9 Integer Partitions

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Def An (integer) partition is a finite sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0.$$

The entries of λ are called the parts of λ.
The set of all partitions is denoted $\mathcal{Y}$.
Define

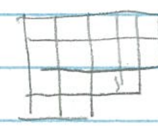
$$n(\lambda) := \lambda_1 + \cdots + \lambda_k \qquad \text{is called the size of } \lambda,$$
$$k(\lambda) := k \qquad \text{is called the length of } \lambda.$$

The Ferrer's diagram (or Young diagram) of λ, denoted F_λ, has λ_i boxes in row i, left justified.

$$d(\lambda) := \#\{i ; \lambda_i \geq i\} \quad \text{is the number of boxes on the main diagonal of } F_\lambda.$$

ex If $\lambda = 5542$ then

$$n(\lambda) = 16, \quad k(\lambda) = 4, \quad d(\lambda) = 3$$



ex The empty partition is denoted ε. We have $n(\varepsilon) = k(\varepsilon) = d(\varepsilon) = 0$.

Let $p(n)$ denote the number of partitions of size n.

ex $p(4) = 5 \quad (1111, 211, 31, 22, 4)$

Theorem:
$$p(n) = [x^n] \prod_{j=1}^{\infty} \frac{1}{1-x^j}.$$

Equivalently,
$$\Phi^n_{\mathcal{Y}}(x) = \prod_{j=1}^{\infty} \frac{1}{1-x^j}.$$

Before we can prove this, we need to discuss the infinite product.

Intuition:
$$\prod_{j=1}^{\infty} (1 + x^j + x^{2j} + \cdots)$$

can choose a certain number of parts of size j, $\forall j$

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Def (FPS Limit) Let $A_1(x), A_2(x), \ldots$ be a sequence of FPS (in $R[[x]]$).
We say that
$$\lim_{k\to\infty} A_k(x) = \sum_{n=0}^{\infty} a_n x^n$$
if there is a function $\varphi: \mathbb{N} \to \mathbb{N}$ such that
$$a_n = [x^n] A_k(x) \quad \forall k \geq \varphi(n).$$

ex $$\lim_{k\to\infty} \frac{1}{1-x^k} = 1 \qquad \text{via } \varphi(n) = n+1$$

ex $$\lim_{k\to\infty} \frac{1}{1-\frac{x}{k}} \quad \text{DNE}$$

Def (Infinite Product) We say that
$$\prod_{j=1}^{\infty} A_j(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (*)$$
if
$$\lim_{k\to\infty} \prod_{j=1}^{k} A_j(x) = \sum_{n=0}^{\infty} a_n x^n.$$
In other words, $(*)$ means that there is a function $\varphi: \mathbb{N} \to \mathbb{N}$ st
$$a_n = [x^n] \prod_{j=1}^{k} A_j(x) \qquad \forall k \geq \varphi(n).$$
Informally, an infinite product makes sense if we can use a finite product to compute any coefficient we want.

ex $$\prod_{j=1}^{k} \left(1 + x^{2^{j-1}}\right) = 1 + x + \cdots + x^{2^k - 1}$$
so $$\prod_{j=1}^{\infty} \left(1 + x^{2^{j-1}}\right) = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

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Let $\mathcal{M}$ be the set of all infinite sequences
$$\rho = \langle r_1, r_2, r_3, \dots \rangle,$$
where $r_i \in \mathbb{N}$, and only finitely many are non-zero.

ex $\rho = \langle 5,3,1,0,2,17,4,0,0,0,0,0,0,0,0,0,\dots \rangle$

Convention: Write $\rho = \langle r_1, r_2, r_3, \dots, r_\ell \rangle$ if $r_{\ell+1} = r_{\ell+2} = \dots = 0$.

ex $\rho = \langle 5,3,1,0,2,17,4 \rangle$

Let $\ell(\rho) = \begin{cases} \max\{j;\ r_j \neq 0\} & \text{if } \rho \neq \langle 0 \rangle \\ 0 & \text{if } \rho = \langle 0 \rangle \end{cases}$

For a partition $\lambda \in \mathcal{Y}$, let
$$m(\lambda) = \langle m_1(\lambda), m_2(\lambda), \dots \rangle$$
where
$$m_j(\lambda) = \#\{i;\ \lambda_i = j\}.$$

ex $\lambda = 644222221$, $m(\lambda) = \langle 1,5,0,2,0,1 \rangle$.

