CO 430 - Algebraic Enumeration

2015 09 14

Binomial Series:

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \qquad \text{for } \alpha \in \mathbb{C}$$

converges for $|x|<1$. Taylor series

$$\binom{\alpha}{k} = \frac{1}{k!}\left(\frac{d}{dx}\right)^k (1+x)^{\alpha}\Big|_{x=0} = \frac{\alpha(\alpha-1)\cdot\dots\cdot(\alpha-k+1)}{k!}$$

Valid for indeterminate α.

Generally,

$$\binom{-\alpha-1}{k} = (-1)^k \binom{k+\alpha}{k}$$

So

$$\frac{1}{(1-x)^{\alpha+1}} = \sum_{k=0}^{\infty} \binom{-\alpha-1}{k}(-x)^k = \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x^k.$$

Useful when $\alpha+1=t$ a positive integer

$$\frac{1}{(1-x)^t} = \sum_{k=0}^{\infty} \binom{m+t-1}{t-1} x^m = \sum_{(m_1,\dots,m_t)\in\mathbb{N}^t} x^{m_1+\dots+m_t}$$

$$= \left(\sum_{k=0}^{\infty} x^k\right)^t$$

<u>Special cases</u>

$$\frac{1}{\sqrt{1-4x}},\ \sqrt{1-4x},\ (1-x)^{-3/2}$$

<u>LIFT</u>

Let $\mathbb{K}$ be a commutative ring containing $\mathbb{Q}$. Let $F(u)$, $G(u)$ be power series in $\mathbb{K}[[u]]$ with $[u^0]G(u)$ invertible in $\mathbb{K}$. Then

(a) There is a unique $R(x) \in \mathbb{K}[[x]]$ such that

$$R(x) = x\,G(R(x)).$$

(b) $[x^0]R(x) = 0$ and for all $n \geq 1$

$$[x^n]F(R(x)) = \frac{1}{n}[u^{n-1}]F'(u)\,G(u)^n.$$

Exponential Generating Functions

A class of structures $\mathcal{A}$
- takes as "input" any finite set X
- returns as "output" another finite set $\mathcal{A}_X$

(Interpretation: $\mathcal{A}_X$ is the set of all $\mathcal{A}$-type structures with vertex-set X.)

Two "phenomenological" axiom
- if $X \neq Y$ then $\mathcal{A}_X \cap \mathcal{A}_Y = \emptyset$.
- if $|X| = |Y|$ then $|\mathcal{A}_X| = |\mathcal{A}_Y|$.

General Problem: Given a class $\mathcal{A}$, determine $|\mathcal{A}_X|$ for any finite set X.
For $\{1, 2, \ldots, n\}$, let $\mathcal{A}_n = \mathcal{A}_{\{1, \ldots, n\}}$. So if $|X| = n$ then $|\mathcal{A}_X| = |\mathcal{A}_n|$.

Examples

- $\mathcal{G}$ class of simple graphs, $\gamma = (G, E)$ (E a set of 2-element subsets of X)
$$|\mathcal{G}_n| = 2^{\binom{n}{2}}.$$
- $\mathcal{S}$ class of permutations $\sigma : X \to X$ (bijection)
$$|\mathcal{S}_n| = n! \qquad S(x) = \sum_{n=0}^{\infty} n! \frac{x^n}{n!} = \frac{1}{1-x}$$
- $\mathcal{C}$ class of cyclic permutations
$$|\mathcal{C}_n| = \begin{cases} 0 & \text{if } n = 0 \\ (n-1)! & \text{if } n \geq 1 \end{cases} \qquad C(x) = \sum_{n=1}^{\infty} (n-1)! \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{n} = \log\left(\frac{1}{1-x}\right)$$
- $\mathcal{E}$ class of sets. A set is a set
$$\mathcal{E}_X = \{X\}$$
$$|\mathcal{E}_n| = 1 \text{ for all } n \in \mathbb{N} \qquad E(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = \exp(x)$$
- $\mathcal{E}_k$ class of k-element sets
$$(\mathcal{E}_k)_X = \begin{cases} \emptyset & \text{if } |X| \neq k \\ \{X\} & \text{if } |X| = k \end{cases} \qquad E_k(x) = \frac{x^k}{k!}$$

For a class $\mathcal{A}$ of structures
$$A(x) = \sum_{n=0}^{\infty} |\mathcal{A}_n| \frac{x^n}{n!}$$

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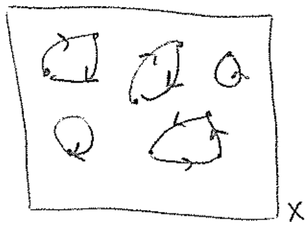
A permutation is a finite set of disjoint cyclic permutations.



$$S \equiv \mathcal{E}[\mathcal{C}]$$
$$S(x) = E(C(x))$$

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$\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ classes of structures

<u>Sums</u>

Given $\mathcal{A}, \mathcal{B}$ define a class $\mathcal{A} \oplus \mathcal{B}$ with exponential generating function $A(x) + B(x)$. More generally, for classes $\mathcal{A}^{(i)}$ $i \in \mathbb{N}$ define a class

$$\bigoplus_{k=0}^{\infty} \mathcal{A}^{(k)}$$

with exponential generating function

$$\sum_{k=1}^{\infty} A^{(k)}(x)$$

when possible.

For any finite set X

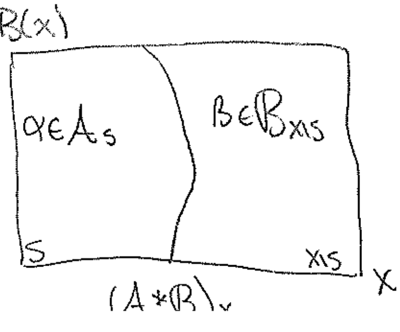
$$\left(\bigoplus_{k=0}^{\infty} \mathcal{A}^{(k)}\right)_X = \bigcup_{k=0}^{\infty} \left(\{k\} \times \mathcal{A}^{(k)}_X\right).$$

This defines a class if and only if for every finite set X, only finitely many of the sets $\mathcal{A}^{(k)}_X$ are not empty.

$(\mathcal{A}^{(k)}, k \in \mathbb{N})$ is <u>locally finite</u>.

<u>Products</u>

Given $\mathcal{A}, \mathcal{B}$ define a class $\mathcal{A} * \mathcal{B}$ with exponential generating function $A(x)B(x)$



For a finite set X

$$(A * B)_X = \bigcup_{S \subseteq X} \left(A_S \times B_{X \setminus S} \right).$$

Given $(\alpha, \beta) \in (A * B)_X$ we can determine $S \subseteq X$.
So this union is disjoint. So for all $n \in \mathbb{N}$

$$\#(A * B)_n = \sum_{k=0}^{\infty} \binom{n}{k} (\# A_k)(\# B_{n-k})$$

Exponential generating function is

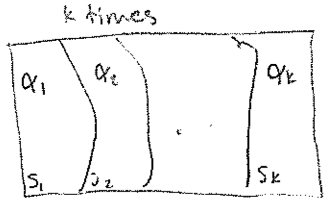
$$\sum_{n=0}^{\infty} \#(A * B)_n \frac{x^n}{n!} = A(x)B(x).$$

$*$ is associative: (more later)

$$(A * B) * C \equiv A * (B * C)$$

k-tuples of A-structures

$$\underbrace{A * \cdots * A}_{k \text{ times}} = A^k$$



strings of A-structures

$$A^* = \bigoplus_{k=0}^{\infty} A^k$$

locally finite if and only if $A_\emptyset = \emptyset$.
If $\alpha \in A_\emptyset$ then

$$\underbrace{(\alpha, \ldots, \alpha)}_{k} \in A^*_\emptyset$$

for all k.
If $A_\emptyset = \emptyset$ and $(\alpha_1, \ldots, \alpha_k) \in A^k_X$ then $\# X \geq k$.
Such classes A are called <u>connected</u>.

<u>k-sets of A-structures</u>

For any finite set X, consider

$$(\mathcal{A}^k)_X \longrightarrow \mathcal{E}_k[\mathcal{A}]_X$$
$$(\alpha_1, \ldots, \alpha_k) \longmapsto \{\alpha_1, \ldots, \alpha_k\}$$

forget the order of the substructures

Each k-set in $\mathcal{E}_k[\mathcal{A}]_X$ is the image of $k!$ k-tuples in $(\mathcal{A}^k)_X$.
So for all $n \in \mathbb{N}$

$$\#(\mathcal{A}^k)_n = k! \, \# \mathcal{E}_k[\mathcal{A}]_n$$

Multiply by $\frac{x^n}{n!}$, sum over n:

$$E_k[A](x) = \frac{A(x)^k}{k!}.$$

<u>Sets of $\mathcal{A}$-structures</u>

$$\mathcal{E}[\mathcal{A}] = \bigoplus_{k=0}^{\infty} \mathcal{E}_k[\mathcal{A}]$$

is locally finite if and only if $\mathcal{A}_\emptyset = \emptyset$.
exponential generating function

$$\sum_{k=0}^{\infty} \frac{A(x)^k}{k!} = \exp(A(x))$$

<u>Exponential Formula</u>

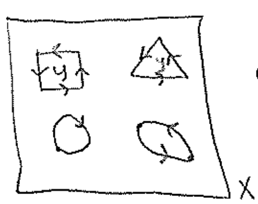
Permutations are sets of cyclic permutations

$$\mathcal{S} = \mathcal{E}[\mathcal{C}]$$

mark each cycle with an indeterminate y.

$$S(x,y) = \sum_{n=0}^{\infty} \left(\sum_{\sigma \in S_n} y^{c(\sigma)} \right) \frac{x^n}{n!}$$

where $c(\sigma)$ is the number of cycles of σ.



