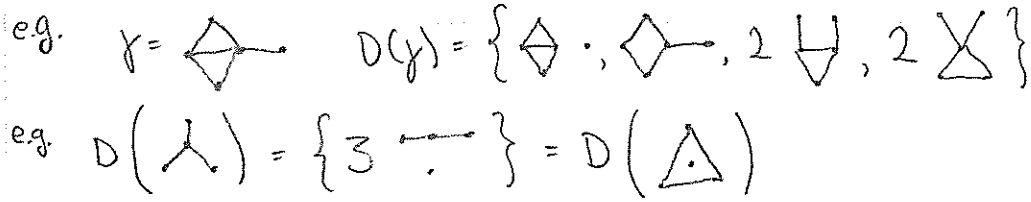
## Application to Graphs

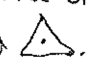
$N = \{1, 2, \dots, n\}$ (vertices), $A$ is 2-element subsets of $N$ (edges).
$\mathcal{P}(A)$ is simple graphs with vertex set $N$. $G = S_N$ acting on vertices.
For $a = \{v, w\}$ in $A$ and $\sigma \in G$: $\sigma(a) = \{\sigma(v), \sigma(w)\}$. Induce this up to acting on $\mathcal{P}(A)$. The orbits of this action of $S_N$ on $\mathcal{P}(A)$ are the isomorphism classes of graphs with $n$ vertices.
Let $g(n,k)$ be the number of isomorphism classes of $n$-vertex graphs with $k$ edges. The action of $X, Y, H$ on $\mathcal{P}(A)$ is $S_N$-

equivariant. So $g(n,k)$ $(0 \le k \le \binom{n}{2})$ is palindromic & unimodal.

## Edge-Reconstruction

Given an (unlabelled) graph $\gamma$. Delete one edge of $\gamma$ in all possible ways, $\{\gamma \setminus e \,;\, e \in E(\gamma)\}$ is a multiset of $|E(\gamma)|$ (isomorphism classes of) graphs. This multiset is the edge-deck $D(\gamma)$ of $\gamma$.



Conjecture: The only failure of injectivity for the map $\gamma \mapsto D(\gamma)$ is for  and .

Linearize the construction. $S_N$ acting on $\{1,2,\dots,n\}$, then on $A$ (2-element subsets) then on $\mathbb{P}(A)$, graphs with vertex-set $N$. Orbits are indexed by isomorphism classes $\gamma$ of graphs.

$$[\mathcal{O}_\gamma] = \sum_{\substack{S \subseteq A \\ (N,S) \simeq \gamma}} [S]$$

$$Y[\mathcal{O}_\gamma] = \sum_{\substack{S \subseteq A \\ (N,S) \simeq \gamma}} \sum_{b \in S} [S \setminus b] = \sum_{\eta \in D(\gamma)} c(\gamma,\eta)[\mathcal{O}_\eta].$$

$Y$ is injective on isomorphism classes of graphs with $k > \frac{1}{2}\binom{n}{2}$ edges.

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Reynolds operator: $R : V \to V$ ($\mathbb{C}\mathbb{P}(A)$ still) G-symmetrizer

$$R(\underline{v}) = \frac{1}{|G|} \sum_{\sigma \in G} \sigma(\underline{v})$$

Exercise: $R^2 = R$. So it is a projection. Image of $R$ is $V^G$.

Let $S \subseteq A$ represent a graph $(N,S)$ isomorphism class $\gamma$.

$$R([S]) = \frac{1}{n!} \sum_{\sigma \in S_N} \bar{\sigma}([S]) = \frac{\#\mathrm{aut}(\gamma)}{n!} \sum_{\substack{Z \subseteq A \\ (N,Z) \simeq \gamma}} [Z] = \frac{\#\mathrm{aut}(\gamma)}{n!} [\mathcal{O}_\gamma].$$

R commutes with X, Y, and H. Since the $sl_2$-representation is G-equivariant. For [S], the edge-deck is Y([S]).

$$RY([S]) = YR([S]) = Y\frac{\#aut(\gamma)}{n!}[\mathcal{O}_\gamma]$$

$$RY([S]) = R\sum_{b\in S}[S\setminus b] = \sum_{\eta} D(\eta,\gamma)\frac{\#aut(\eta)}{n!}[\mathcal{O}_\eta]$$

where $D(\eta,\gamma)$ is the multiplicity that the isomorphism class $\eta$ occurs in the edge deck of $\gamma$.