Proposition: The function m defines a bijection $\mathcal{Y} \rightleftarrows \mathcal{M}$. Furthermore, if $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathcal{Y}$ corresponds to $\rho = \langle r_1, \dots, r_\ell \rangle$ then
(a) $k(\lambda) = r_1 + \dots + r_\ell = r_1 + r_2 + r_3 + \dots$
(b) $n(\lambda) = r_1 + 2r_2 + \dots + \ell r_\ell = r_1 + 2r_2 + 3r_3 + \dots$
(c) $\lambda_1 = \ell(\rho)$
(d) $d(\lambda) = \max\{j;\ j \le r_j + r_{j+1} + \dots + r_\ell\}$.

Proof: See notes. $\square$

Proof of Theorem:
$$\Phi^n_{\mathcal{Y}}(x) = \sum_{\lambda \in \mathcal{Y}} x^{n(\lambda)} = \sum_{\rho \in \mathcal{M}} x^{r_1 + 2r_2 + \dots} \overset{(*)}{=} \Big(\sum_{r_1 \in \mathbb{N}} x^{r_1}\Big)\Big(\sum_{r_2 \in \mathbb{N}} x^{2r_2}\Big) \cdots$$
$$= \Big(\frac{1}{1-x}\Big)\Big(\frac{1}{1-x^2}\Big)\Big(\cdots \Big) = \prod_{j=1}^{\infty} \frac{1}{1-x^j}.$$

We must justify the equality ⊛, using the definition of infinite product. Specifically, we need to show that if $k \geq n$,

$$[x^n]\sum_{\rho\in\mathcal{M}} x^{r_1+2r_2+\cdots} = [x^n]\left(\sum_{r_1\in\mathbb{N}} x^{r_1}\right)\left(\sum_{r_2\in\mathbb{N}} x^{2r_2}\right)\cdots\left(\sum_{r_k\in\mathbb{N}} x^{kr_k}\right).$$

Well,

$$RHS = \#\{(r_1,\ldots,r_k)\in\mathbb{N}^k ;\ r_1+\cdots+kr_k = n\}$$

$$LHS = \#\{\rho\in\mathcal{M} ;\ r_1+2r_2+\cdots = n\}$$

$$= \#\{\langle r_1,\ldots,r_k\rangle\in\mathcal{M} ;\ r_1+2r_2+\cdots+kr_k = n\} = RHS$$

because if $r_1+2r_2+\cdots = n$ then $k \geq n \geq \lambda_1 = \ell(e) \Rightarrow \rho = \langle r_1,\ldots,r_k\rangle$. ∎

ex $\Phi^{(n,k)}_{\mathcal{Y}}(x,y) = \prod_{j=1}^{\infty} \frac{1}{1-x^j y}$

Proof:
$$\Phi^{(n,k)}_{\mathcal{Y}}(x,y) = \sum_{\lambda\in\mathcal{Y}} x^{n(\lambda)} y^{k(\lambda)}$$

$$= \sum_{\rho\in\mathcal{M}} x^{r_1+2r_2+\cdots} y^{r_1+r_2+\cdots}$$

$$= \left(\sum_{r_1\in\mathbb{N}} x^{r_1}y^{r_1}\right)\left(\sum_{r_2\in\mathbb{N}} x^{2r_2}y^{r_2}\right)\cdots$$

$$= \left(\frac{1}{1-xy}\right)\left(\frac{1}{1-x^2y}\right)\cdots$$ ∎

ex A partition λ is said to ~~be~~ have <u>distinct parts</u> (also called <u>strict</u>) if $m_j(\lambda)\in\{0,1\}$ $\forall j\in\mathbb{N}$.

ex 7542 has *distinct parts* but 6642 does not

Let $\mathcal{D}\subseteq\mathcal{Y}$ denote the set of partitions with distinct parts.
Let $\mathcal{M}'\subseteq\mathcal{M}$ be the set of sequences $\rho=\langle r_1,r_2,\ldots\rangle\in\mathcal{M}$ such that $r_j\in\{0,1\}$ $\forall j\in\mathbb{N}$.
By definition, $\lambda\in\mathcal{D} \Leftrightarrow m(\lambda)\in\mathcal{M}'$.

