II Symmetric Functions

Def) Let $\underline{x} := \{x_1, x_2, x_3, \dots\}$ be commuting indeterminants over $\mathbb{Q}$. A symmetric function is a sum

$$f := \sum_{\alpha} c_\alpha \underline{x}^\alpha$$

where c_α are coefficients (in $\mathbb{Z}$, $\mathbb{Q}$, some $\mathbb{Q}$-algebra) and $\alpha : \mathbb{P} \to \mathbb{N}$ is an "exponent vector":

- $|\alpha| = \alpha(1) + \alpha(2) + \cdots$ is finite, so
$$\underline{x}^\alpha := \prod_{j=1}^{\infty} x_j^{\alpha(j)}$$
is a finite product
- f has finite degree: there is $d \in \mathbb{N}$ such that if $c_\alpha \neq 0$, then $|\alpha| \leq d$.
- f is invariant under all permutations of the variables: for any bijection $\sigma : \mathbb{P} \to \mathbb{P}$, let $\sigma(x_j) := x_{\sigma(j)}$, extended algebraically,
$$\sigma(\underline{x}^\alpha) = \prod_{j=1}^{\infty} x_{\sigma(j)}^{\alpha(j)}, \quad \sigma(f) = \sum_{\alpha} c_\alpha \sigma(\underline{x}^\alpha)$$
then for all $\sigma \in S_{\mathbb{P}}$, $\sigma(f) = f$.

Notation: $\mathbb{P} = \{1, 2, 3, \dots\}$, $\mathbb{N} = \{0, 1, 2, \dots\}$, Λ is the ring of symmetric functions over $\mathbb{Z}$.

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Note that S_∞ is the set of all bijections $\sigma : \mathbb{P} \to \mathbb{P}$, of cardinality $2^{\aleph_0}$, while

$$S_{\mathbb{P}} := \bigcup_{n=0}^{\infty} S_n$$

is the set of permutations $\sigma : \mathbb{P} \to \mathbb{P}$ which move only finitely many elements, of cardinality $\aleph_0$.

Note that Λ is indeed a ring, since

$$\sigma(f+g) = \sigma(f) + \sigma(g), \quad \sigma(f \cdot g) = \sigma(f) \cdot \sigma(g).$$

Let Λ^k be the set of all symmetric functions which are homogeneous of degree k. Then

$$\Lambda = \bigoplus_{k=0}^{\infty} \Lambda^k.$$

Bases for Λ^n

Let $\alpha: \mathbb{P} \to \mathbb{N}$ with $|\alpha| = n$. So
$$\underline{x}^\alpha = \prod_{j=1}^{\infty} x_j^{\alpha(j)}$$
has degree n. If $\underline{x}^\alpha$ occurs in f with coefficient c_α then $\sigma(\underline{x}^\alpha)$ occurs in f with the same coefficient c_α.

A <u>partition</u> is an exponent vector $\lambda: \mathbb{P} \to \mathbb{N}$ such that
$$\lambda(1) \geq \lambda(2) \geq \cdots \geq \lambda(k) > 0$$
and $\lambda(j) = 0$ for all $j > k$. We write
$$\lambda: \lambda_1 \lambda_2 \cdots \lambda_k, \quad |\lambda| = \lambda_1 + \cdots + \lambda_k = n, \quad \lambda \vdash n.$$
For every $\alpha: \mathbb{P} \to \mathbb{N}$, there is exactly one partition $\lambda: \mathbb{P} \to \mathbb{N}$ in the same orbit of S_∞ as α. This λ is the shape of α, $sh(\alpha)$.

Monomial Symmetric Functions

For every partition $\lambda: \mathbb{P} \to \mathbb{N}$, let
$$m_\lambda = \sum_{\substack{\alpha: \mathbb{P} \to \mathbb{N} \\ sh(\alpha) = \lambda}} \underline{x}^\alpha.$$
Note $m_\lambda \in \Lambda$. If $f \in \Lambda^n$ then
$$f = \sum_{\lambda \vdash n} c_\lambda m_\lambda,$$
and this expression is unique. So Λ^n is the free $\mathbb{Z}$-module generated by $\{m_\lambda ; \lambda \vdash n\}$. So the dimension of $\Lambda^n \otimes \mathbb{Q}$ over $\mathbb{Q}$ is $p(n)$, the number of partitions of the integer n.

n	0	1	2	3	4
$p(n)$	1	1	2	3	5
	()	(1)	(2),(1,1)	(3),(2,1),(1,1,1)	(4),(3,1),(2,2),(2,1,1),(1,1,1,1)

From CO 330 chapter 9 we have
$$\sum_{n=0}^{\infty} p(n) x^n = \prod_{j=1}^{\infty} (1 - x^j)^{-1}.$$
Λ is a ring, so $m_\lambda m_\mu = \sum_\theta \omega_{\lambda\mu}^\theta m_\theta$. What do $\omega_{\lambda\mu}^\theta \in \mathbb{N}$ count?

Reverse Lexicographical (Revlex) Order

An order on multiplicity vectors $\alpha : \mathbb{P} \to \mathbb{N}$. We say $\alpha < \beta$ in revlex order if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and $\alpha \neq \beta$ and for the first index $j \in \mathbb{P}$ such that $\alpha(j) \neq \beta(j)$, we have $\alpha(j) > \beta(j)$.
Revlex restricts to partitions. Let $\alpha : \mathbb{P} \to \mathbb{N}$. The revlex earliest element in the $S_{\mathbb{P}}$-orbit of α is $sh(\alpha)$.

Revlex order on partitions of 4:

$$4 < 31 < 22 < 211 < 1111$$

Revlex

Elementary Symmetric Functions

For $k \in \mathbb{P}$,

$$e_k = \sum_{1 \le i_1 < \cdots < i_k} x_{i_1} \cdots \cdot x_{i_k}.$$

The sum is over all k-subsets of $\mathbb{P}$. We have

$$e_k = m_{\underbrace{1 \ldots 1}_{k \text{ 1's}}}.$$

For $\lambda \vdash n$, $\lambda : \lambda_1 \lambda_2 \cdots \lambda_k$, let

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots \cdot e_{\lambda_k}.$$

Note $e_\lambda \in \Lambda^n$.

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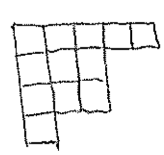
Ferrers diagram

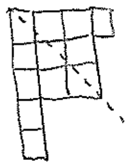
- λ_i boxes in row i
- rows are left justified
- F_λ

Example: $\lambda = 5331$



F_{5331}

The conjugate of λ is λ', where $F_{\lambda'}$ is the "transpose" of F_λ.



$$(5331)' = 43311$$

Note $(\lambda')' = \lambda$. We have $\lambda'_1 = k(\lambda)$ is the number of parts of λ, $m_j(\lambda)$ is the number of times j occurs as a part in λ, and $\underline{m}(\lambda) = \langle m_1, m_2, \dots \rangle$ is the multiplicity vector of λ. We have

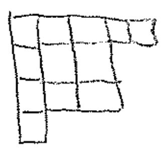
$$m_1 + m_2 + \cdots = k(\lambda), \text{ length of } \lambda$$
$$m_1 + 2m_2 + 3m_3 + \cdots = |\lambda|, \text{ size of } \lambda$$

Example $\lambda = 5331$ $\qquad \lambda' = 43311$

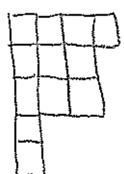
$\underline{m}(\lambda') = \langle 2, 0, 2, 1 \rangle$

in general, $m_j(\lambda') = \lambda_j - \lambda_{j+1}$

$$m_1(\lambda') = \lambda_1 - \lambda_2$$
$$m_2(\lambda') = \lambda_2 - \lambda_3$$
$$m_3(\lambda') = \lambda_3 - \lambda_4$$
$$m_4(\lambda') = \lambda_4 - \lambda_5$$