$\sigma \in S_X$

$y^{c(\sigma)}$

exponential generating function for cycles is

$$C(x,y) = y \sum_{n=1}^{\infty} (n-1)! \frac{x^n}{n!} = y \sum_{n=1}^{\infty} \frac{x^n}{n} = y \log\left(\frac{1}{1-x}\right).$$

So

$$S(x,y) = \exp\left(y \log\left(\frac{1}{1-x}\right)\right)$$

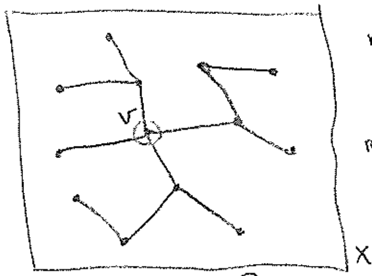
Set Problem 2 [illegible]
use example 11.19
to solve exercise 11.15(a)
[illegible]
Example 2.3 Exercises 2.8, 2.11, 2.12, [illegible]

$$= \frac{1}{(1-x)^y}$$

$$= \sum_{n=0}^{\infty} \binom{n+y-1}{n} x^n$$

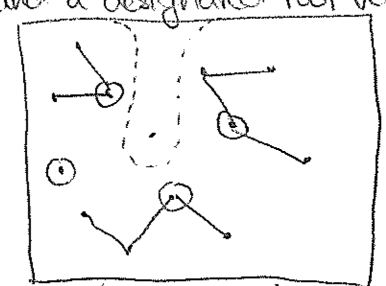
$$n!\,[x^n]S(x,y) = \sum_{\sigma \in S_n} y^{c(\sigma)} = n!\,\frac{(y-1+n)\cdots(y)}{n!} = (y-1+n)_{(n)}$$ falling factorial

$\mathcal{R}$ class of rooted trees
$\mathcal{R}_X$ set of trees with vertex set X and a designated root vertex $v \in X$.



$(T,v) \in \mathcal{R}_X$

remove the root & recurse



$(\mathcal{E}_1 * \mathcal{E}[\mathcal{R}])_X$

This gives an equivalence

$$\mathcal{R} \equiv \mathcal{E}_1 * \mathcal{E}[\mathcal{R}]$$

so

$$R(x) = x\exp(R(x))$$

Apply LIFT with $\mathbb{K} = \mathbb{Q}$, $F(u) = u$, $G(u) = \exp(u)$.

$$\#\mathcal{R}_n = n!\,[x^n]R(x)$$

$$= n!\,\frac{1}{n}[u^{n-1}]\exp(u)^n$$

$$= (n-1)!\,[u^{n-1}]\sum_{j=0}^{\infty} \frac{(nu)^j}{j!} = n^{n-1}$$

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Rooted structures

For any class $\mathcal{A}$, define a class $\mathcal{A}^\circ$ as follows. For any finite set X, $(\mathcal{A}^\circ)_X$ is the set of pairs (α, v) with $\alpha \in \mathcal{A}_X$ and $v \in X$.

$$(\mathcal{A}^\circ)_X = \mathcal{A}_X \times X$$

$$\#(\mathcal{A}^\circ)_n = n \cdot \#\mathcal{A}_X$$

Exponential generating function of $\mathcal{A}^\circ$ is

$$x\frac{d}{dx}A(x).$$

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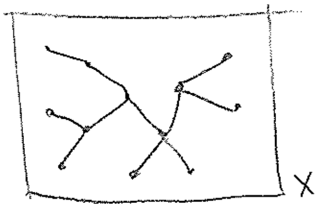
Trees $\mathcal{T}$

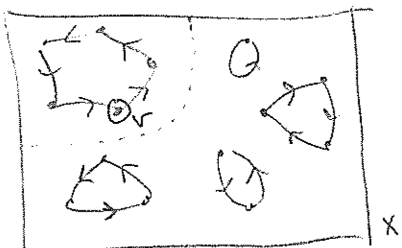


in $\mathcal{T}_X$

$\mathcal{T}^\circ = \mathcal{R}$ rooted trees

$\# \mathcal{T}_n = \frac{1}{n} \# \mathcal{T}_n^\circ = n^{n-2}$

Rooted permutation



in $\mathcal{S}_X^\circ$

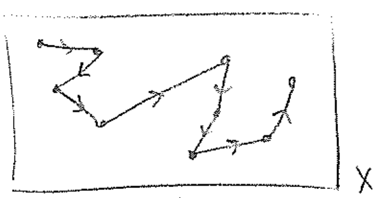
$\mathcal{S}^\circ \equiv \mathcal{C}^\circ * \mathcal{E}[\mathcal{C}] \equiv \mathcal{C}^\circ * \mathcal{S}$

$\mathcal{C}^\circ \equiv \mathcal{X} * \mathcal{X}^*$

where $\mathcal{X} \equiv \mathcal{E}_1$ is the class of singletons (1-element sets)

$\Rightarrow$ identities for power series (exponential generating functions)

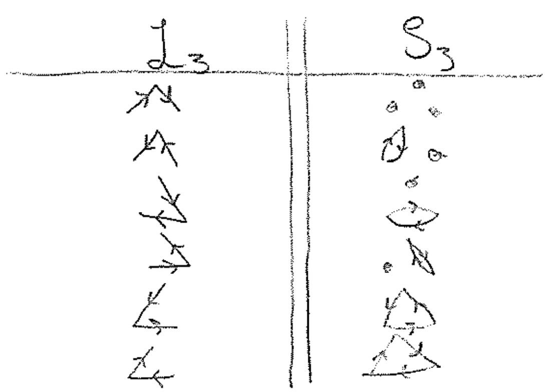
$\mathcal{L} = \mathcal{X}^*$ is the class of total (linear) orders, or lists



in $\mathcal{X}_n^*$

$\# \mathcal{L}_n = n!$ $\quad \# \mathcal{S}_n = n!$

$L(x) = \frac{1}{1-x} = S(x)$



Composition of classes

Given classes $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{A}$ connected ($\mathcal{A}_\emptyset = \emptyset$).
Define the class of $\mathcal{B}[\mathcal{A}]$-structures as follows
For any finite set X, a $\mathcal{B}[\mathcal{A}]$-structure is a pair (ξ, β) where
$\xi \in \mathcal{E}[\mathcal{A}]_X$ (a finite set of $\mathcal{A}$-structures partitioning X)
and $\beta \in \mathcal{B}_\xi$.
That is

$$\mathcal{B}[\mathcal{A}]_X = \bigcup_{\xi \in \mathcal{E}[\mathcal{A}]_X} \left(\{\xi\} \times \mathcal{B}_\xi \right)$$

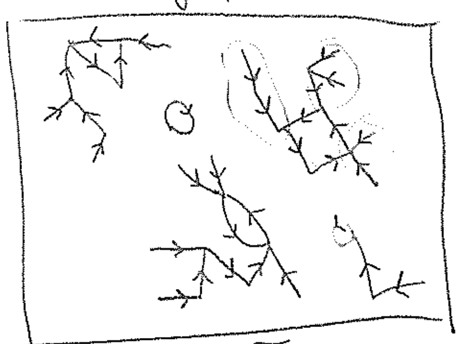
Then

$$\#\mathcal{B}[\mathcal{A}]_X = \sum_{k=0}^{\infty} \left(\# \mathcal{E}_k[\mathcal{A}]_X \right) \left(\# \mathcal{B}_k \right)$$

Exponential generating function of $\mathcal{B}[\mathcal{A}]$ is
$B(A(x))$.

Endofunctions $\mathcal{F}$

$\mathcal{F}_X$ is the set of all functions $\phi : X \to X$
functional digraph $v \to \phi(v)$ for all $v \in X$



X

in $\mathcal{F}_X$

- it is a set of components $\mathbb{C}$ <u>connected</u> endofunctions
- there is exactly one cycle in each component (Lemma)
- the cut-edges of each component form a set of trees, each rooted at a vertex on the cycle

$\mathcal{F} \equiv \mathcal{E}[\mathbb{C}]$ an endofunction is a set of components
$\mathbb{C} \equiv \mathcal{C}[\mathcal{T}^\bullet]$ each component is a cyclic permutation of rooted trees
$\mathcal{T}^\bullet \equiv X * \mathcal{E}[\mathcal{T}^\bullet]$

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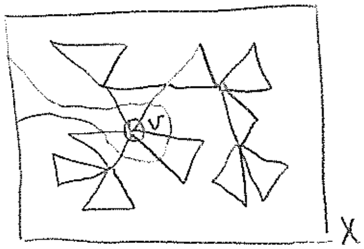
Triangle-Trees Δ



- graph $G=(X,E)$
- connected, simple
- each block is a K_3

introduce a root vertex, remove it, and recurse

$$\Delta^\circ \equiv \mathcal{X} * \mathcal{E}[Q]$$

Q are the components of the triangle tree with the root deleted

$$Q \equiv \mathcal{E}_2[\Delta^\circ]$$

$$\Delta^\circ \equiv \mathcal{X} * \mathcal{E}[\mathcal{E}_2[\Delta^\circ]]$$

$$P(x) = \sum_{n=0}^{\infty} (\# \Delta^\circ_n) \frac{x^n}{n!}$$

$$P(x) = x \exp\left(\frac{P(x)^2}{2} \right)$$

LIFT for $n \geq 1$

$$\# \Delta_n = \frac{1}{n} \# \Delta^\circ_n = \frac{n!}{n} [x^n] P(x) = \frac{n!}{n^2} [u^{n-1}] \exp\left(\frac{u^2}{2}\right)^n$$

$$\cancel{= \frac{1}{n^2} [u^{n-1}] \sum_{j=0}^{\infty} \frac{(nu^2)^j}{j!}} = \cancel{\begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{(2j+1)^{j-2}}{j!} & \text{if } n \text{ is odd} \quad n=2j+1 \end{cases}}$$

correction: 2015 09 2

$$= \frac{(n-1)!}{n} [u^{n-1}] \exp\left(\frac{nu^2}{2}\right)$$

$$= \frac{(n-1)!}{n} [u^{n-1}] \sum_{j=0}^{\infty} \frac{(nu^2/2)^j}{j!} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{(2j)!\,(2j+1)^{j-1}}{\cancel{(2j+1)}\, j!\, 2^j} & \text{if } n = 2j+1 \end{cases}$$

Aside: In $\mathbb{Q}[[t,q]]$

$$\exp(t)\exp(q) = \exp(t+q)$$

Consequence of Vandermonde convolution

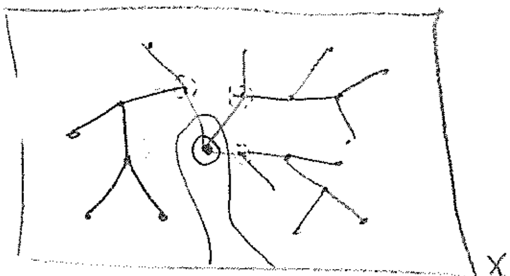
$$\binom{n+m}{k} = \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j}$$

$\exp(q)^n = \exp(nq)$.