$$\Phi_{\mathcal{D}}^{(n,k)}(x,y) = \sum_{\lambda\in\mathcal{D}} x^{n(\lambda)} y^{k(\lambda)}$$
$$= \sum_{\rho\in\mathcal{M}'} x^{r_1+2r_2+\cdots}\, y^{r_1+r_2+\cdots}$$
$$= \left(\sum_{r_1\in\{0,1\}} x^{r_1}y^{r_1}\right)\left(\sum_{r_2\in\{0,1\}} x^{2r_2}y^{r_2}\right)\cdots$$
$$= (1+xy)(1+x^2y)(1+x^3y)\cdots$$
$$= \prod_{j=1}^{\infty}(1+x^jy).$$
□

Notation: If $\varphi(\lambda)$ is a statement about a partition λ, then let $\mathcal{Y}_{\varphi(\lambda)}$ denote the set of all partitions for which $\varphi(\lambda)$ is true.

ex $\mathcal{Y}_{\lambda_1\le\ell}$ is the set of partitions λ such that $\lambda_1\le\ell$ (ie first part at most ℓ)

$$\Phi^{(n,k)}_{\mathcal{Y}_{\lambda_1\le\ell}}(x,y) = \prod_{j=1}^{\ell}\left(\frac{1}{1-x^jy}\right)$$
$$\Phi^{(n,k)}_{\mathcal{Y}_{\lambda_1=\ell}}(x,y) = \left(\prod_{j=1}^{\ell-1}\frac{1}{1-x^jy}\right)\left(\frac{x^\ell y}{1-x^\ell y}\right)$$
$$\Phi^{(n,k)}_{\mathcal{D}_{\lambda_1\le\ell}}(x,y) = \prod_{j=1}^{\ell}(1+x^jy)$$
$$\Phi^{(n,k)}_{\mathcal{D}_{\lambda_1=\ell}}(x,y) = \left(\prod_{j=1}^{\ell}(1+x^jy)\right)x^\ell y$$

see 9.12 in course notes

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Partitions with a fixed number of parts.

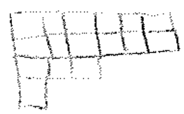
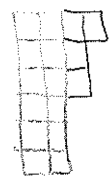
How many partitions on n with k parts?

Method 1: Last time,
$$\Phi^{(n,k)}_{\mathcal{Y}}(x,y) = \prod_{j=1}^{\infty}\frac{1}{1-x^jy},$$
so answer is
$$[x^ny^k]\prod_{j=1}^{\infty}\frac{1}{1-x^jy}.$$

Method 2: We need to compute

$$[x^n]\,\Phi^n_{\mathcal{Y}_{k(\lambda)=k}}(x).$$

For $\lambda \in \mathcal{Y}$ the conjugate of λ, denoted $\hat{\lambda}$ is the partition whose Ferrers diagram is obtained by reflecting F_λ in the main diagonal.

ex $\lambda = 6631$ $F_\lambda =$  $F_{\hat{\lambda}} =$  $\hat{\lambda} =$ 433222

Note: $n(\lambda) = n(\hat{\lambda})$

$k(\lambda) = \hat{\lambda}_1$

$\lambda_1 = k(\hat{\lambda})$

$d(\lambda) = d(\hat{\lambda})$

Conjugation gives a bijection

$$\mathcal{Y}_{k(\lambda)=k} \rightleftarrows \mathcal{Y}_{\lambda_1=k}$$

which is weight preserving for "size". Therefore

$$\Phi^n_{\mathcal{Y}_{k(\lambda)=k}}(x) = \Phi^n_{\mathcal{Y}_{\lambda_1=k}}(x)$$

last time

$$\overset{\downarrow}{=} x^k \prod_{j=1}^{k} \frac{1}{1-x^j}$$

Answer:

$$[x^n]\, x^k \prod_{j=1}^{k} \frac{1}{1-x^j}$$

Since we have two methods of answering this question we get the identity

$$[x^n y^k] \prod_{j=1}^{\infty} \frac{1}{1-x^j y} = [x^n]\, x^k \prod_{j=1}^{k} \frac{1}{1-x^k}$$

$$\iff \prod_{j=1}^{\infty} \frac{1}{1-x^j y} = \sum_{k=0}^{\infty} \left(x^k \prod_{j=1}^{k} \frac{1}{1-x^k} \right) y^k$$

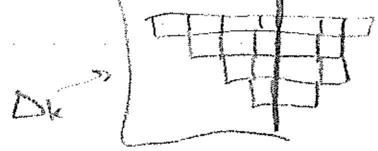
How many partitions on n with k distinct parts?