λ



λ'

Example $n = 3$ $\qquad 3 < 21 < 111$

$$e_3 = \sum_{1 \le i_1 < i_2 < i_3} x_{i_1} x_{i_2} x_{i_3} = m_{111}$$

$$e_{21} = e_2 e_1 = \left(\sum_{1 \le i_1 < i_2} x_{i_1} x_{i_2} \right) \left(\sum_{1 \le i_3} x_{i_3} \right) = m_{21} + 3m_{111}$$

$$e_{111} = e_1 e_1 e_1 = \left(\sum_{1 \le i_1} x_{i_1} \right) \left(\sum_{1 \le i_2} x_{i_2} \right) \left(\sum_{1 \le i_3} x_{i_3} \right) = m_3 + 3m_{21} + 6m_{111}$$

	m_3	m_{21}	m_{111}
e_3	0	0	1
e_{21}	0	1	3
e_{111}	1	3	6

	m_3	m_{21}	m_{111}
$e_{3'}$	1	3	6
$e_{21'}$	0	1	3
$e_{111'}$	0	0	1

Theorem: Fix n. For any $\lambda \vdash n$,

$$e_{\lambda'} = m_\lambda + \sum_{\lambda < \mu} a_{\lambda\mu} m_\mu$$

for some nonnegative integers $a_{\lambda\mu}$ where $\lambda < \mu$ in revlex order.

Corollary: The set $\{e_\lambda ; \lambda \vdash n\}$ is a basis for Λ^n.

Proof (of corollary): The matrix $A = (a_{\lambda\mu})$ is invertible over $\mathbb{Z}$.

Corollary: Every symmetric function in Λ can be written uniquely as a polynomial in $\{e_1, e_2, \dots\}$,

$$\Lambda = \mathbb{Z}[e_1, e_2, \dots]$$

and the e_i are algebraically independent and $\deg(e_i) = i$.

Proof (of theorem): What is the revlex earliest monomial in

$$\begin{aligned} e_{\lambda'} &= e_{\lambda'_1} e_{\lambda'_2} \cdots e_{\lambda'_k} \\ &= e_1^{m_1(\lambda')} e_2^{m_2(\lambda')} \cdots e_k^{m_k(\lambda')} \\ &= e_1^{\lambda_1 - \lambda_2} e_2^{\lambda_2 - \lambda_3} \cdots e_{k-1}^{\lambda_{k-1} - \lambda_k} e_k^{\lambda_k} \ ? \qquad (\lambda_{k+1} = 0) \end{aligned}$$

The revlex earliest monomial in $f \cdot g$ is the product of the revlex earliest monomials of f and of g, for any $f, g \in \Lambda^n$.
So the answer is

$$\begin{aligned} &(x_1)^{\lambda_1 - \lambda_2} (x_1 x_2)^{\lambda_2 - \lambda_3} \cdots (x_1 x_2 \cdots x_{k-1})^{\lambda_{k-1} - \lambda_k} (x_1 x_2 \cdots x_k)^{\lambda_k} \\ &= x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}, \end{aligned}$$

a monomial of shape λ, and every other monomial in $e_{\lambda'}$ is revlex later. Hence

$$e_{\lambda'} = m_\lambda + \sum_{\lambda < \mu} a_{\lambda\mu} m_\mu,$$

and all $a_{\lambda\mu} \in \mathbb{N}$ since each e_j has nonnegative coefficients. ■

Fun Fact:

$$E(t) = \sum_{j=0}^{\infty} e_j t^j = \sum_{j=0}^{\infty} t^j \sum_{1 \le i_1 < \dots < i_j} x_{i_1} x_{i_2} \cdots x_{i_j} = \prod_{j=1}^{\infty} (1 + x_j t).$$

Note $e_0 = 1$.

sometimes I say
things I don't mean

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Complete symmetric functions:

one-parts: $h_0 = 1$
for $k \geq 1$:

$$h_k = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

homogeneous of degree k.
For any partition $\lambda: \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$,

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_r}.$$

Then $\{h_\lambda ; \lambda \vdash n\}$ are $p(n)$ symmetric functions in Λ^n.

$$\begin{aligned} H(t) &= \sum_{j=0}^{\infty} h_j t^j \\ &= \prod_{i=1}^{\infty} \left(1 + x_i t + x_i^2 t^2 + x_i^3 t^3 + \cdots \right) \\ &= \prod_{i=1}^{\infty} \frac{1}{1 - x_i t} \\ &= \frac{1}{\prod_{i=1}^{\infty} (1 - x_i t)} \\ &= E(-t)^{-1}. \end{aligned}$$

Duality

Recall $\Lambda = \mathbb{Z}[e_1, e_2, e_3, \dots]$ and the e_j are algebraically independent.
For any ring R we can define a ring homomorphism $\varphi: \Lambda \to R$ by specifying $\varphi(e_j)$ for each positive integer j and extending via $+$ and $\cdot$ to all of Λ (with $\varphi(e_0) = \varphi(1) = 1$).
Define $\omega: \Lambda \to \Lambda$ by $\omega(e_j) = h_j$ for all $j \in \mathbb{N}$. What is $\omega(h_j)$?

$$\begin{aligned} 1 &= \omega(1) \\ &= \omega(E(-t) H(t)) \\ &= \omega(E(-t)) \omega(H(t)) \\ &= \left(\sum_{j=0}^{\infty} \omega(e_j)(-t)^j \right) \left(\sum_{k=0}^{\infty} \omega(h_k) t^k \right) \\ &= \left(\sum_{j=0}^{\infty} h_j (-t)^j \right) \left(\sum_{k=0}^{\infty} \omega(h_k) t^k \right) \\ &= H(-t) Q(t). \end{aligned}$$

So $Q(t) = H(-t)^{-1} = E(-t)$. So $\omega(h_j) = e_j$ for all $j \in \mathbb{N}$.

Consequently, ω is an involution: $\omega^2 = 1$.
Also, $\{h_1, h_2, h_3, \dots\}$ are algebraically independent.

$$\mathbb{Z}[e_1, e_2, \dots] \xleftrightarrow[\text{isomorphism}]{\omega} \mathbb{Z}[h_1, h_2, \dots]$$

For all $n \in \mathbb{N}$, $\{h_\lambda ; \lambda \vdash n\}$ is a basis for Λ^n as a $\mathbb{Z}$-module.

Power Sum symmetric functions

For positive integer $r \geq 1$:

$$p_r = x_1^r + x_2^r + x_3^r + \cdots = \sum_{i=1}^{\infty} x_i^r.$$

Note p_0 is not defined. For any partition $\lambda: \lambda_1 \lambda_2 \cdots \lambda_k$,

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}.$$

Example:

$$\begin{aligned}
p_1 &= x_1 + x_2 + x_3 + \cdots = e_1 = h_1 \\
p_2 &= x_1^2 + x_2^2 + x_3^2 + \cdots = h_2 - e_2 \\
h_1^2 &= p_2 + 2e_2 \\
e_2 &= (h_1^2 - p_2)/2 \\
h_2 &= p_2 + e_2 = (h_1^2 - p_2)/2 + p_2 \\
h_2 &= p_2/2 + p_{11}/2.
\end{aligned}$$

Generating function

$$P(t) = \sum_{j=1}^{\infty} p_j t^{j-1} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} x_i^j t^{j-1} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_i^j t^{j-1} = \sum_{i=1}^{\infty} \frac{d}{dt} \sum_{j=1}^{\infty} \frac{x_i^j t^j}{j}$$

$$= \sum_{i=1}^{\infty} \frac{d}{dt} \log\left(\frac{1}{1 - x_i t}\right) = \frac{d}{dt} \log\left(\prod_{i=1}^{\infty} \frac{1}{1 - x_i t}\right) = \frac{d}{dt} \log(H(t)) = \frac{H'(t)}{H(t)}.$$