Trivalent Trees

Let $\mathcal{A}$ be the class of trees in which every vertex has degree one or three. $\#\mathcal{A}_n = ?$

n	0	1	2	3	4	5	6
$\#\mathcal{A}_n$	0	0	1	0	4	0	90



in $\mathcal{A}^\circ_X$

$$\mathcal{A}^\circ \not\equiv \mathcal{X} * \underset{\text{wrong}}{\mathcal{E}[\mathcal{A}^\circ]}$$

Let (T,v) be a rooted trivalent tree. v has degree one or three. Delete v. $T\setminus v$ has either one or three components, rooted at the unique neighbour of v. Each component is a rooted tree in which each vertex has either 0 or 2 children. Let $\mathbb{B}$ be the class of such rooted trees.

$$\mathcal{A}^\circ \equiv \mathcal{X} * (\mathcal{E}_1 \oplus \mathcal{E}_3)[\mathbb{B}]$$

$$\mathbb{B} \equiv \mathcal{X} * (\mathcal{E}_0 \oplus \mathcal{E}_2)[\mathbb{B}]$$

$$A^\circ(x) = \sum_{n=0}^{\infty} (\#\mathcal{A}^\circ_n) \frac{x^n}{n!}$$

$$B(x) = \sum_{n=0}^{\infty} (\#\mathbb{B}_n) \frac{x^n}{n!}$$

$$A^\circ(x) = x\left(B(x) + \frac{B(x)^3}{6}\right)$$

$$B(x) = x\left(1 + \frac{B(x)^2}{2}\right).$$

$$B(x) = xG(B(x)) \quad \text{where}$$

$$G(u) = \left(1 + \frac{u^2}{2}\right)$$

$$A^\circ(x) = xF(B(x)) \quad \text{where}$$

$$F(u) = \left(u + \frac{u^3}{6}\right.$$

$$F'(u) = \frac{d}{du}F(u) = \left(1 + \frac{u^2}{2}\right) = G(u)$$

for $n \geq 2$

$$\# \mathcal{A}_n = \frac{1}{n} \# \mathcal{A}^\circ_n = \frac{n!}{n}[x^n] A^\circ(x)$$

$$= (n-1)! [x^n] x F(B(x))$$

$$= (n-1)! [x^{n-1}] F(B(x))$$

By LIFT,

$$\# \mathcal{A}_n = (n-1)! \frac{1}{n-1} [u^{n-2}] F'(u) G(u)^{n-1}$$

$$= (n-2)! [u^{n-2}] \left(1 + \frac{u^2}{2}\right)^n$$

$$= (n-2)! [u^{n-2}] \sum_{k=0}^{\infty} \binom{n}{k} \frac{u^{2k}}{2^k}$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{(2k)!}{2^k} \binom{2k+2}{k} & \text{if } n = 2k+2 \end{cases}$$

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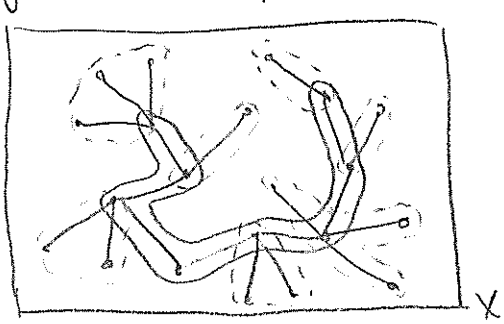
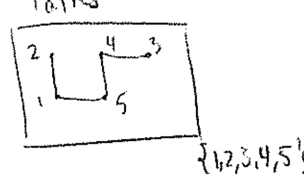
Caterpillars

A tree with at least one leaf such that after deleting all leaves, what remains is a path. A path is a tree with all vertices of degree at most two. Denote their classes $\mathcal{K}$ and $\mathbb{P}$ respectively.



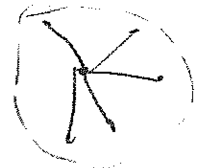
Paths



$$\# \mathbb{P}_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ n!/2 & \text{if } n \geq 2 \end{cases}$$

$$P(x) = x + \frac{1}{2}\left(\frac{x^2}{1-x}\right)$$

· A caterpillar consists of a path of "bags".
· Each bag contains one vertex on the path and a (possibly empty) set of leaves attached to it by edge



a bag $\mathcal{B}$

$\mathcal{K} \neq \mathcal{P}[\mathcal{B}]$

(***):

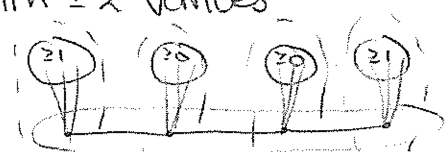
impossible

not catepillars

we need ≥ 2 leaves in this bag

$\leftarrow$ $\mathcal{X} * \mathcal{E}_{\geq 2}$

paths with ≥ 2 vertices



$\leftarrow$ $\frac{1}{2}\Big[(\mathcal{X}*\mathcal{E}_{\geq 1}) * \underbrace{(\mathcal{X}*\mathcal{E}) * \cdots * (\mathcal{X}*\mathcal{E})}_{\text{arbitrary length} \geq 0} * (\mathcal{X}*\mathcal{E}_{\geq 1}$

$\frac{1}{2}\Big[(\mathcal{X}*\mathcal{E}_{\geq 1})*(\mathcal{X}*\mathcal{E})^{*}*(\mathcal{X}*$

now back to (****)

Generally, a "bag" is a finite set with one special element

different

$$\mathcal{E}^{\circ} \equiv \mathcal{X} * \mathcal{E}$$

exponential generating function of $\mathcal{E}^{\circ}$ is

$$x \frac{d}{dx} \exp(x) = x\exp(x)$$

Let

$$\mathcal{E}_{\geq m} = \bigoplus_{k=m}^{\infty} \mathcal{E}_k,$$

the class of sets of size $\geq m$. Now to (***). (****): So

$$\mathcal{K} \equiv (\mathcal{X} * \mathcal{E}_{\geq 2}) \oplus \frac{1}{2}\Big[(\mathcal{X} * \mathcal{E}_{\geq 1})^2 * (\mathcal{X} * \mathcal{E})^{*}\Big]$$

informal shorthand

$$e^x = \exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \in \mathbb{Q}[[x]]$$

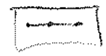
$$K(x) = x(e^x - 1 - x) + \frac{1}{2}\left(\frac{x^2(e^x-1)^2}{1 - xe^x}\right)$$

2015 09 23 C2

$$\# \mathcal{K}_n = n!\,[x^n]K(x) \quad \text{(exercise!)}$$

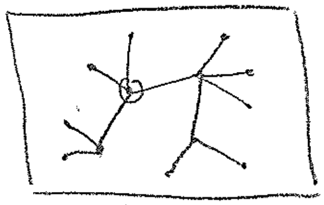
$$\# \mathcal{K}_2 = 0 \checkmark$$

$$\# \mathcal{K}_3 = 3! \cdot \frac{1}{2!} + 3! \cdot \frac{1}{2} \cdot 0 = 3 \checkmark$$



Even Rooted Trees

Rooted trees in which every vertex has an even number of children, $\mathcal{Q}$.



$$\mathcal{Q} \equiv \mathcal{X} * \mathcal{E}_{\text{even}}[\mathcal{Q}] \qquad Q(x) = x\cosh(Q(x))$$

$$\mathcal{E}_{\text{even}} = \bigoplus_{j=0}^{\infty} \mathcal{E}_{2j}$$

$$E_{\text{even}}(x) = \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$\# \mathcal{Q}_0 = 0$ and for $n \geq 1$:

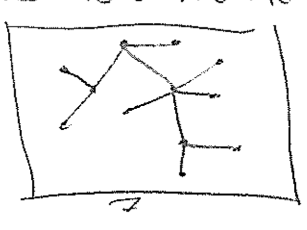
$$
\begin{aligned}
\# \mathcal{Q}_n &= n!\,[x^n]Q(x) \\
&= (n-1)!\,[u^{n-1}]\cosh(u)^n \\
&= (n-1)!\,[u^{n-1}]\left(\frac{e^u + e^{-u}}{2}\right)^n \\
&= \frac{(n-1)!}{2^n}[u^{n-1}]\sum_{j=0}^{n}\binom{n}{j} e^{ju} e^{-(n-j)u} \\
&= \frac{(n-1)!}{2^n}[u^{n-1}]\sum_{j=0}^{n}\binom{n}{j} e^{(2j-n)u} \\
&= \frac{(n-1)!}{2^n}\sum_{j=0}^{n}\binom{n}{j}\frac{(2j-n)^{n-1}}{(n-1)!}.
\end{aligned}
$$

So, for $n \geq 1$

$$\# \mathcal{Q}_n = \frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}(2j-n)^{n-1}$$

n even: j term cancels $n-j$ term and $\# \mathcal{Q}_n = 0$.

Unrooted trees in which every vertex has odd degree, $\mathcal{Z}$.



$$\mathcal{Z}^{\bullet} \equiv \mathcal{X} * \mathcal{E}_{\text{odd}}[\mathcal{Q}] \qquad E_{\text{odd}}(x) = \sinh(x)$$

$$\begin{cases} Z^{\bullet} = x\sinh(Q) \\ Q = x\cosh(Q) \end{cases} \rightarrow \text{LIFT} \quad \begin{aligned} F(u) &= \sinh(u) \\ G(u) &= \cosh(u) = F'(u \end{aligned}$$

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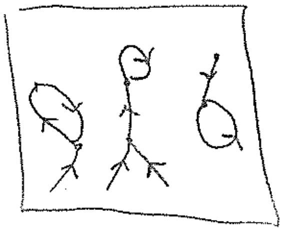
Components of Endofunctions

$\phi : X \to X$ an endofunction, $c(\phi) = \#$ of connected components of ϕ

$c(\phi) = 3$



What is the average value of $c(\phi)$ among all $\phi \in \mathcal{F}_n$? Call it $\bar{c}(n)$. We know $\# \mathcal{F}_n = n^n$.