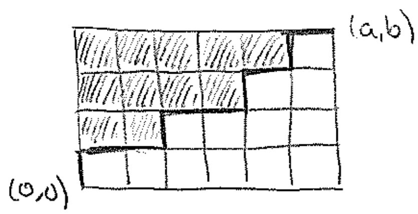
$$Y\frac{\#aut(\gamma)}{n!}[\mathcal{O}_\gamma] = \sum_{\eta} D(\eta,\gamma)\frac{\#aut(\gamma)}{n!}[\mathcal{O}_\gamma].$$

Represent everything as matrices indexed by isomorphism classes of graphs, $\gamma, \eta$, etc. Let $\Delta$ be the diagonal matrix with $\Delta_{\gamma\gamma} = \#aut(\gamma)$. Let D be the matrix with entries $D(\eta,\gamma)$.

$$Y\Delta[\mathcal{O}_\gamma] = \Delta D[\mathcal{O}_\gamma]$$

Then $D = \Delta^{-1}Y\Delta$, $\Delta$ is invertible, Y is injective for graphs $\gamma$ with $k > \frac{1}{2}\binom{n}{2}$. So D is injective for graphs $\gamma$ with $k > \frac{1}{2}\binom{n}{2}$ edges.

<u>q-binomial coefficients</u>




$\mathcal{L}(a,b)$: all lattice paths from (0,0) to (a,b)

$\simeq$ Ferrers diagrams of all partitions $\lambda$ with $\lambda_1 \le a$ and $\ell(\lambda) \le b$.

$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q = \sum_{\lambda} q^{|\lambda|}$$

sum over all partitions $\lambda$ with $\lambda_1 \le a$ and $\ell(\lambda) \le b$. At $q=1$ this specializes to

$$\binom{a+b}{b} = \frac{(a+b)!}{a!\,b!}.$$

Formula:

$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q = \frac{[a+b]_q}{[a]_q[b]_q}$$

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$$[m]_q! = [m]_q[m-1]_q \cdot \cdots \cdot [2]_q[1]_q$$
$$[m]_q = 1 + q + q^2 + \cdots + q^{m-1}$$

Coefficients of $\left[{a+b \atop b}\right]_q$ are palindromic. Replace $\lambda$ by $\check{\lambda}$. (Rotate the picture $\pi$ radians to get $\check{\lambda}$.)

To prove unimodality, we need:

- A set $A$
- A group action of $G$ on $\mathbb{P}(A)$
- so that the orbits of $G$ on $\mathbb{P}(A)$ are indexed by partitions $\lambda$ with $\lambda_1 \leq a$ and $\ell(\lambda) \leq b$.
- Boolean representation of $sl_2$ on $\mathbb{C}\mathbb{P}(A)$ is $G$-equivariant.

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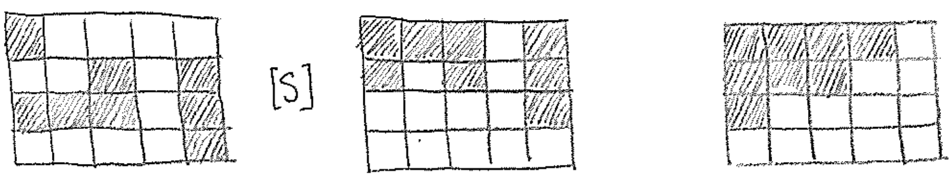
$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q = \sum_{\lambda : \{ {\lambda_1 \leq a \atop \ell(\lambda) \leq b} } q^{|\lambda|} = \sum_{k=0}^{ab} c_k q^k$$

Then $c_k$ is palindromic and unimodal. (We show unimodality now.)
Let $A$ be the set of boxes in the $a \times b$ rectangle $\mathbb{P}(A)$ set of all subsets of $A$.




In each column, move all occupied boxes to the top. Then permute the columns so that the number of occupied boxes in each column decreases from left to right.
We have one copy of the symmetric group $S_b$ for each column.

$$S_b^a = \underbrace{S_b \times \cdots \times S_b}_{a \text{ copies}}$$

We can also permute the $a$ columns ($S_a$) leaving the rows fixed.
This is the <u>wreath product</u>

$$S_a \wr S_b$$

generated by the subgroups $H = S_b^a$ and $K = S_a$.

Claim: • Every orbit of $S_a \wr S_b$ acting on $\mathbb{P}(A)$ contains exactly one Ferrers diagram.

- The Boolean representation of $sl_2$ on $\mathbb{C}\mathbb{P}(A)$ is $G$-equivariant.