Hence $H'(t) = P(t) H(t)$. So

$$\sum_{n=0}^{\infty} n h_n t^{n-1} = \left(\sum_{j=1}^{\infty} p_j t^{j-1}\right)\left(\sum_{k=0}^{\infty} h_k t^k\right).$$

Looking at the coefficient of t^{n-1}:

$$n h_n = \sum_{j=1}^{n} p_j h_{n-j}$$

The RHS only depends on $h_0, \dots, h_{n-1}$. So

$$h_n = \frac{1}{n} \sum_{j=1}^{n} p_j h_{n-j}$$

for each $n \in \mathbb{N}$. So each h_n is a polynomial in $\{p_1, p_2, \dots, p_n\}$ with coefficients in $\mathbb{Q}$. Every $f \in \Lambda^n$ is a polynomial in the power series, so $\{p_\lambda ; \lambda \vdash n\}$ spans Λ^n over $\mathbb{Q}$. We have

$$\dim_{\mathbb{Q}}(\Lambda^n) = p(n).$$

So $\{p_\lambda ; \lambda \vdash n\}$ is a basis for $\Lambda^n_{\mathbb{Q}}$. It follows that

$$\Lambda_{\mathbb{Q}} = \mathbb{Q}[p_1, p_2, \dots]$$

and the p_j are algebraically independent.

Remark: $\omega(e_n) = h_n$, $\omega(h_n) = e_n$, $\omega(p_n) = ?$ use $P(t) = H'(t)/H(t)$.
$\omega(p_\lambda) = ?$

Schur Functions

For any partition $\lambda \vdash n$, s_λ is a generating function for some objects.
Semi-standard Young Tableaux (SSYT)

Example $\lambda = 422$

$$F_{422} \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 3 & 3 \\ \cline{1-2} 4 & 6 \\ \cline{1-2} \end{array} \quad T \qquad \underline{x}^T = x_1 x_2^2 x_3^3 x_4 x_6$$

put positive integers in the boxes, $\leq$ left-to-right, $<$ top-to-bottom

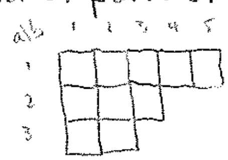
$$s_\lambda = \sum_{\substack{\text{SSYT } T \\ \text{of shape } \lambda}} \underline{x}^T .$$

Schur Functions

$\lambda \vdash n$ is a partition of n
$F_\lambda = \{(a,b) ; 1 \leq a \leq \ell(\lambda), 1 \leq b \leq \lambda_a\}$ } Ferrers diagram
$\ell(\lambda)$ is the number of parts of λ.

eg.

F_{532}:

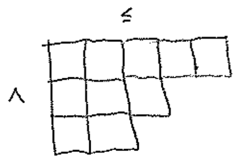


SSYT: $T : F_\lambda \to \mathbb{P}$ such that

$$T(a,b) \leq T(a,b+1) \quad \text{and}$$
$$T(a,b) < T(a+1,b)$$

whenever both boxes are in F_λ.

eg.



$$\underline{x}^T = x_1^3 x_2^3 x_3^2 x_4 x_6$$

$$\underline{x}^T = \prod_{(a,b)\in F_\lambda} x_{T(a,b)}$$

<u>Schur Function</u>:

$$s_\lambda = \sum_T \underline{x}^T,$$

sum over all SSYT $T: F_\lambda \to \mathbb{P}$.

s_λ is homogeneous of degree n if $\lambda \vdash n$.

Proposition (Bender-Knuth): s_λ is a symmetric function

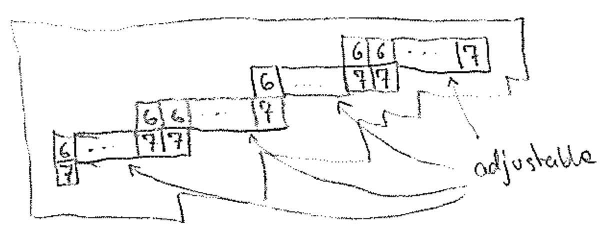
Proof?:
- s_λ is invariant under S_∞
- s_λ is invariant under $S_\mathbb{P}$
- $S_\mathbb{P}$ is generated by adjacent transpositions $(i\ i{+}1)$ $i \in \mathbb{P}$
- It suffices to show that if $\sigma = (i\ i{+}1)$ for some $i \in \mathbb{P}$ then $\sigma(s_\lambda) = s_\lambda$.

Consider a SSYT $T: F_\lambda \to \mathbb{P}$. $\sigma(\underline{x}^T)$ is some monomial.
We'll define an involution ψ on the set of all SSYT of shape λ such that $\sigma(\underline{x}^T) = \underline{x}^{\psi(T)}$.
This proves that $s_\lambda \in \Lambda$, since for any $\sigma = (i\ i{+}1)$

$$\sigma(s_\lambda) = \sigma\left(\sum_T \underline{x}^T\right) = \sum_T \sigma(\underline{x}^T) = \sum_T \underline{x}^{\psi(T)} = \sum_T \underline{x}^T = s_\lambda.$$

eg. $(i\ i{+}1) = (6\ 7)$

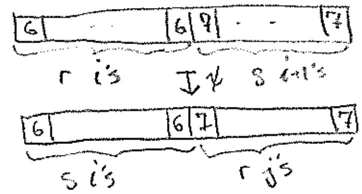
F_λ



"Adjustable boxes" contain either an i or an $i{+}1$, and they are the only one in their column. Each row contains at most one string of adjustable boxes.

Consider the adjustable boxes in a row.

Do this ($\longrightarrow$) for every row.



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$T \mapsto \psi(T)$ is an involution, $\psi(\psi(T)) = T$. $\underline{x}^{\psi(T)} = \sigma(\underline{x}^T)$. Done. ■

Since $s_\lambda \in \Lambda$, it is a linear combination of m_μ's.

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu \qquad (\underline{\text{Kostka numbers}})$$

For each $n \in \mathbb{N}$, $(K_{\lambda\mu})$ is the <u>Kostka matrix</u> indexed by $\lambda, \mu \vdash n$.

revlex leading terms of m_μ

s_{32}:

$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2\\\cline{1-2}\end{array}$ $1 m_{32}$

$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&3\\\cline{1-2}\end{array}$ $1 m_{311}$

$\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3\\\cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2\\\cline{1-2}\end{array}$ $2 m_{221}$

$\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 3&4\\\cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&4\\\cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&1&4\\\hline 2&3\\\cline{1-2}\end{array}$ $3 m_{2111}$

$\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}$ $5 m_{11111}$

$$s_{32} = m_{32} + m_{311} + 2m_{221} + 3m_{2111} + 5m_{11111}$$

revlex order on partitions of 5:

$$5 < 41 < 32 < 311 < 221 < 2111 < 11111$$

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$\lambda: \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \qquad \lambda \vdash n$

$\mu: \mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell > 0 \qquad \mu \vdash n$

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu \qquad \text{each } K_{\lambda\mu} \in \mathbb{N}$$

$\underline{x}^\mu$ is the revlex earliest monomial in m_μ.
So $K_{\lambda\mu}$ is the coefficient of $\underline{x}^\mu$ in s_λ.