The average is

$$\frac{1}{n^n}\left(\begin{array}{c}\text{total \# of components} \\ \text{among all } \phi \in \mathcal{F}_n\end{array}\right).$$

Let

$$\Phi(x,y) = \sum_{n=0}^{\infty}\left(\sum_{\phi \in \mathcal{F}_n} y^{c(\phi)}\right)\frac{x^n}{n!}.$$

Note:

$$\Phi(x,1) = \sum_{n=0}^{\infty}\left(\# \mathcal{F}_n\right)\frac{x^n}{n!}$$

$$\frac{\partial}{\partial y}\Phi(x,y)\Big|_{y=1} = \sum_{n=0}^{\infty}\left(\sum_{\phi \in \mathcal{F}_n} c(\phi)\right)\frac{x^n}{n!}.$$

So

$$\bar{c}(n) = \frac{1}{n^n}\, n!\, [x^n]\, \frac{\partial}{\partial y}\Phi(x,y)\Big|_{y=1}.$$

Recall

$$\begin{cases} \mathcal{F} \equiv \mathcal{E}[\mathbb{C}] \\ \mathbb{C} \equiv C[\mathbb{R}] \quad \text{where } \mathbb{R} = \mathcal{T}^{\bullet} \\ \mathbb{R} \equiv \mathcal{E}[\mathbb{R}] \end{cases}$$

$R(x) = x\exp(R(x))$

$\mathbb{C}(x,y) = y\log\left(\frac{1}{1-R}\right)$

$$\Phi(x,y) = \exp(\mathbb{C}(x,y)) = \exp\left(y\log\left(\frac{1}{1-R}\right)\right) = \frac{1}{(1-R)^y}.$$

$$\frac{\partial}{\partial y}\Phi(x,y)\Big|_{y=1} = \frac{\partial}{\partial y}\exp\left(y\log\left(\frac{1}{1-R}\right)\right)\Big|_{y=1}$$
$$= \frac{1}{1-R}\cdot\log\left(\frac{1}{1-R}\right)$$
$$= \left(\sum_{i=0}^{\infty}R^i\right)\left(\sum_{j=1}^{\infty}\frac{R^j}{j}\right)$$
$$= \sum_{p=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{p}\right)R^p$$

$H_p = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p}$ Harmonic numbers

$$\lim_{p\to\infty}\left(H_p - \log(p)\right) \longrightarrow \gamma \approx 0.577\ldots \quad \text{Euler's constant}$$

Apply LIFT with $G(u)=\exp(u)$ and $F(u)=u^p$ $(p\geq 1)$. $F'(u)=pu^{p-1}$

$$n!\,[x^n]R(x)^p = (n-1)!\,[u^{n-1}]\,pu^{p-1}\exp(u)^n$$
$$= (n-1)!\,p\,[u^{n-p}]\exp(nu)$$
$$= \frac{p(n-1)!\,n^{n-p}}{(n-p)!}, \quad \text{for } 1\leq p\leq n, \quad 0 \text{ if } p>n$$

$$\bar{c}(n) = \frac{1}{n^n}n!\,[x^n]\frac{\partial}{\partial y}\Phi(x,y)\Big|_{y=1}$$
$$= \frac{(n-1)!}{n^n}\sum_{p=1}^{n}pH_p\,\frac{n^{n-p}}{(n-p)!}$$

n	$n^n\bar{c}(n)$	$\bar{c}(n)$
1	1	1
2	5	1.24
3	38	$1.\overline{407}$
4	390	1.523...
5	5049	1.616...
6	78960	1.692...

Vertices with $p \geq 0$ children in a rooted tree

$$\mathcal{R} = \mathcal{T}^{\bullet} \qquad \mathcal{R} \equiv \mathcal{X} * \mathcal{E}[\mathcal{R}]$$

Given $(T,v) \in \mathcal{R}_n$, let $c(T,v)$ be the number of vertices of T with exactly p children. What is $\bar{c}(n)$, the average value $c(T,v)$ among all of $\mathcal{R}_n$? We know $\# \mathcal{R}_n = n^{n-1}$.

$$R(x,y) = \sum_{n=0}^{\infty} \left(\sum_{(T,v) \in \mathcal{R}_n} y^{c(T,v)} \right) \frac{x^n}{n!}$$

$$= x \left(\exp(R(x,y)) - \frac{R(x,y)^p}{p!} + y \frac{R(x,y)^p}{p!} \right)$$

$$= x \left(\exp(R) + (y-1) R^p / p! \right)$$

$$\bar{c}(n) = \frac{1}{n^{n-1}} n! \, [x^n] \frac{\partial}{\partial y} R(x,y) \Big|_{y=1}$$

$$= \frac{1}{n^{n-1}} n! \frac{\partial}{\partial y} [x^n] R(x,y) \Big|_{y=1}$$

$$= \frac{1}{n^{n-1}} n! \frac{\partial}{\partial y} \frac{1}{n} [u^{n-1}] \left(\exp(u) + (y-1) \frac{u^p}{p!} \right)^n \Big|_{y=1}$$

$$= \frac{(n-1)!}{n^{n-1}} [u^{n-1}] \frac{\partial}{\partial y} \left(\exp(u) + (y-1) \frac{u^p}{p!} \right)^n \Big|_{y=1}$$

$$= \frac{n!}{n^{n-1}} [u^{n-1}] \exp(u)^{n-1} \frac{u^p}{p!}$$

$$= \frac{n!}{p! \, n^{n-1}} [u^{n-1-p}] \exp((n-1)u)$$

$$= \begin{cases} \dfrac{n! \, (n-1)^{n-1-p}}{p! \, (n-1-p)! \, n^{n-1}} & \text{for } 0 \leq p \leq n-1 \\ 0 & \text{if } p > n-1 \end{cases}$$

For $0 \leq p \leq n$

$$\bar{c}(n) = \frac{n}{(n-1)^p} \binom{n-1}{p} \left(\frac{n-1}{n} \right)^{n-1}$$

$$= n \cdot \underbrace{\frac{1}{(n-1)^p} \binom{n-1}{p}}_{\frac{n(n-1)\cdots(n-p+1)}{p!(n-1)^p} \to \frac{1}{p!}} \underbrace{\left(1 - \frac{1}{n} \right)^n}_{\to e^{-1}} \underbrace{\frac{n}{n-1}}_{\to 1}$$

With p fixed, as $n \to \infty$

$$\bar{c}(n) \sim \frac{n}{p! \, e}, \qquad \frac{\bar{c}(n)}{n} \longrightarrow \frac{1}{p! \, e}$$

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Note

$$\sum_{p=0}^{\infty} \frac{1}{p!e} = 1$$

$S \equiv \mathcal{E}[C]$ permutations

$c(\phi)$ be the number of even length cycles in $\phi: N_n \to N_n$

$$S(x,y) = \sum_{n=0}^{\infty} \left(\sum_{\phi \in S_n} y^{c(\phi)} \right) \frac{x^n}{n!}$$

$$= \exp(C(x,y))$$

$$C = \bigoplus_{k=1}^{\infty} C_k \qquad C_k(x,y) = \begin{cases} (k-1)!\,\frac{x^k}{k!} & k \geq 1 \text{ odd} \\ (k-1)!\,\frac{x^k}{k!}\,y & k \geq 2 \text{ even} \end{cases}$$

$$C(x,y) = \sum_{j=1}^{\infty} \left(\frac{x^{2j-1}}{2j-1} + y\frac{x^{2j}}{2j} \right)$$

$$= \underbrace{\sum_{j=1}^{\infty} \left(\frac{x^{2j-1}}{2j-1} \right)}_{\log\left(\frac{1}{1-x}\right) - \frac{1}{2}\log\left(\frac{1}{1-x^2}\right)} + y \underbrace{\left(\sum_{j=1}^{\infty} \frac{x^{2j}}{2j} \right)}_{\frac{1}{2}\log\left(\frac{1}{1-x^2}\right)}$$

$$= \log\left(\frac{\sqrt{1-x^2}}{1-x}\right) = \log\left(\frac{\sqrt{1+x}}{\sqrt{1-x}}\right) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$C(x,y) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) + \frac{y}{2}\log\left(\frac{1}{1-x^2}\right)$$

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Natural Classes of Structures

A natural class $\mathcal{A}$ associates to each finite set X another finite set $\mathcal{A}_X$ satisfying the following axioms:

[*V] Given a structure α from $\mathcal{A}$ one can compute $V_{\mathcal{A}}(\alpha) = X$, where X is the set of vertices of α: $\alpha \in \mathcal{A}_X$.

(Note: $\alpha \in \mathcal{A}_X \cap \mathcal{A}_Y$ then $X = V_{\mathcal{A}}(\alpha) = Y$. So if $X \neq Y$ then $\mathcal{A}_X \cap \mathcal{A}_Y = \emptyset$.)

[*C1] For any bijection $f: X \to Y$ there is an induced bijection $f_*: \mathcal{A}_X \to \mathcal{A}_Y$. (Sometimes denoted $f_{\mathcal{A}}$ if necessary.)

[*C2] For $id_X: X \to X$ we have $(id_X)_{\mathcal{A}} = id_{\mathcal{A}_X}: \mathcal{A}_X \to \mathcal{A}_X$.

[*C3] For $f: X \to Y$ and $g: Y \to Z$ bijections, $g_* \circ f_* = (g \circ f)_*: \mathcal{A}_X \to \mathcal{A}_Z$.

Note that [*C1] implies that if $\#X = \#Y$ then $\#\mathcal{A}_X = \#\mathcal{A}_Y$.

Examples

Graphs $\mathcal{G}$ (simple graphs)

For a finite set X, $\mathcal{G}_X$ is the set of all pairs $\gamma = (X, E)$ with E a subset of 2-element subsets of X.

[*V]: Let $V_{\mathcal{G}}(\gamma) = X$.

[*C1]: For a bijection $f: X \to Y$ let $f_*(\gamma) = (Y, F)$ in which
$$F = \{\{f(u), f(v)\} ; \{u,v\} \in E\} \quad \text{(is a bijection)}$$

[*C2]: Obvious.

[*C3]: Obvious.

Endofunctions

For X, Y finite sets, and a bijection $f: X \to Y$ and an endofunction $\phi \in \mathcal{F}_X$, let
$$f_*(\phi) = f \circ \phi \circ f^{-1}.$$
Check the axioms.