Method 1:

$$[x^n y^k]\, \Phi_{\mathcal{D}}^{(n,k)}(x,y) \overset{\text{last time}}{=} [x^n y^k] \prod_{j=1}^{\infty} (1+x^j y)$$

Method 2: For $\lambda \in \mathcal{D}$ the shifted diagram of λ is obtained by shifting the i^{th} row of F_λ ~~by~~ $(i-1)$ boxes to the right.

ex $\lambda = 7542$



If $k(\lambda)=k$, there are $\frac{1}{2}k(k+1)$ boxes in the first k columns of the shifted diagram. Call this shape Δ_k. The remaining columns form the Ferrer diagram of a partition μ with $k(\mu) \le k$.
Therefore

$$\mathcal{D}_{k(\lambda)=k} \rightleftharpoons \{\Delta_k\} \times \mathcal{Y}_{k(\lambda)\le k}$$

with $n(\lambda) = n(\Delta_k) + n(\mu)$. Therefore

$$\begin{aligned}\Phi^n_{\mathcal{D}_{k(\lambda)=k}}(x) &= x^{\frac{k(k+1)}{2}}\, \Phi^n_{\mathcal{Y}_{k(\lambda)\le k}}(x) \\ &= x^{\frac{k(k+1)}{2}}\, \Phi^n_{\mathcal{Y}_{\lambda_1 \le k}} \\ &= x^{\frac{k(k+1)}{2}} \prod_{j=1}^{k} \frac{1}{1-x^j \cancel{y}}\end{aligned}$$

Answer:

$$[x^n]\, x^{\frac{k(k+1)}{2}} \prod_{j=1}^{k} \frac{1}{1-x^j \cancel{y}}$$

Combining two methods:

$$\prod_{j=1}^{\infty}(1+x^j y) = \sum_{k=0}^{\infty} x^{\frac{k(k+1)}{2}} \left(\prod_{j=1}^{k} \frac{1}{1-x^j \cancel{y}} \right) y^k .$$

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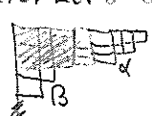
Euler Identities

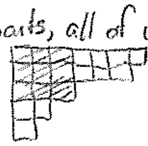
Theorem:
$$\prod_{j=1}^{\infty}\frac{1}{1-x^jy}=\sum_{d=0}^{\infty}x^{d^2}y^d\left(\prod_{i=1}^{d}\frac{1}{1-x^iy}\right)\left(\prod_{k=1}^{d}\frac{1}{1-x^k}\right).$$
Proof: The left-hand side is $\Phi_{\mathcal{Y}}^{(n,k)}(x,y)$. We must find a bijection. There is a $d\times d$ square ($d=d(\lambda)$) in the upper left corner of F_λ, called the <u>Durfee square</u>.
To the right is a partition α with at most d parts.
Below is a partition β with all parts at most d.
This gives a bijection
$$\mathcal{Y}\rightleftarrows\bigcup_{d=0}^{\infty}\{d\}\times\mathcal{Y}_{k(\lambda)\le d}\times\mathcal{Y}_{\lambda_1\le d},$$
$$\lambda\longmapsto(d,\alpha,\beta).$$

Note
$$n(\lambda)=d^2+n(\alpha)+n(\beta),$$
$$k(\lambda)=d+0+k(\beta).$$
Therefore
$$\Phi_{\mathcal{Y}}^{(n,k)}(x,y)=\sum_{d=0}^{\infty}x^{d^2}y^d\,\Phi^{n}_{k(\lambda)\le d}(x)\,\Phi^{(n,k)}_{\lambda_1\le d}(x,y).\ \blacksquare$$