$$s_\lambda = \sum_T \underline{x}^T,$$

sum over SSYT of shape λ.
$K_{\lambda\mu}$ is the number of SSYT of shape λ and "content" μ:

$$\left.\begin{array}{ll} \mu_1 & 1\text{'s} \\ \mu_2 & 2\text{'s} \\ \vdots & \\ \mu_\ell & \ell\text{'s} \end{array}\right\} \text{ in } T$$

ex. $\lambda = 42$
$\mu = 321$

$\begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&3\\\cline{1-2}\end{array}$

$K_{42,321}$

$\begin{array}{|c|c|c|c|}\hline 1&1&1&3\\\hline 2&2\\\cline{1-2}\end{array}$

Notice that if $K_{\lambda\mu} \neq 0$

$$\lambda_1 \geq \mu_1$$
$$\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$$
$$\vdots$$

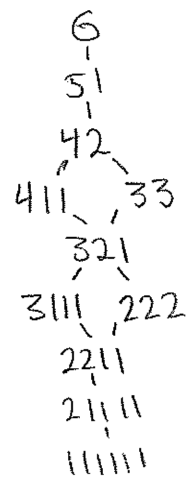
dominance (partial) order on partitions of n, $\lambda \trianglerighteq \mu$

$$\left\{ \begin{array}{l} \sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i \quad \text{for all } j \\ \lambda, \mu \vdash n \end{array} \right.$$

If $\lambda \trianglerighteq \mu$ then $\lambda \leq \mu$ in revlex order.
Contrapositive. If $\mu < \lambda$ in revelex order then $\lambda \not\trianglerighteq \mu$.
Let j be the first index with $\mu_j \neq \lambda_j$. Then $\mu_j > \lambda_j$. So

$$\sum_{i=1}^{j} \mu_i > \sum_{i=1}^{j} \lambda_i$$

so $\lambda \not\trianglerighteq \mu$.



$K_{\lambda\lambda} = 1$.
It follows that

$$s_\lambda = m_\lambda + \sum_{\lambda < \mu} K_{\lambda\mu} m_\mu$$

for some $K_{\lambda\mu} \in \mathbb{N}$. So for any $n \in \mathbb{N}$, the Kostka matrix $(K_{\lambda\mu})$ is invertible over $\mathbb{Z}$. So $\{s_\lambda ; \lambda \vdash n\}$ is a basis for symmetric functions of degree n.

<u>The Jacobi-Trudy Formula</u>

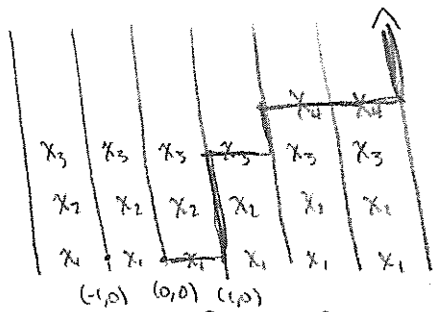
$$h_r = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

"geometric" picture

$\mathcal{H} = \{(a,b) \in \mathbb{Z}^2 ; b \geq 0\}$. with unit edges directed N and E

$(a,b) \xrightarrow{1} (a,b+1)$ vertical edges get weight 1

$(a,b) \xrightarrow{x_{b+1}} (a+1,b)$ horizontal edges get weight x_{b+1}



A lattice path P with finitely many horizontal steps is encoded by $1 \leq i_1 \leq i_2 \leq \cdots \leq i_r$ for some $r \in \mathbb{N}$. Correspond monomial:

$$\underline{x}^{\mathcal{H}(P)} = x_{i_1} x_{i_2} \cdots x_{i_r},$$

$$h_r = \sum_P \underline{x}^{\mathcal{H}(P)},$$

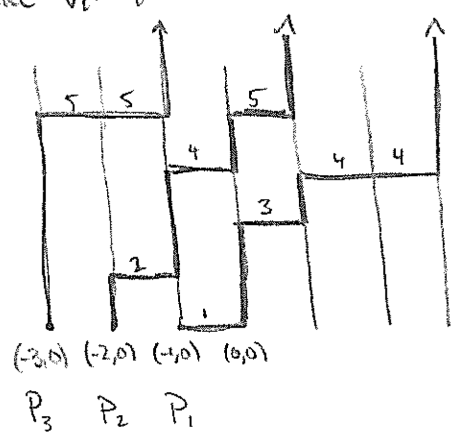
sum over all paths from $(0,0)$ to (r,∞).

$\mathcal{H}$ has automorphisms $(a,b) \mapsto (a+j, b)$ for all $j \in \mathbb{Z}$.

Schur Function

Consider a SSYT T of shape λ: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. Let P_i be a lattice path in $\mathcal{H}$ corresponding to the i^{th} row of T. P_i starts at vertex $(v_i, 0)$ $(v_i \in \mathbb{Z})$

Take $v_i = -i$



1	3	4	4
2	4	5	
5	5		

The < condition for T corresponds exactly to $(P_1, \dots, P_k)$ being vertex-disjoint paths P_i from $(-i, 0)$ to $(\lambda_i - i, \infty)$.
$(P_1, \dots, P_k)$ contributes $\underline{x}^{\mathcal{H}(P_1)} \cdots \underline{x}^{\mathcal{H}(P_k)}$.

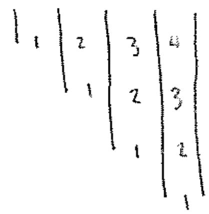
$$S_\lambda = \sum_{(P_1, \dots, P_k)} \underline{x}^{\mathcal{H}(P_1)} \cdots \underline{x}^{\mathcal{H}(P_k)} = \det(h_{\lambda_i - i + j}).$$

$\ell(\lambda)$ by $\ell(\lambda)$
$h_r = 0$ if $r < 0$
$h_0 = 1$

vertex-disjoint $P_i : (-i, 0) \to (\lambda_i - i, \infty)$

$$S_{432} = \begin{bmatrix} h_4 & h_5 & h_6 \\ h_2 & h_3 & h_4 \\ 1 & h_1 & h_2 \end{bmatrix}$$

Elementary Grid $\mathcal{E} = \{(a,b) \in \mathbb{Z}^2 ; a + b \geq 0\}$
$(a,b) \to (a+1, b)$ gets weight x_{a+b+1}



$$S_\lambda = \det(e_{\lambda_i' - i + j})$$

$$S_{432} = \begin{bmatrix} e_3 & e_4 & e_5 & e_6 \\ e_2 & e_3 & e_4 & e_5 \\ 1 & e_1 & e_2 & e_3 \\ 0 & 0 & e_1 & e_1 \end{bmatrix}$$

2015 11 02

Gessel-Viennot, Lindström, Karlim MacGregor:

We have a finite graph $G = (V, E)$, directed and acyclic.
Commutative ring R: every edge $e \in E$ has an element $x_e \in R$ attached.
Let A, Z be vertices. For any path P from A to Z let

$$\underline{x}^P = \prod_{e \in E(P)} x_e$$

$$M(A, Z) = \sum_{\substack{\text{all paths } P \\ A \text{ to } Z}} \underline{x}^P,$$

a finite sum. It is in R.
Let $A_1, A_2, \dots, A_k, Z_1, Z_2, \dots, Z_k$ be vertices in G.