Comparing Classes of Structures

Natural Transformations

Given natural classes $\mathcal{A}, \mathcal{B}$, a natural transformation $\tau: \mathcal{A} \Rightarrow \mathcal{B}$ is as follows. For every finite set X, there is a function $\tau_X : \mathcal{A}_X \to \mathcal{B}_X$

[*T] For any bijection $f: X \to Y$
$$\begin{array}{ccc} \mathcal{A}_X & \xrightarrow{f_{\mathcal{A}}} & \mathcal{A}_Y \\ {\scriptstyle \tau_X}\downarrow & & \downarrow{\scriptstyle \tau_Y} \\ \mathcal{B}_X & \xrightarrow[f_{\mathcal{B}}]{} & \mathcal{B}_Y \end{array}$$
commutes. For any $\alpha \in \mathcal{A}_X$, $(\tau_Y \circ f_{\mathcal{A}})(\alpha) = (f_{\mathcal{B}} \circ \tau_X)(\alpha)$.

Examples

A natural transformation $\tau: \mathcal{F} \Rightarrow \mathcal{G}$. Given $\phi \in \mathcal{F}_X$, let
$$\tau(\phi) = (X, E)$$
with

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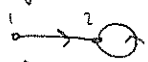
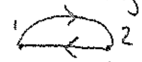
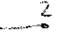
$E = \{\{u,v\} \subseteq X;\ u \neq v \text{ and either } u = \phi(v) \text{ or } v = \phi(u)\}.$

Note: In general τ_X is neither injective nor surjective:

-  and  both map to ;
-  is not the image of any endofunction.

If τ_X is injective for all X we can say that $\mathcal{A}$ is a (natural) <u>subclass</u> of $\mathcal{B}$.

Example: $\mathcal{T}^{\cdot}$ is a natural subclass of $\mathcal{F}$.

Exercise: Define $\tau_X : \mathcal{T}^{\cdot}_X \to \mathcal{F}_X$ explicitly.

Then $(\mathcal{B} \setminus \mathcal{A})_X = \mathcal{B}_X \setminus \{\tau_X(\alpha);\ \alpha \in \mathcal{A}_X\}$ has exponential generating function $B(x) - A(x)$.

<u>Natural Equivalence</u>

$\mathcal{A} \equiv \mathcal{B}$ means there are natural transformations

$$\tau : \mathcal{A} \Rightarrow \mathcal{B} \quad \text{and} \quad \rho : \mathcal{B} \Rightarrow \mathcal{A}$$

such that for any finite set X

$$\tau_X : \mathcal{A}_X \to \mathcal{B}_X \quad \text{and} \quad \rho_X : \mathcal{B}_X \to \mathcal{A}_X$$

are mutually inverse bijections.

Example: $\mathcal{R}$ rooted trees

$\mathcal{R} \equiv \mathcal{X} * \mathcal{E}[\mathcal{R}]$ is a natural equivalence

For a finite set X define $\tau_X : \mathcal{R}_X \to (\mathcal{X} * \mathcal{E}[\mathcal{R}])_X$ as follows. Let $(T, v) \in \mathcal{R}_X$. Let the components of $T \setminus v$ be $S_1, \ldots, S_k$. Let w_i be the unique neighbour of v in S_i.

$$\tau_X(T, v) = (v, \{(S_1, w_1), \ldots, (S_k, w_k)\}).$$

Definition of ρ_X ... exercise.

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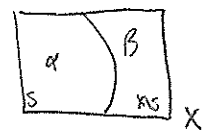
<u>Natural Classes - Examples</u>

$\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ natural classes

• $(\mathcal{A} * \mathcal{B})$ is natural

For a finite set X and $(\alpha, \beta) \in (\mathcal{A} * \mathcal{B})_X$



$$V_{\mathcal{A}*\mathcal{B}}(\alpha, \beta) = V_{\mathcal{A}}(\alpha) \cup V_{\mathcal{B}}(\beta)$$

Induced bijection of $f: X \to Y$ is

$$f_*(\alpha, \beta) = (f_{\mathcal{A}}(\alpha), f_{\mathcal{B}}(\beta))$$

• Any two ways of parenthesizing a multiple product are naturally equivalent

natural classes } $(\mathcal{A} * \mathcal{B}) * (\mathcal{C} * \mathcal{D}) \equiv (\mathcal{A} * (\mathcal{B} * \mathcal{C})) * \mathcal{D}$

sets & bijections } $((\mathcal{A} * \mathcal{B}) * (\mathcal{C} * \mathcal{D}))_X \rightleftharpoons ((\mathcal{A} * (\mathcal{B} * \mathcal{C})) * \mathcal{D})_X$

elements 1-1 correspondence } $((\alpha, \beta), (\gamma, \delta)) \longleftrightarrow ((\alpha, (\beta, \gamma)), \delta)$

• $\mathcal{L} = \mathcal{X}^* = \bigoplus_{k=0}^{\infty} \mathcal{X}^k$ class of total orders, $\mathcal{X} = \mathcal{E}_1$.

On a set X, a typical $\mathcal{X}^*$-structure naturally is a sequence

$$(\{v_1\}, \{v_2\}, \dots, \{v_k\})$$

in which $\{v_1\} \cup \{v_2\} \cup \dots \cup \{v_k\} = X$ and the entries are pairwise disjoint, naturally equivalent to

$$(v_1, v_2, \dots, v_k)$$

• $\mathcal{S}$ permutations
$\mathcal{L} = \mathcal{X}^*$ total orders
$\left\{ \begin{array}{l} S(x) = \frac{1}{1-x} = L(x) \\ \# S_n = n! = \# L_n \end{array} \right.$

superposition: $(\mathcal{A} \& \mathcal{B})_X = \mathcal{A}_X * \mathcal{B}_X$ for any finite set X

$$(\alpha, \beta): \alpha \in \mathcal{A}_X, \beta \in \mathcal{B}_X$$

$V(\alpha, \beta) = V_{\mathcal{A}}(\alpha)$

For $f: X \to Y$ a bijection $f_*(\alpha, \beta) = (f_{\mathcal{A}}(\alpha), f_{\mathcal{B}}(\beta))$ is natural

Claim: $\mathcal{S} \& \mathcal{L} \equiv \mathcal{L} \& \mathcal{L}$ naturally

Verification: Fix X. $(\mathcal{S} \& \mathcal{L})_X \rightleftharpoons (\mathcal{L} \& \mathcal{L})_X$

$$(\sigma, \vec{v}) \longleftrightarrow (\vec{u}, \vec{\omega})$$

natural transformations

$$\tau_X : (\mathcal{S} \& \mathcal{L})_X \to (\mathcal{L} \& \mathcal{L})_X$$
$$\rho_X : (\mathcal{L} \& \mathcal{L})_X \to (\mathcal{S} \& \mathcal{L})_X$$

mutually inverse bijections

Given $(\sigma, \vec{v})$, $\sigma: X \to X$ bijection, $\vec{v} = (v_1, v_2, \dots, v_k) \in \mathcal{L}_X$,

Let

$$\vec{u} = (\sigma(v_1), \sigma(v_2), \dots, \sigma(v_k)) \in \mathcal{L}_X, \quad \vec{\omega} = \vec{v}$$

$$\tau_X(\sigma,\vec{v}) = (\vec{u},\vec{w})$$

Given $(\vec{u},\vec{w})$, $\vec{u} = (u_1,u_2,\dots,u_k)$, $\vec{w} = (w_1,w_2,\dots,w_k)$

Let

$$\sigma(w_i) = u_i \quad \text{for all } 1 \le i \le k, \text{ so } \sigma \in S_X$$
$$\vec{v} = \vec{w}$$
$$\rho_X(\vec{u},\vec{w}) = (\sigma,\vec{v}).$$

In general, the operation $\mathcal{A} \rightsquigarrow \mathcal{A}\&\mathcal{L}$ is essentially numbering the elements of X in order $1,2,\dots,n$ where $n = \#X$.

What about $\mathcal{S} \equiv \mathcal{L}$?

We need natural transformations

$$\tau : \mathcal{S} \Rightarrow \mathcal{L} \quad \text{and} \quad \rho : \mathcal{L} \Rightarrow \mathcal{S}$$

that are mutually inverse bijections on any set X.

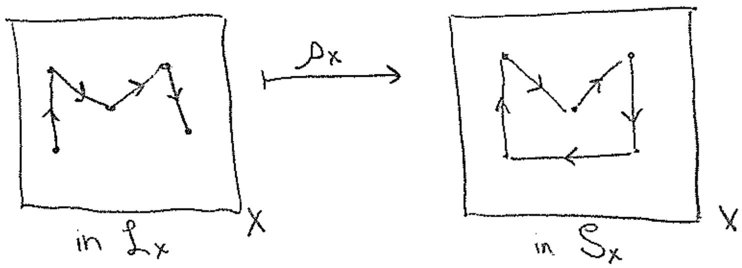


Image is $\mathcal{S}_0 \circledast \mathcal{C}$. Each $\sigma \in \mathcal{C}_n$ is the image of n lists in $\mathcal{L}_n$.

This $\rho : \mathcal{L} \Rightarrow \mathcal{S}$ is natural.

In fact: there are no natural transformations from $\mathcal{S}$ to $\mathcal{L}$.

Automorphisms

Let $\mathcal{A}$ be a natural class. Let X be a finite set. Let $\alpha \in \mathcal{A}_X$. A permutation $\sigma : X \to X$ is an automorphism of α if $\sigma_*(\alpha) = \alpha$.

Example: $\mathcal{G}$: class of graphs

Let

$$\operatorname{aut}(\alpha) = \{\sigma \in S_X \, ; \, \sigma_*(\alpha) = \alpha\}.$$

Claim: $\mathrm{aut}(\alpha)$ is a subgroup of S_X.

Proof: $\mathrm{aut}(\alpha)$ is a subset of S_X. $\iota = id_X : X \to X$ is in $\mathrm{aut}(\alpha)$ since $(id_X)_* = id_{A_X}$ so $\iota(\alpha) = \alpha$, by [*C2]. For permutations $\sigma, \phi \in \mathrm{aut}(\alpha)$,

$$(\sigma \circ \phi)_*(\alpha) = (\sigma_* \circ \phi_*)(\alpha) = \sigma_*(\phi_*(\alpha)) = \sigma_*(\alpha) = \alpha.$$

$$\left.\begin{array}{l} \cdot\ \mathrm{aut}(\alpha) \subseteq S_X \\ \cdot\ \text{contains } id_X \\ \cdot\ \text{closed by composition} \end{array}\right\} \ \mathrm{aut}(\alpha) \text{ is a group}$$

□

Let $\mathcal{P}$ be the class of permutation groups. For a finite set X, $\mathcal{P}_X$ is the set of all subgroups of S_X.

Claim: For any natural class $\mathcal{A}$,

$$\mathrm{aut} : \mathcal{A} \Rightarrow \mathcal{P}$$

is a natural transformation.