Theorem:
$$\prod_{j=1}^{\infty}(1+x^jy)=1+\sum_{d=1}^{\infty}x^{d^2}y^d\left(\left(\prod_{i=1}^{d}(1+x^iy)\right)\left(x^{\frac{d(d+1)}{2}}\prod_{k=1}^{d}\frac{1}{1-x^k}\right)+\left(\prod_{i=1}^{d-1}(1+x^iy)\right)\left(x^{\frac{d(d-1)}{2}}\prod_{k=1}^{d-1}\frac{1}{1-x^k}\right)\right).$$
Proof: The LHS is $\Phi_{\mathcal{D}}^{(n,k)}(x,y)$. For the RHS, there are two cases. Again we draw F_λ. 2014 11 12
Look at the Durfee square. Let $d=d(\lambda)$.
Case 1: $\lambda_d>d$.



In this case, α is a partition with exactly d distinct parts, β is a partition with distinct parts, all of which are at most d.
Case 2: $\lambda_d=d$.



In this case, α has exactly $d-1$ parts and β has distinct parts, all of which are at most $d-1$.

Hence we have a bijection

$$\mathcal{D} \rightleftarrows \{\varepsilon\} \cup \bigcup_{d=1}^{\infty} \left(\{d\} \times \mathcal{D}_{k(\lambda)=d} \times \mathcal{D}_{\lambda_1 \le d} \cup \{d\} \times \mathcal{D}_{k(\lambda)=d-1} \times \mathcal{D}_{\lambda_1 \le d-1} \right).$$

Continue as before. ∎

Something exciting happens if we substitute $y=-1$. We get

$$\begin{aligned}
\prod_{j=1}^{\infty} (1-x^j) &= 1 + \sum_{d=1}^{\infty} (-1)^d x^{d^2} \left(x^{\frac{d(d+1)}{2}} + x^{\frac{(d-1)d}{2}} \right) \\
&= 1 + \sum_{d=1}^{\infty} (-1)^d x^{d^2 + \frac{d(d+1)}{2}} + \sum_{d=1}^{\infty} (-1)^d x^{d^2 + \frac{(d-1)d}{2}} \\
&= 1 + \sum_{d=1}^{\infty} (-1)^d x^{(3d^2+d)/2} + \sum_{d=1}^{\infty} (-1)^d x^{(3d^2-d)/2} \\
&= \sum_{d=0}^{\infty} x^{(3d^2+d)/2} (-1)^d + \sum_{h=-\infty}^{-1} (-1)^h x^{(3h^2+h)/2} \\
&= \sum_{h=-\infty}^{\infty} (-1)^h x^{(3h^2+h)/2}
\end{aligned}$$

Euler's Pentagonal Number Theorem:

$$\prod_{j=1}^{\infty} (1-x^j) = \sum_{h=-\infty}^{\infty} (-1)^h x^{(3h^2-h)/2}$$

The numbers $\frac{3h^2-h}{2}$ are called the pentagonal numbers.

triangular numbers | square numbers | pentagonal numbers

$\frac{n^2+n}{2}$ | n^2 | $\frac{3n^2-n}{2}$

This calculation is happening in $\mathbb{Q}[y][[x]]$, which is why we can substitute $y=-1$.

Jacobi Triple Product Formula:

$$\sum_{h=-\infty}^{\infty} x^{h^2} y^h = \prod_{j=1}^{\infty} (1+x^{2j-1}y)(1+x^{2j-1}y^{-1})(1-x^{2j}) \quad \text{in } \mathbb{Q}(y)[[x]].$$

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ex For $n \in \mathbb{N}$, the number of solutions to $a^2+b^2=n$, $a,b \in \mathbb{Z}$

$$[x^n]\sum_{(a,b)\in\mathbb{Z}^2} x^{a^2+b^2} = [x^n]\left(\sum_{a\in\mathbb{Z}} x^{a^2}\right)^2$$

take $y=1$ in JTPF

$$= [x^n]\left(\prod_{j=1}^{\infty}(1+x^{2j-1})^2(1-x^{2j})\right)^2$$

$$= [x^n]\prod_{j=1}^{\infty}(1+x^{2j-1})^4(1-x^{2j})^2$$