Theorem: Let G be directed & acyclic, properly embedded in the plane, with ring elements $\{x_e; e \in E\}$ for each edge. Let $A_1, A_2, \ldots A_k, Z_k, Z_{k-1}, \ldots, Z_1$ be distinct vertices on the boundary of the outer face of G, which occur in that order going around the outer face. Let

$$G(\vec{A}, \vec{Z}; \underline{x}) = \sum_{(P_1, \ldots, P_k)} \underline{x}^{P_1} \underline{x}^{P_2} \cdots \underline{x}^{P_k}$$

sum overall k-tuples of vertex-disjoint paths P_i from A_i to Z_i. Then

$$G(\vec{A}, \vec{Z}; \underline{x}) = \det(M(A_i, Z_j)).$$

Proof:

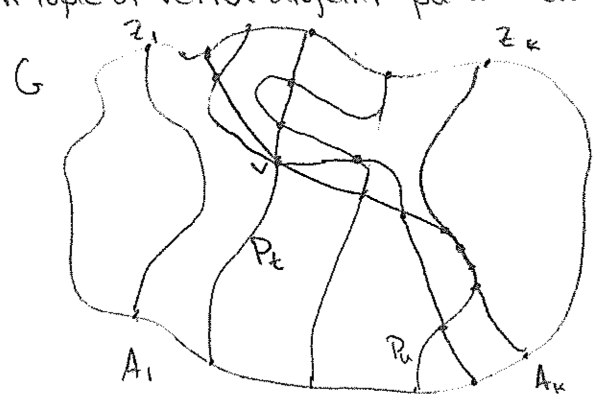
$$\det(M(A_i, Z_j)) = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \prod_{i=1}^{k} M(A_i, Z_{\sigma(i)})$$

$$= \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \sum_{\substack{(P_1, \ldots, P_k) \\ P_i : A_i \text{ to } Z_{\sigma(i)}}} \underline{x}^{P_1} \underline{x}^{P_2} \cdots \underline{x}^{P_k}.$$

We cancel off a bunch of these k-tuples in pairs. Let $\mathbb{P}$ be the set of these k-tuples. Define an involution $\psi : \mathbb{P} \to \mathbb{P}$ so that either

- $\psi(\Phi) = \Phi$ or
- $\underline{x}^{\Psi(\Phi)} = \underline{x}^{\Phi}$ and $\operatorname{sign}(\psi(\Phi)) = -\operatorname{sign}(\Phi)$.

If Φ is a k-tuple of vertex disjoint paths then let $\Psi(\Phi) = \Phi$.



Assume $\Phi = (P_1, \ldots, P_k)$ are not vertex disjoint.
Let P_t be the first path that has a vertex on some other path. Let V be the first vertex on P_t (from A_t to $Z_{\sigma(t)}$) that is also on another path. Let P_u be the first path after P_t that also contains V. $(t<u)$.
Let P_t' be P_t from A_t to V, then P_u from V to $Z_{\sigma(u)}$.

Let P_u' be P_u from A_u to V, then P_t from V to $Z_{\sigma(t)}$.
Let $P_i' = P_i$ for all $i \notin \{t, u\}$.
Let $\Psi(\Phi) = (P_1', P_2', \ldots, P_t')$.

- $\underline{x}^{\psi(\Phi)} = \underline{x}^{\Phi}$ ✓
- $\psi^2 = id$ ✓
- $\sigma(\psi(\Phi)) = \sigma(\Phi)(t\ u) \Rightarrow sign(\sigma(\psi(\Phi))) = -sign(\sigma(\Phi))$ ✓

2015 11 04

$$H(\underline{x}, t) = \sum_{r=0}^{\infty} h_r t^r = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t}.$$

<u>Cauchy Product</u>

$$\begin{aligned}
\prod(x, y) &= \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{1 - x_i y_j} \\
&= \prod_{j=1}^{\infty} H(\underline{x}, y_j) \\
&= \prod_{j=1}^{\infty} \left(\sum_{r=0}^{\infty} h_r y_j^r \right) \\
&= \sum_{\alpha} h_\alpha y^\alpha
\end{aligned}$$

where the sum is over all exponent vectors $\alpha : \mathbb{P} \to \mathbb{N}$ and $h_\alpha = h_{\alpha(1)} h_{\alpha(2)} \cdots$, $y^\alpha = y_1^{\alpha(1)} y_2^{\alpha(2)} \cdots$. Note $h_\alpha = h_\lambda(\underline{x})$ if $sh(\alpha) = \lambda$.

$$\begin{aligned}
&= \sum_{\lambda} h_\lambda(\underline{x}) m_\lambda(\underline{y}) \\
&= \sum_{\lambda} m_\lambda(\underline{x}) h_\lambda(y).
\end{aligned}$$

Recall

$$P(t) = \sum_{j=1}^{\infty} p_j t^{j-1} = \frac{d}{dt} \log H(t).$$

Integrate

$$\sum_{j=1}^{\infty} \frac{p_j t^j}{j} = \log H(t).$$

So

$$H(t) = \exp\left(\sum_{j=1}^{\infty} \frac{p_j t^j}{j} \right)$$

$$= \prod_{j=1}^{\infty} \exp\left(\frac{p_j t^j}{j}\right)$$
$$= \prod_{j=1}^{\infty}\left(\sum_{m_j=0}^{\infty} \frac{p_j^{m_j} t^{j m_j}}{m_j!\, j^{m_j}}\right)$$
$$= \sum_{\lambda} \frac{p_\lambda t^{|\lambda|}}{z_\lambda}$$

where

$$z_\lambda = \prod_{j=1}^{\infty}\left(m_j!\, j^{m_j}\right).$$

Let $\underline{x}\underline{y} = \{x_i y_j ;\ i,j \geq 1\}$.

$$\Pi(\underline{x},\underline{y}) = \prod_{i,j} \frac{1}{1-x_i y_j} = H(\underline{x}\underline{y}, 1) = \sum_{\lambda} \frac{p_\lambda(\underline{x}\underline{y})\, 1^{|\lambda|}}{z_\lambda}.$$

For any $r \geq 1$,

$$p_r(\underline{x}\underline{y}) = \sum_{i,j} (x_i y_j)^r = \sum_{i=1}^{\infty} x_i^r \sum_{j=1}^{\infty} y_j^r = p_r(\underline{x}) p_r(\underline{y}).$$

So for any partition λ: $p_\lambda(\underline{x}\underline{y}) = p_\lambda(\underline{x}) p_\lambda(\underline{y})$. In conclusion,

$$\sum_{\lambda} h_\lambda(\underline{x}) m_\lambda(\underline{y}) = \Pi(\underline{x},\underline{y}) = \sum_{\lambda} \frac{p_\lambda(\underline{x}) p_\lambda(\underline{y})}{z_\lambda}.$$

Inner Product on Λ

$\{h_\lambda\}$ and $\{m_\lambda\}$ are dual bases. For any partitions λ, μ:

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu} = \mathrm{tv}[\lambda = \mu].$$

Proposition: Let $\{v_\lambda\}$ and $\{w_\lambda\}$ be bases for $\Lambda_{\mathbb{Q}}$ indexed by partitions, with v_λ and w_λ homogeneous of degree $|\lambda|$. The following are equivalent:

(a) $\{v_\lambda\}$ and $\{w_\lambda\}$ are dual bases via $\langle \cdot, \cdot \rangle$;

(b)
$$\Pi(\underline{x},\underline{y}) = \sum_{\lambda} v_\lambda(\underline{x}) w_\lambda(\underline{y}).$$

Proof: Fix attention on partitions of size n for some $n \in \mathbb{N}$. Write

$$v_\lambda = \sum_{\nu \vdash n} a_{\lambda\nu} h_\nu \qquad \text{and} \qquad w_\mu = \sum_{\theta \vdash n} b_{\mu\theta} m_\theta$$

for $p(n)$-by-$p(n)$ matrices $A = (a_{\lambda\nu})$ and $B = (b_{\mu\theta})$.

(a) $\langle v_\lambda, w_\mu \rangle = \sum_\nu \sum_\theta a_{\lambda\nu} b_{\mu\theta} \langle h_\nu, m_\theta \rangle = \sum_\nu a_{\lambda\nu} b_{\mu\nu} = (AB^T)_{\lambda\mu}$.

So $\{v_\lambda\}$ and $\{w_\mu\}$ are dual bases if and only if $AB^T = I$.