(Note $\mathcal{P}$ is a natural class:
for $f : X \to Y$ bijection, $\Gamma \leq S_X$,
$$f_*(\Gamma) = \{ f \circ \phi \circ f^{-1} ;\ \phi \in \Gamma \} \leq S_Y.)$$

Proof: Check this commutes: (for any bijection of finite sets $f : X \to Y$)

$$\begin{tikzcd} \mathcal{A}_X \arrow[r, "f_\mathcal{A}"] \arrow[d, "\mathrm{aut}_X"'] & \mathcal{A}_Y \arrow[d, "\mathrm{aut}_Y"] \\ \mathcal{P}_X \arrow[r, "f_\mathcal{P}"'] & \mathcal{P}_Y \end{tikzcd}$$

For any $\alpha \in \mathcal{A}_X$
$$\mathrm{aut}_Y(f_\mathcal{A}(\alpha)) = f_\mathcal{P}(\mathrm{aut}_X(\alpha)),$$
as subgroups of S_Y.

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For $\sigma \in S_Y$,

- $\sigma \in \mathrm{aut}(f_\mathcal{A}(\alpha))$ if and only if $\sigma_\mathcal{A}(f_\mathcal{A}(\alpha)) = f_\mathcal{A}(\alpha)$
- $\sigma \in f_\mathcal{P}(\mathrm{aut}(\alpha))$ if and only if there is a $\pi \in \mathrm{aut}(\alpha)$ such that $\sigma = f \circ \pi \circ f^{-1}$.

Exercise: Show that these conditions on $\sigma \in S_Y$ are equivalent. ⊠

Note: $(f^{-1})_* = (f_*)^{-1}$ for any bijection f and natural class. Follows from [*C2,3].

Examples X a finite set

- Endomorphisms

  $\phi \in \mathcal{F}_X$ an endomorphism $\phi: X \to X$

  $\sigma \in S_X$ is in $\mathrm{aut}(\phi)$ if and only if $\sigma_*(\phi) = \phi$,

  i.e. $\sigma \circ \phi \circ \sigma^{-1} = \phi$ i.e. $\sigma \circ \phi = \phi \circ \sigma$

  $\mathrm{aut}(\phi)$ is the set of all $\sigma \in S_X$ that commute with ϕ.

- Lists $\mathcal{L} = X^*$

  Given $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathcal{L}_X$ and $\sigma \in S_X$, when is $\sigma \in \mathrm{aut}(\vec{v})$

  $\sigma_{\mathcal{L}}(\vec{v}) = \vec{v}$

  $\sigma_{\mathcal{L}}(\vec{v}) = (\sigma(v_1), \sigma(v_2), \ldots, \sigma(v_n)) = \vec{v}$

  if and only if $\sigma(v_i) = v_i$ for all $1 \le i \le n$

  if and only if $\sigma = id_X$ is the identity

  For any list $\vec{v} \in \mathcal{L}_X$: $\mathrm{aut}(\vec{v}) = \{id_X\}$

Proposition: Let $\mathcal{A}, \mathcal{B}$ be natural classes and let $\tau: \mathcal{A} \Rightarrow \mathcal{B}$ be a natural transformation. Fix a finite set X and $\alpha \in \mathcal{A}_X$.

$$\mathrm{aut}(\alpha) \le \mathrm{aut}(\tau(\alpha))$$

is an inclusion of permutation groups in $\mathcal{F}_X$.

Proof: Consider any $\sigma \in \mathrm{aut}(\alpha)$. So $\sigma \in S_X$ and $\sigma_{\mathcal{A}}(\alpha) = \alpha$. Now

$$\tau_X(\alpha) = \tau_X(\sigma_{\mathcal{A}}(\alpha)) = \sigma_{\mathcal{B}}(\tau_X(\alpha))$$

since $\sigma_{\mathcal{A}}(\alpha) = \alpha$. So $\sigma \in \mathrm{aut}(\tau_X(\alpha))$. So $\mathrm{aut}(\alpha) \le \mathrm{aut}(\tau_X(\alpha))$. ∎

Corollary: There are no natural transformations $\tau: S \Rightarrow \mathcal{L}$.

Proof: Let X be a set with $\#X \ge 2$. $id_X \in S_X$ and $\mathrm{aut}(id_X) = S_X$. If τ exists then $\tau_X(id_X) = \vec{v} \in \mathcal{L}_X$ and $\mathrm{aut}(\vec{v}) = \{id_X\}$. Since $\#X \ge 2$, $\# S_X > 1$. So τ doesn't exist. ∎

Question $\quad S(x) = (1-x)^{-1} = L(x)$

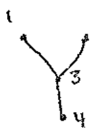
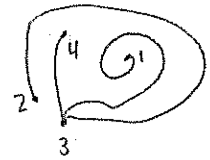
but $\mathcal{S} \not\equiv \mathcal{L}$ naturally

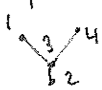
but $\mathcal{S} \& \mathcal{L} \equiv \mathcal{L} \& \mathcal{L}$ naturally

Given any two natural classes $\mathcal{A}, \mathcal{B}$ such that $A(x) = B(x)$, does it follow that $\mathcal{A} \& \mathcal{L} \equiv \mathcal{B} \& \mathcal{L}$ naturally?

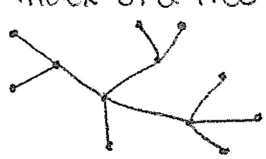
Homework Problems from Chapter 12:

- Labelled plane trees $\mathcal{Q}$
  - Tree with vertex set $\{1,2,\ldots,n\}$ embedded in the plane $\mathbb{R}^2$.
  - up to "ambient isotopy"

 is ambient isotopic to 

but not to 

- Wiener index of a tree



$$W(T) = \frac{1}{2} \sum_{v,w \in V} \mathrm{dist}_T(v,w)$$

What is

$\overline{W}(n)$ = average value of $W(T)$ among all n^{n-2} trees in $\mathcal{T}_n$?

$$\overline{W}(n) = \frac{1}{n^{n-2}} \frac{1}{2} \sum_{T \in \mathcal{T}_n} \sum_{v=1}^{n} \sum_{w=1}^{n} \mathrm{dist}_T(v,w)$$

(T,v,w) is a doubly-rooted tree in $\mathcal{T}_n^{\bullet\bullet}$.

- Productivity

$\pi(T) = 4 \cdot 3 \cdot 2 = 24$

$$\pi(T) = \prod_{v \in T} \deg_T(v)$$

"It's good to think about hard problems even if you don't solve them. That's basically the story of my career."

2015 10 02(3)

What is

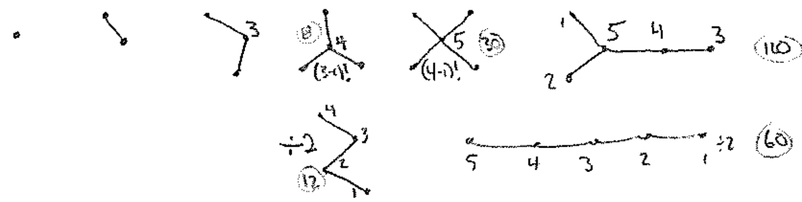
$$\bar{\pi}(n) = \frac{1}{n^{n-2}} \sum_{T \in \mathcal{T}_n} \pi(T)\,?$$

2015 10 05

<u>Labelled Plane Trees</u>

$h(n) = \#$ of labelled plane trees on $\{1, 2, \ldots, n\}$

n	0	1	2	3	4	5
$h(n)$	0	1	1	3	20	210



13. Combinatorial Proof of LIFT

Combinatorial interpretation of
$$\begin{cases} [x^n]F(R(x)) = \frac{1}{n}[u^{n-1}]F'(u)G(u)^n \\ \quad \text{when } R(x) = xG(R(x)) \\ \quad\quad \text{and } n \geq 1 \text{ and } [x^0]G(u) \neq 0 \end{cases}$$

Exponential Generating Functions
$$F(u) = \sum_{n=0}^{\infty} f_n \frac{u^n}{n!} \quad \text{and} \quad G(u) = \sum_{n=0}^{\infty} g_n \frac{u^n}{n!}$$

Assume nothing about $\{f_n\}$ and $\{g_n\}$: these and u are algebraically independent indeterminates comuting over a field $\mathbb{K}$. Anything provable for generic $F(u)$ and $G(u)$ holds for any specialization. Eg. for a class $\mathcal{A}$,
$$f_n \mapsto \#\mathcal{A}_n \quad F(u) \mapsto A(x).$$

A <u>generic class</u> is just the class $\mathcal{E}$ of sets. We mark a set of size n with f_n and we get
$$F(u) = \sum_{n=0}^{\infty} f_n \frac{u^n}{n!}.$$

$R(x) = x \cdot G(R(x))$ with
$$G(u) = \sum_{n=0}^{\infty} g_n \frac{u^n}{n!} \quad \text{and} \quad g_0 \neq 0.$$

On the level of classes,

$$R \equiv X * \mathcal{E}[R].$$

$$R(x) = \sum_{n=0}^{\infty} r_n \frac{x^n}{n!}$$

Claim:

☺ —— $r_n = \sum_{(T,v)\in \mathcal{T}_n^\bullet} g^{(T,v)}$ where $g^{(T,v)} = \prod_{w\in N_n} g_{c(T,v;w)}$

and $c(T,v;w) =$ # of children of w in (T,v)

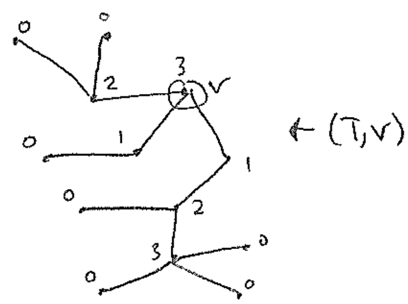
"Proof": By induction on n.