(b)
$$\sum_\lambda v_\lambda(\underline{x}) w_\lambda(\underline{y}) = \sum_\lambda \sum_\nu \sum_\theta a_{\lambda\nu} b_{\lambda\theta} h_\nu(\underline{x}) m_\theta(\underline{y})$$
$$= \sum_\nu \sum_\theta \left(\sum_\lambda a_{\lambda\nu} b_{\lambda\theta} \right) h_\nu(\underline{x}) m_\theta(\underline{y})$$
$$\Pi(\underline{x}, \underline{y}) = \sum_\rho h_\rho(\underline{x}) m_\rho(\underline{y})$$

these are equal if and only if
$$\sum_\lambda a_{\lambda\nu} b_{\lambda\theta} = \delta_{\nu\theta}$$
if and only if $A^T B = I$. ∎

$$\Pi(\underline{x}, \underline{y}) = \sum_\lambda \frac{p_\lambda(\underline{x}) p_\lambda(\underline{y})}{z_\lambda}$$

$\{p_\lambda\}$ and $\{\frac{1}{z_\lambda} p_\lambda\}$ are dual bases,
$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} \cdot z_\lambda$$
$\{p_\lambda\}$ is an orthogonal (but not orthonormal) basis with respect to $\langle \cdot, \cdot \rangle$. So $\langle \cdot, \cdot \rangle$ is positive definite: $\langle f, f \rangle > 0$ for any non-zero $f \in \Lambda$.

$$\omega : e_k \longmapsto h_k$$
$$\omega(p_j) = (-1)^{j-1} p_j$$
$$\omega(p_\lambda) = (-1)^{|\lambda| - \ell(\lambda)} p_\lambda$$
$$\langle \omega(p_\lambda), \omega(p_\mu) \rangle = (-1)^{|\lambda| - \ell(\lambda)} (-1)^{|\mu| - \ell(\mu)} \langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda = \langle p_\lambda, p_\mu \rangle.$$

So ω is an isometry with respect to $\langle \cdot, \cdot \rangle$.

We'll see
$$\Pi(\underline{x}, \underline{y}) = \sum_\lambda s_\lambda(\underline{x}) s_\lambda(\underline{y}),$$
$\{s_\lambda\}$ is an orthonormal basis for Λ.

2015 11 06

Theorem: (a)
$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda(\underline{x}) s_\lambda(\underline{y})$$
(b) $\{s_\lambda\}$ is an orthonormal basis for Λ with respect to $\langle \cdot, \cdot \rangle$.

These are equivalent, because Wednesday. We prove (a) bijectively.

Proof (Robinson-Schensted-Knuth): LHS:

$$\prod_{i,j} \frac{1}{1-x_i y_j} = \prod_{j=1}^{\infty} H(\underline{x}; y_j) = \prod_{j=1}^{\infty} \left(\sum_{r=0}^{\infty} h_r(\underline{x}) y_j^r \right).$$

A biword is a sequence of pairs

$$(i_1, j_1), (i_2, j_2), (i_3, j_3), \ldots, (i_n, j_n)$$

so that $j_1 \le j_2 \le \cdots \le j_n$ and if $j_a = j_{a+1}$ then $i_a \le i_{a+1}$

example

$$\begin{array}{c|cccccccccccc} i & \multicolumn{4}{c}{\le} & \multicolumn{4}{c}{\le} & \multicolumn{2}{c}{\le} & \multicolumn{3}{c}{\le} \\ j & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 4 \end{array}$$

The biword β contributes

$$(\underline{x}\,\underline{y})^{\beta} = \prod_{a=1}^{n} x_{i_a} y_{j_a}$$

to the LHS.

RHS: Sum over all pairs (T, U) of SSYT of the same shape, each contributes $\underline{x}^T \underline{y}^U$.

Easy special cases

$\beta: ((i_a, j_a) : 1 \le a \le n)$ all $j_a = 1$, $i_1 \le i_2 \le \cdots \le i_n$

T: $\boxed{i_1}\boxed{i_2}\cdots\boxed{i_n}$ U: $\boxed{1}\boxed{1}\cdots\boxed{1}$

$j_1 = j_2 = \cdots = j_{n-1} = 1$ and $j_n = 2$, $i_1 \le i_2 \le \cdots \le i_{n-1}$ and then some i_n

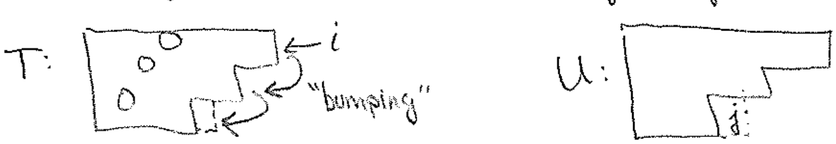
example $i: 1, 1, 3, 3, 4, 4$ $\qquad$ $i = 1, 1, 3, 3, 4, 2$

$j: 1, 1, 1, 1, 1, 2$ $\qquad$ $j = 1, 1, 1, 1, 1, 2$

T: $\boxed{1}\boxed{1}\boxed{3}\boxed{3}\boxed{4}\boxed{4}$ $\qquad$ T: $\boxed{1}\boxed{1}\boxed{3}\boxed{3}\boxed{4} \leftarrow 2$ $\downarrow 3$ $\qquad$ $\begin{array}{|c|c|c|c|c|}\hline 1&1&2&3&4\\ \hline 3 \\ \cline{1-1}\end{array}$

U: $\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}$ $\qquad$ U: $\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}$ $\qquad$ $\begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\ \hline 2 \\ \cline{1-1}\end{array}$

Inserting one pair (i,j) into a pair of SSYT of the same shape (T, U) with j at least as big as every entry in U.



Bumping

First row of T looks like $t_1 \le t_2 \le \cdots \le t_b$. Insert the number u.
If $t_b \le u$ then we can put it at the end, and stop, $t_1 \le \cdots \le t_b \le u$.
If $u < t_b$ then let z be the first index such that $u < t_z$. Note z exists.
Replace t_z by u, and insert t_z into the next row.

example

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 4 \\ \hline 2 & 3 & 3 & 4 & \\ \hline 4 & 4 & & & \\ \hline \end{array} \;\leftarrow 3 \quad \downarrow 4$$

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 4 \\ \hline 2 & 3 & 3 & & \\ \hline 4 & 4 & & & \\ \hline 4 & & & & \\ \hline \end{array} \;\leftarrow 1 \quad \downarrow 2 \quad \downarrow 3 \quad \downarrow 4$$

RSK Algorithm

Given a biword $\beta = ((i_a, j_a) : 1 \le a \le n)$.
Start with $(T, U) = (\emptyset, \emptyset)$
Construct the pair for $((i_a, j_a) : 1 \le a \le n-1)$ to get (T', U')
Insert i_n into T' to get T
Add a box to U' to get U the same shape as T and put j_n in the new box.

example β

$$\begin{array}{c|ccccccccccccc} i & 1 & 1 & 3 & 2 & 2 & 4 & 4 & 1 & 2 & 2 & 5 & 3 & 3 \\ \hline j & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 4 & 4 & 4 & 5 & 5 \end{array}$$

T	U
$\begin{array}{\|c\|}\hline 1 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline\end{array}$
$\begin{array}{\|c\|c\|}\hline 1 & 1 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 1 \\ \hline\end{array}$
$\begin{array}{\|c\|c\|c\|}\hline 1 & 1 & 3 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 1 & 1 \\ \hline\end{array}$
$\begin{array}{ccc} 1 & 1 & 2 \\ 3 & & \end{array}$	$\begin{array}{ccc} 1 & 1 & 1 \\ 2 & & \end{array}$
$\begin{array}{cccc} 1 & 1 & 2 & 2 \\ 3 & & & \end{array}$	$\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & & & \end{array}$
(skip 1 step)	
$\begin{array}{cccccc} 1 & 1 & 2 & 2 & 4 & 4 \\ 3 & & & & & \end{array}$	$\begin{array}{cccccc} 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & & & & & \end{array}$
$\begin{array}{cccccc} 1 & 1 & 1 & 2 & 4 & 4 \\ 2 & & & & & \\ 3 & & & & & \end{array}$	$\begin{array}{cccccc} 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & & & & & \\ 3 & & & & & \end{array}$
$\vdots$	$\vdots$
$\begin{array}{cccccccc} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 2 & 4 & 4 & 5 & & & & \\ 3 & & & & & & & \end{array}$	$\begin{array}{cccccccc} 1 & 1 & 1 & 2 & 2 & 2 & 4 & 5 \\ 2 & 4 & 4 & 5 & & & & \\ 3 & & & & & & & \end{array}$

2015 11 09

Let

$$\begin{array}{ccccccccccccc} u_1 & \le & u_2 & \le & u_3 & \le & \cdots & \le & u_t & \le & \cdots & \le & u_s \\ \wedge & & \wedge & & \wedge & & & & \wedge & & & & \\ v_1 & \le & v_2 & \le & v_3 & \le & \cdots & \le & v_t & & & & \end{array}$$

be two consecutive rows in a tableau $(t \le s)$.

Insert w into the 1^{st} row.

Let: If $u_s \le w$ then let $a = s+1$

- If $w < u_s$ then let $1 \le a \le s$ be such that $u_{a-1} \le w < u_a \le u_{a+1}$.

  In this case w displaces u_a into 2^{nd} row.
- If v_a exists then $u_a < v_a$, so u_a will be inserted into a position $1 \le b \le a$ with $v_b \le u_a < v_a$. This still is true if u_a is appended to the end of the 2^{nd} row.
- Note $u_a > w$.

Let T be a SSYT of shape λ with rows $R_1, R_2, \ldots, R_\ell$.

$$u_1 \leftsquigarrow (R_1 \leftarrow w)$$
$$u_2 \leftsquigarrow (R_2 \leftarrow u_1) \qquad\qquad \frac{\emptyset \leftsquigarrow (R \leftarrow \emptyset)}{R}$$
$$\vdots$$
$$u_\ell \leftsquigarrow (R_\ell \leftarrow u_{\ell-1})$$
$$R_{\ell+1} = (u_\ell) \text{ if } u_\ell \neq \emptyset$$

Then $w < u_1 < u_2 < \cdots < u_\ell$ and the "bumping path" where these are inserted moves weakly to the left (from top to bottom).

Corollary: If T is a tableau then the result of inserting w into T is also a tableau.

First insertion	Second insertion	
$\leftarrow w$	$\leftarrow z$	$(w \le z)$
$\leftarrow u_a$	$\leftarrow u_c$	$(u_a \le u_c)$

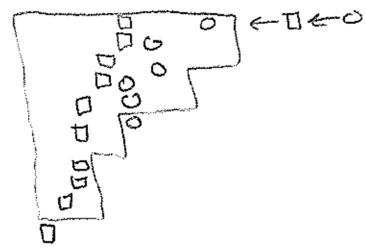
$1 \le c \le s$, $u_{c-1} \le z < u_c$ so z bumps u_c out. Since $w \le z$, z inserts strictly to the left of w and displaces u_c. Note $u_a \le u_c$ by the $\le$ row condition.

Proposition: Let T be a SSYT and $w \le z$. Let $T' = (T \leftarrow w)$ and $T'' = (T' \leftarrow z)$. The bumping path for z into T' stays strictly to the right of the bumping path for w into T.

Corollary: Let T be a SSYT. Let $w_1 \le w_2 \le \cdots \le w_r$.

$$\hat{T} = (\cdots((T \leftarrow w_1) \leftarrow w_2) \leftarrow \cdots) \leftarrow w_r$$

Then the shape of $\hat{T}$ minus the shape of T has at most one box per column. The last boxes of these bumping paths are ordered strictly left to right and weakly bottom to top.



Let $\beta = ((i_1, j_1), \ldots, (i_n, j_n))$ be a biword.

$$i_1 \le \cdots \le i_r$$
$$1 = \cdots = 1$$

T:

i_1	i_2		i_r

U:

1	1		1

T':

x	x	x	x	x		

1	1	1	1	1	2	2
2	2	2				

Corollary: In RSK, the recording tableau U is column strict

$$\{\text{biwords}\} \xrightarrow{RSK} \{(T, U);\ sh(T) = sh(U)\}$$
$$(xy)^\beta = \underline{x}^T y^U$$

Inverse Bijection (example)

(T, U)

1	1	3	3	5
2	4	6		
5	6			

1	1	1	2	4
2	2	6		
4	4			

β:

i							1	3	6	5
j	1	1	1	2	2	2	4	4	4	6

$\nearrow 5$

1	1	3	3	6
2	4			
5	6			

$\vdots$

and so on

$$\underset{\beta}{\text{biwords}} \rightleftharpoons (T, U) \quad sh(T) = sh(U)$$

RHS: Let f_λ be the number of SYT of shape λ, ie T contains $\{1, 2, \ldots, n\}$ once each. $f_\lambda = K_{\lambda, 1^n}$, shape λ, content 1^n

Both T, U are SYT: $\sum_{\lambda \vdash n} f_\lambda^2 = \text{LHS}: n!$

$$\beta: \begin{array}{c|c} i & \text{any permutation} \\ \hline j & 1\ 2\ 3 \cdots n \end{array}$$

2015 11 11

Littlewood-Richardson coefficients

Let λ, μ be partitions.

$$s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu$$

Turns out these are in $\mathbb{N}$. (non-negativity is the issue)
Littlewood-Richardson Rule gives a combinatorial interpretation
(Appendix A1 of Stanley)

$s_\lambda s_1$. $\quad s_1 = e_1 = h_1 = p_1$

F_λ 

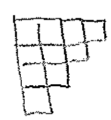
Tableau of shape λ. $F_\mu \setminus F_\lambda$ has exactly one box

$$\sum_{\substack{(T,a) \\ T \text{ SSYT},\ a \geq 1}} \underline{x}^T x_a = \sum_{\substack{T \text{ SSYT}(\mu) \\ F_\mu \setminus F_\lambda \text{ has 1 box}}} x^T$$

$$s_\lambda s_1 = \sum_{\substack{\mu \geq \lambda \\ |\mu/\lambda| = 1}} s_\mu$$

Alternants (§7.15 of Stanley)

Λ set, $x_i = 0$ for all $i > n$
$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$
alternant $\quad a_\alpha = \det(x_i^{\alpha_j})_{1 \leq i,j \leq n}$
We may assume $\alpha_1 > \alpha_2 > \dots > \alpha_n \geq 0$, $\alpha_i = \lambda_i + n - i$ for $1 \leq i \leq n$. Then $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. λ is a partition with at most n parts.
Staircase: $\delta = (n-1, n-2, \dots, 2, 1, 0)$. So $\alpha = \lambda + \delta$
Fact:



$$s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\delta}}{a_\delta}$$

Hook length formula

$\lambda \vdash n$. Look at F_λ. hook of (a,b) in F_λ is (a,b) and all boxes to the right in the same row and below in the same column. The hook length of (a,b) is $h(a,b)$ = size of the hook of (a,b).