$[x^0]R(x) = 0$ and $[x^1]R(x) = g_0$

corresponds to $\odot\ g_0$

Assume ☺ for all values strictly less than n.

$$\begin{aligned} r_n &= n!\,[x^n]R(x) \\ &= n!\,[x^n]\,xG(R(x)) \\ &= n!\,[x^{n-1}]\sum_{k=0}^{\infty} g_k \frac{R(x)^k}{k!} \\ &= n\cdot(n-1)!\,[x^{n-1}]\sum_{k=0}^{\infty} g_k \frac{R(x)^k}{k!}. \end{aligned}$$



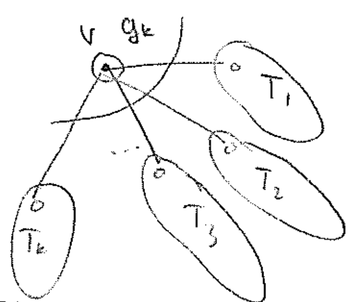
$$g^{(T,v)} = g_0^7 g_1^2 g_2^2 g_3^2$$

By induction,

$$m!\,[x^m]R(x) = \sum_{(T,v)\in\mathcal{T}_m^\bullet} g^{(T,v)}$$

for all $0 \le m < n$.

$$\frac{R(x)^k}{k!} = \sum_{\Phi\in\mathcal{E}_k[\mathcal{T}^\bullet]} \prod_{(T,v)\in\Phi} g^{(T,v)}.$$



<u>Formally</u>

on a set X (of size n)

$$R \equiv X * \mathcal{E}[R]$$

$$R_X \rightleftharpoons (X * \mathcal{E}[R])_X$$

$$(T,v) \leftrightarrow (\{v\}, \{(S_1,w_1),\ldots,(S_k,w_k)\})$$

$$g^{(T,v)} = g_k \prod_{i=1}^{k} g^{(S_i,w_i)}$$

$$R = \sum_{n=0}^{\infty}\left(\sum_{(T,v)\in\mathcal{T}_n^\bullet} g^{(T,v)}\right)\frac{u^n}{n!}$$ satisfies $R = xG(R)$.

2015 10 07

e.g. Let $g_0 = g_1 = g_2 = 1$ and $g_k = 0$ for $k \geq 3$. Then every vertex has at most two children.

$$\mathcal{A} \equiv \mathcal{X} * (\mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \mathcal{E}_2)[\mathcal{A}]$$
$$A(x) = x(1 + A(x) + A(x)^2/2).$$

For $n \geq 1$: $n!\,[x^n]F(R(x)) = n!\,\frac{1}{n}[u^{n-1}]F'(u)G(u)^n$.

Consider $\mathcal{F}$ and $\mathcal{G}$ as "generic" (natural) classes of structures ($\mathcal{F} = \mathcal{G} = \mathcal{E}$, sets). Then $\mathcal{R} \equiv \mathcal{X} * \mathcal{G}[\mathcal{R}]$ has exponential generating function $R(x) = x\,G(R(x))$.

LHS: $n!\,[x^n]F(R(x))$ is tracking sets of rooted trees with n vertices.
A forest $\varphi = \{(T_1,v_1),\dots,(T_k,v_k)\}$ of rooted trees contributes
$$M(\varphi) = f_k \prod_{i=1}^{k} g^{(T_i,v_i)}$$

RHS: $F'(u) = \frac{d}{du}F(u) = u^{-1}F^{\bullet}(u)$

$G(u)^n$ is the exponential generating function of $\mathcal{G}^n$

$$n!\,\frac{1}{n}[u^{n-1}]F'(u)G(u)^n = \frac{1}{n}\,n!\,[u^n]F^{\bullet}(u)G(u)^n$$
$$= \frac{1}{n}\cdot \text{sum over all structures in } \mathcal{F}^{\bullet} * \mathcal{G}^n \text{ on the set } \{1,\dots,n\}$$

$\sigma = ((A,v), B_1,\dots,B_n) \in (\mathcal{F}^{\bullet} * \mathcal{G}^n)_n$ contributes
$$m(\sigma) = f_{\#A} \prod_{i=1}^{n} g_{\#B_i}$$

Dividing by n is unpleasant.
$$n \cdot n!\,[x^n]F(R(x)) = n!\,[u^n]F^{\bullet}(u)G(u)^n$$
$$\shortparallel$$
$$n!\,[x^n](F(R(x)))^{\bullet}$$

To prove this we want a bijection
$$(\mathcal{F}[\mathcal{R}])^{\bullet}_n \rightleftharpoons (\mathcal{F}^{\bullet} * \mathcal{G})_n$$

missing root!!!

$$\cancel{\varphi = \{(T_1,v_1),\dots,(T_k,v_k)\} \Leftrightarrow ((A,v),B_1,B_2,\dots,B_n) = \sigma}$$
$$\cancel{M(\varphi) = m(\sigma)}$$

2015 10 09

On the LHS we have a forest of rooted trees φ and a "special" vertex w:
$$(\varphi, w) \text{ with } \varphi = \{(T_1,v_1),\dots,(T_k,v_k)\} \in \mathcal{F}[\mathcal{R}]_n,\ w \in \{1,\dots,n\},$$

(φ, w) is counted using the monomial

$$M(\varphi) = f_{\#\varphi} \prod_{(T,v)\in\varphi} g^{(T,v)}.$$

On the RHS we have

$$\sigma = ((A,v),\ B_1, B_2, \ldots, B_n)$$

$A, B_1, B_2, \ldots, B_n$ are sets that together cover every element of $\{1, \ldots, n\}$ exactly once each.

Also $v \in A$. σ is counted with the monomial

$$m(\sigma) = f_{\#A} \prod_{i=1}^{n} g_{\#B_i}.$$

$$\mathcal{F}[\mathcal{R}]^{\circ}_n \rightleftarrows (\mathcal{F}^{\circ} * \mathcal{G}^{n})_n$$

$$(\varphi, w) \longleftrightarrow \sigma = (A, v, B_1, \ldots, B_n)$$

$$M(\varphi) = m(\sigma)$$

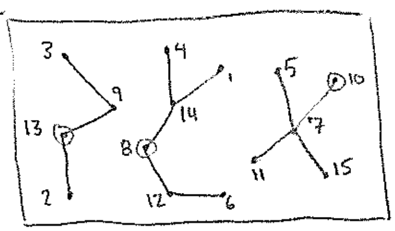
To begin with, define a function

$$\mathcal{F}[\mathcal{R}]_n \longrightarrow (\mathcal{F} * \mathcal{G}^{n})_n$$

$$\varphi \longmapsto (A, C_1, \ldots, C_n)$$

$$f_{\#\varphi} \prod_{(T,v)\in\varphi} g^{(T,v)} = f_{\#A} \prod_{i=1}^{n} g_{\#C_i}$$

Example



N_{15}

φ

$$M(\varphi) = f_3 (g_0^3 g_1^1 g_2^2)(g_0^3 g_1^1 g_3^1)(g_0^2 g_1^1 g_2^1)$$

Let A be the set of roots of the trees in φ. Then $f_{\#\varphi} = f_{\#A}$.

Define $(C_1, \ldots, C_n)$ by Breadth First Search on φ

Let $A = \{v_1, \ldots, v_k\}$ be the roots of the components of φ in ascending numerical order $v_1 < v_2 < \cdots < v_k$. ⟵ Let L be an empty list

For each $1 \le i \le k$ do:

put v_i at the start of a queue Q.

While Q is not empty do:

2015 10 09(2)

Let C be the set of children of the first vertex on Q.
Append the vertices of C to Q in ascending numerical order.
Append the set C to the list L.
Delete the first vertex of Q.

Output: (A, L) $\leftarrow$ $L: C_1, C_2, \ldots, C_n$

Example (continued)

$A = \{8, 10, 13\}$

$L: \{12,14\}, \{6\}, \{1,4\}, \ldots$

$Q: \cancel{8}, \cancel{12}, \cancel{14}, \cancel{6}, \cancel{1}, \cancel{4}$
$\quad \{12,14\}\ \{6\}\ \{1,4\}\ \emptyset\ \emptyset\ \emptyset$ $\qquad g_0^3 g_1 g_2^2$

$Q: 10, \cancel{7}, 5, \cancel{11}, \cancel{15}$
$\quad \{7\}\ \{5,11,15\}\ \emptyset\ \emptyset\ \emptyset$ $\qquad g_0^3 g_1 g_3$

$Q: 13, \cancel{2}, \cancel{9}, \cancel{3}$
$\quad \{2,9\}\ \emptyset\ \{3\}\ \emptyset$ $\qquad g_0^2 g_1 g_2$

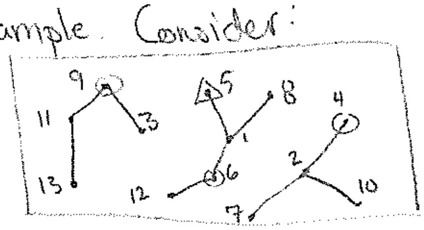
Every vertex of $\{1, 2, \ldots, n\}$ is in exactly one of the sets $A, C_1, \ldots, C_n$.
So $(A, C_1, \ldots, C_n)$ is in $(\mathcal{F} * \mathcal{G}^n)_n$.

$$\mathcal{F}[R]_n \longrightarrow (\mathcal{F} * \mathcal{G}^n)_n$$

is injective, but not surjective.

2015 10 14

Example. Consider:



φ
roots $\circ$
ω $\triangle$

Let A be the set of roots of components of φ.
Let v be the root of the component of φ containing ω.
Do breadth first search on components of φ always going in ascending numerical order, recording the set of children of each node.

$A: \cancel{4}, \cancel{6}, \cancel{9}$

$Q: \cancel{4}, \cancel{2}, \cancel{7}, \cancel{10}$

$\{2\}, \{7,10\}, \emptyset, \emptyset$ $\qquad C_1, \ldots, C_4$

$Q: \cancel{6}, \cancel{1}, \cancel{12}, \cancel{5}, \cancel{8}$

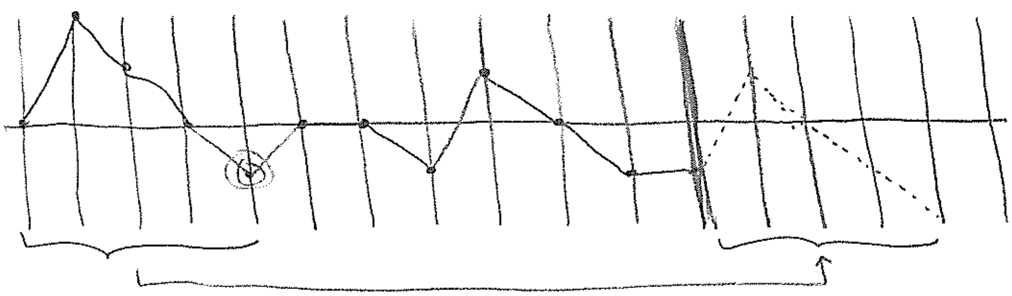
$\{1,12\}, \{5,8\}, \emptyset, \emptyset, \emptyset$ $\qquad C_5, \ldots, C_9$

$Q: 9, 3, 11, 13$

$\{3,11\}, \emptyset, \{13\}, \emptyset$ $\qquad C_{10}, \ldots, C_{13}$

Initially, the length of the queue is 1. Let $c_i = \#C_i - 1$. After the i^{th} vertex is examined, the length of Q changes by c_i. The queue stays non-empty until it's empty. When it's empty, it's empty. Each c_i is an integer, $c_i \geq -1$, $c_1 + \cdots + c_i \geq 0$ for all $1 \leq i < k$, $c_1 + \cdots + c_k = -1$. A sequence with this property is called a simple Raney sequence.