Frame-Robinson-Thrall

$$\#SYT(\lambda) = f_\lambda = K_{\lambda,1^n} = \frac{n!}{\prod_{(a,b)\in F_\lambda} h(a,b)}.$$

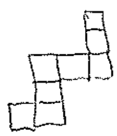
<u>Murnaghan-Nakayama</u>

$$s_\mu p_r = \sum_\lambda (-1)^{ht(\lambda/\mu)} s_\lambda$$

sum over λ such that λ/μ is a border strip of size r,

$$ht(\lambda/\mu) = \#\text{rows}(\lambda/\mu) - 1$$

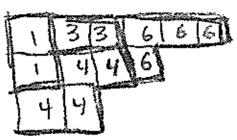
border-strip: connected skew shape with no 2×2 block

ex  $ht = 4$

$$s_\lambda = \sum_\nu z_\nu^{-1} \chi^\lambda(\nu) p_\nu$$

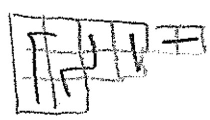
$$\chi^\lambda(\nu) = \sum_T (-1)^{ht(T)} \qquad ht(T) = \sum_{\substack{B \text{ border} \\ \text{strips in } T}} ht(B)$$

border strip tableau of shape λ and type ν.

ex 

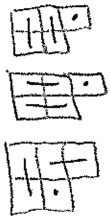
shape 642

type $(2,0,2,4,0,4)$

P_{4422} 

$(4,4,2,2)$

ex shape 32 type 32  $\longleftrightarrow$ shape 32 type 23 

ex shape 32 type 221 

shape 32 type 122 

Hence

$$\langle s_\lambda, p_\mu \rangle = \sum_\nu \chi^\lambda(\nu) z_\nu^{-1} \langle p_\nu, p_\mu \rangle = \chi^\lambda(\nu)$$

Grassmanians

Fix n, r. Set of all r-dimensional subspaces of an n-dimensional vector space over $\mathbb{C}$, $G(n,r)$.
matrix A $r \times n$ of rank r. rowspace(A) is in $G(n,r)$.
If M is $r \times r$ invertible then $\text{row}(MA) = \text{row}(A)$.

$A \mapsto \underbrace{[\det A|_S \,;\ S \text{ is an } r\text{-element subset of } \{1, \dots, n\}]}_{P(A)}$

$MA \mapsto \det(M) P(A)$

This embeds $G(n,r) \hookrightarrow \mathbb{P}^{\binom{n}{r}-1}$ projective space.

Cohomology


S^1 $\quad H^\cdot(S^1) = \underset{0}{\mathbb{Z}} \oplus \underset{1}{\mathbb{Z}}$ degree


S^2 $\quad H^\cdot(S^2) = \underset{0}{\mathbb{Z}} \oplus \underset{2}{\mathbb{Z}}$ degree

2015 11 13

X $\left.\begin{array}{l}\text{topological space} \\ \text{algebraic variety}\end{array}\right\}$ dimension d

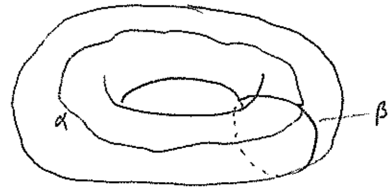
cohomology

$$H^\cdot(X) = \bigoplus_{j=0}^{d} H^j(X)$$

Each $H^j(X) \simeq \mathbb{Z}^{\beta_j}$ (generally, it's an abelian group)
cup product $\cup : H^i(X) \times H^j(X) \to H^{i+j}(X)$
every codimension j subspace Y yields a class $[Y]$ in $H^j(X)$.
the class of X is $[X]$ in $H^0(X) = \mathbb{Z}$ & $[X] \to 1$
$[X] \cup [Y] = [Y]$ for any class Y

eg. Torus $S^1 \times S^1$

j	0	1	2
H^j	$\mathbb{Z}$	$\mathbb{Z}\alpha \oplus \mathbb{Z}\beta$	$\mathbb{Z}$



$\cup$	1	α	β	pt
1	1	α	β	pt
α	α	0	pt	0
β	β	pt	0	0
pt	pt	0	0	0

Grassmannian $G(n,r)$ of r-planes in n-space $\mathbb{C}^n$
A $r \times n$ matrix of rank r.
Plücker embedding $A \mapsto [\det A|_S ;\ S \subseteq [n];\ |S| = r] \in \mathbb{P}^{\binom{n}{r}-1}$

Schubert Calculus

Given A, let $p_j(A)$ be the rank of $A|_{\{1,2,\ldots,j\}}$, and $p_0 = 0$.

$$0 = p_0 \leq p_1 \leq p_2 \leq p_3 \leq \cdots \leq p_n = r.$$

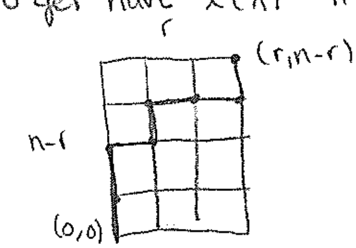
and $0 \leq p_{j+1} - p_j \leq 1$ for each $1 \leq j \leq n$.
Each of $1,2,3,\ldots,r$ occurs at least once. Ignore zeros & remove one part of size j for $1 \leq j \leq r$. The result is a partition λ.

eg. $n=8$ $r=4$

j	1	2	3	4	5	6	7	8
p_j	0	0	1	1	2	3	4	4
	*	*	*		*	*	*	

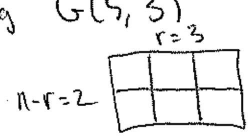
$\lambda = 41$

The partitions you get have $\ell(\lambda) \leq n-r$ and $\lambda_1 \leq r$.



For each partition μ inside the $(n-r) \times r$ rectangle. Let X_μ be the set of all points $G(n,r)$ represented by a matrix A such that μ is contained in $\lambda(A)$ (in terms of Ferrers diagram).

eg $G(5,3)$



ε
1
$2, 11$
$3, 21$
$31, 22$
32
33

Note X_μ is contained in X_λ if and only if $F_\lambda \subseteq F_\mu$. $X_\varepsilon = G(n,r)$ is everything.

$[X_\lambda] \in H^j(G(n,r))$ for each λ with $\ell(\lambda) \le n-r$, $\lambda_1 \le r$, and the codimension is $j = |\lambda|$.

$[X_{r^{n-r}}]$ is the class of a point

$[X_1]$ is the class of a "generic hyperplane section"

Facts:

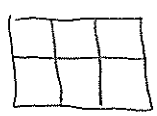
① $\{[X_\lambda];\ \ell(\lambda) \le n-r \text{ and } \lambda_1 \le r\}$ is a basis for $H^\cdot(G(n,r))$

② $[X_\lambda] \mapsto s_\lambda$ is an isomorphism of rings from $H^\cdot(G(n,r))$ with $\Lambda / I(n,r)$, where $I(n,r)$ is the ideal generated by $\{s_\lambda;\ \ell(\lambda) > n-r \text{ or } \lambda_1 > r\}$.

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$$

$G(5,3)$

3

2 

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]!_q}{[r]!_q [n-r]!_q}$$

$$[m]!_q = [m]_q [m-1]_q \cdots [3]_q [2]_q [1]_q$$

$$[m]_q = 1 + q + q^2 + \cdots + q^{m-1} = \frac{1-q^m}{1-q}$$

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix}_q = \frac{[5]_q [4]_q [3]_q \cancel{[2]_q} \cancel{[1]_q}}{\cancel{[3]_q} [2]_q [1]_q \cancel{[2]_q} \cancel{[1]_q}} = \frac{(1-q^5)}{(1-q)} \frac{(1-q^4)}{\cancel{(1-q)}} \frac{\cancel{(1-q)}}{(1-q^2)}$$

$$= \frac{(1-q^5)}{1-q}(1+q^2) = (1+q+q^2+q^3+q^4)(1+q^2)$$

$$= 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6$$