Lemma: Let $(b_1, \ldots, b_k)$ be integers, $b_i \geq -1$, with $b_1 + \cdots + b_k = -1$. Then exactly one cyclic shift of $(b_1, \ldots, b_k)$ is a simple Raney sequence.



Breadth first search on a tree gives a simple Raney sequence. Breadth first search on φ gives r simple Raney sequences, if φ has r components.

$(C_1, \ldots, C_r)$ is a concatenation of r simple Raney sequence

$c_i = \#C_i - 1$ $\qquad \rho_1 \cdots \rho_r$ $\quad$ r-fold simple Raney sequence

Lemma: If $(b_1, \ldots, b_n)$ are integers such that each $b_i \geq -1$ and $b_1 + \cdots + b_n = -r$, then "that's how not to run the algorithm"

2015 10 14(2)

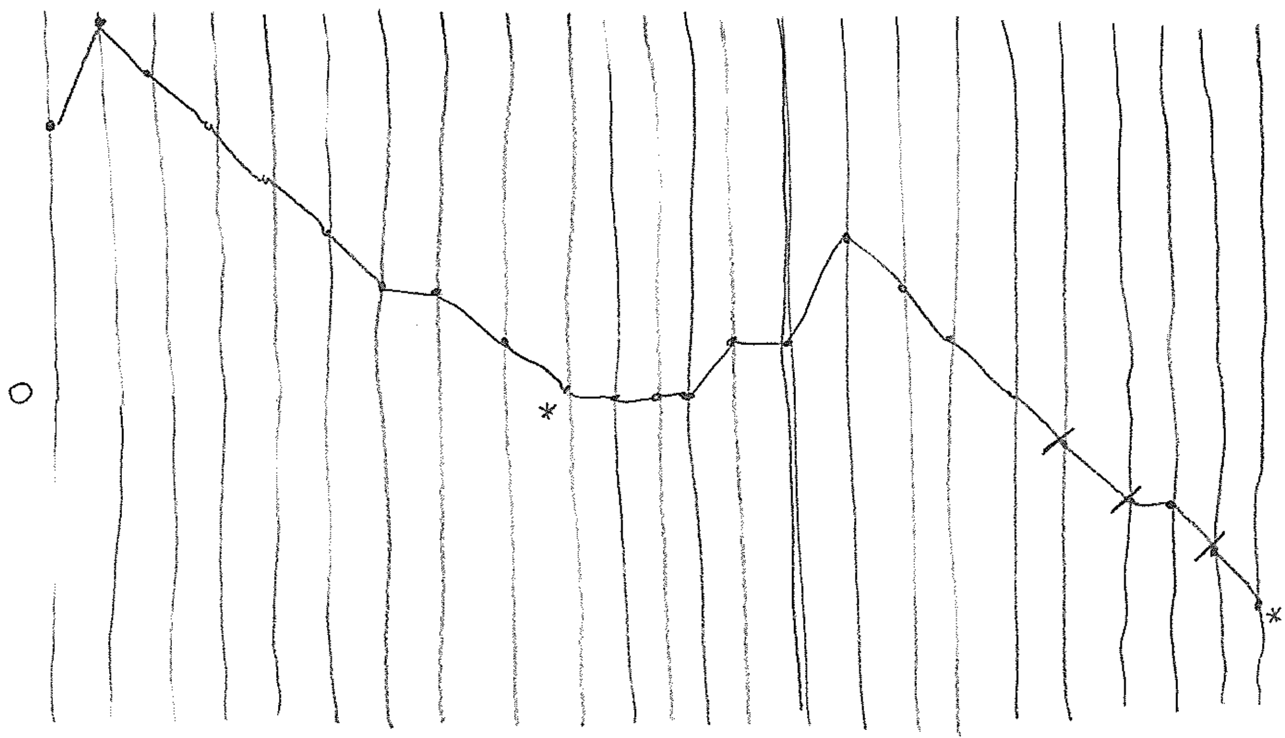
- Exactly r cyclic shifts of the b_i's are r-fold simple Raney sequences;
- These are obtained by cyclically shifting the blocks $\rho_1 \cdots \rho_r$ of any one of them. $\rho_j \rho_{j+1} \cdots \rho_r \rho_1 \cdots \rho_{j-1}$.

See book for final details.

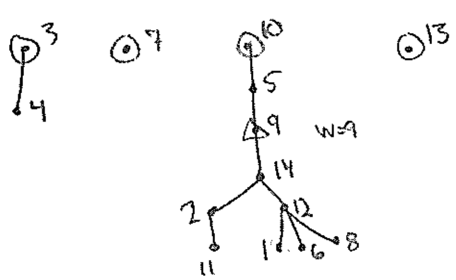
2015 10 16

Example: $n = 14$, $A = \{3, 7, 10, 13\}$, $r = 10$

$b_i = \#B_i - 1$	2	-1	-1	-1	-1	-1	0	-1	-1	0	0	0	1	0
	1,6,8						4			5	9	14	2,12	11
	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}



ρ_3										ρ_4	ρ_1		ρ_2
B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
$C_4 \cdots$											C_1	C_2	$\cdots$



B_1 is the p^{th} set in the s^{th} block

w is the p^{th} vertex in the s^{th} tree

The number of rooted trees on $\{1,2,\dots,n\}$ with exactly k terminal vertices is

$$(n-k)!\binom{n}{k}S(n-1,n-k).$$

Discussion.

2015 10 19

Nested Set Systems

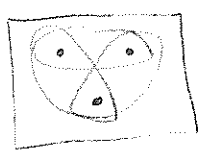
Def] A <u>nested set system</u> is a pair (X,Δ) where:

- X is a finite set;
- Δ is a collection of subsets of X such that if $A,B\in\Delta$ then either $A\subseteq B$, $B\subseteq A$, or $A\cap B=\emptyset$

For any finite set X, $\mathcal{N}_X$ is the set of all nested set systems on X.

This defines a natural class $\mathcal{N}$. For all $n\in\mathbb{N}$, what is $\#\mathcal{N}_n$?

- $n=0$: $\#\mathcal{N}_0=2$, $(\emptyset,\emptyset)$, $(\emptyset,\{\emptyset\})$
- $n=1$: $\#\mathcal{N}_1=4$, $(\{1\},\emptyset)$, $(\{1\},\{\emptyset\})$, $(\{1\},\{\{1\}\})$, $(\{1\},\{\emptyset,\{1\}\})$
- $n=2$: $\#\mathcal{N}_2=2^4=16$
- $n=3$: $\#\mathcal{N}_3=2^8-2^6-2^6-2^6+2^5+2^5=128$



not allowed

Let

$$N(x,y):=\sum_{n=0}^{\infty}\left(\sum_{(X,\Delta)\in\mathcal{N}_n}y^{\#\Delta}\right)\frac{x^n}{n!}=\sum_{n=0}^{\infty}N_n(y)\frac{x^n}{n}.$$

From the above,

$$N_0=1+y,\ N_1=(1+y)^2,\ N_2=(1+y)^4.$$

But what about N_3?

Reductions for $\mathcal{N}$: Sets of size zero or one are no problem: If $A\in\Delta$ and $\#A\leq 1$, then $(X,\Delta\setminus\{A\})$ is nested if and only if (X,Δ) is nested. Thus we say Δ is <u>proper</u> if $\#A\geq 2$ for all $A\in\Delta$, and denote the <u>proper part</u> of Δ by $\Delta^\circ:=\{A\in\Delta;\#A\geq 2\}$.

2015 10 19(2)

Let $\mathcal{M}$ be the class of proper nested set systems

Consider any $\mathcal{N}_X$. We have three functions:

- $\mathcal{N}_\emptyset$ is the set of $\mathcal{N}$-structures on $\emptyset$

$$\mathcal{N}_X \to \mathcal{N}_\emptyset$$
$$(X,\Delta) \mapsto (\emptyset, \{A \in \Delta ; A = \emptyset\})$$

- Let $\mathcal{P}_X := \{(X, A) ; A \subseteq X\}$, a natural class with $\#\mathcal{P}_X = 2^{\#X}$

$$\mathcal{N}_X \to \mathcal{P}_X$$
$$(X,\Delta) \mapsto (X, \{v \in X ; \{v\} \in \Delta\})$$

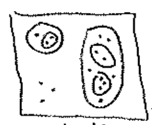
- Finally,

$$\mathcal{N}_X \to \mathcal{M}_X$$
$$(X,\Delta) \mapsto (X, \Delta^\circ).$$

Together, this is one direction of a natural equivalence
$$\mathcal{N} \equiv \mathcal{N}_\emptyset * (\mathcal{P} \,\&\, \mathcal{M}).$$

Let (X,Δ) be a proper nested set system. Note

- it is a disjoint union of components,
- each component is either a vertex or a "cell".



X

in $\mathcal{M}_X$

(Let $\mathcal{Q}$ be the class of cells. Then $\mathcal{M} \equiv \mathcal{E} * \mathcal{E}[\mathcal{Q}]$.)

- a cell in $\mathcal{Q}_X$ is a proper nested set system (X,Δ) with $X \in \Delta$.

Consider $(X, \Delta \setminus \{X\})$, which is a nested set system, and $\#X \geq 2$.
We have (note $\mathcal{E}_0 \equiv (\emptyset, \emptyset)$ and $\mathcal{E}_1 \equiv (\{1\}, \emptyset)$ are proper)
$$\mathcal{Q} \equiv \mathcal{M} \setminus (\mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \mathcal{Q})$$
$$(X,\Delta) \mapsto (X, \Delta \setminus \{X\}).$$