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These are inspired by Taylor series for functions but are technically different.
Doesn't make sense to compare FPS and functions directly.
But they satisfy the same identities.

ex $\exp(x+y) = \exp(x)\exp(y)$
Verification:

$$\exp(x+y) = \sum_{n\geq 0}^{\infty} \frac{(x+y)^n}{n!}$$

$$= \sum_{n\geq 0} \sum_{k=0}^{n} \frac{1}{n!} \binom{n}{k} x^k y^{n-k}$$

$$= \sum_{n\geq 0} \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{l=0}^{\infty} \frac{y^l}{l!} = \exp(x)\exp(y)$$

Many others, we could prove them all using methods like this.
But won't.

Chapter 2?

Lagrange's Implicit Function Theorem

Let $\mathbb{K}$ be a commutative ring containing $\mathbb{Q}$, or an integral domain containing $\mathbb{Z}$. Let $G(u) \in \mathbb{K}[[u]]$.

(a) There is a unique FPS $W(x) \in \mathbb{K}[[x]]$ such that

$$W(x) = xG(W(x)).$$

(b) $[x^0]W(x) = 0$,

$$[x^n]W(x) = \frac{[u^{n-1}]G(u)^n}{n} \quad \text{for } n \geq 1.$$

ex In the first BRT example, we had A satisfying the equation

$$A = x(1+A)^2.$$

Notice that if $G(u) = (1+u)^2$, then A is the solution to

$$A = xG(A).$$

Therefore we can apply LIFT.

Let's redo: What's the average number of terminals among all BRTs with n nodes.

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Recall

$$A - xy = x(1+A)^2 - x$$

and we want

$$[x^n] \frac{\partial}{\partial y} A(x,y)\Big|_{y=1}.$$

Note

$$A = x((1+A)^2 - 1 + y).$$

This is in the right form for using LIFT in the ring $\mathbb{K} = \mathbb{Q}[y]$, with $G(u) = (1+u)^2 - 1 + y$. By LIFT,

$$[x^n] A(x,y) = \frac{1}{n} [u^{n-1}] G(u)^n$$
$$= \frac{1}{n} [u^{n-1}] ((1+u)^2 - 1 + y)^n.$$

So

$$[x^n] \frac{\partial}{\partial y} A(x,y) = \frac{1}{n} [u^{n-1}] \frac{\partial}{\partial y} ((1+u)^2 - 1 + y)^n$$

$$= \cancel{\frac{1}{n}} [u^{n-1}] \cancel{n} ((1+u)^2 - 1 + y)^{n-1}$$

$$= [u^{n-1}] ((1+u)^2 - 1 + y)^{n-1}.$$

So

$$[x^n] \frac{\partial}{\partial y} A(x,y)\Big|_{y=1} = [u^{n-1}] ((1+u)^2 - 1 + y)^{n-1}\Big|_{y=1}$$

$$= [u^{n-1}] (1+u)^{2(n-1)}$$

$$= \binom{2n-2}{n-1}.$$

<u>One more example.</u> Compute the number of BRTs with n nodes such that there are exactly k nodes without a left child.

Let $\mathcal{T}$ be the set of all BRTs, $n(T)$ be the number of nodes in T, and $\alpha(T)$ be the number of nodes without a left child, and

$$\omega : \mathcal{T} \to \mathbb{N}^2$$
$$\omega(T) = (n(T), \alpha(T))$$

We showed

$$\mathcal{T} \rightleftarrows \{\odot\} \times (\mathcal{T} \cup \{\emptyset\}) \times (\mathcal{T} \cup \{\emptyset\}).$$

If $T \longleftrightarrow (\odot, L, R)$ then

$$n(T) = 1 + n(L) + n(R),$$

$$\alpha(T) = \begin{cases} \alpha(L) + \alpha(R) & \text{if } L \neq \emptyset, \\ 1 + \alpha(R) & \text{if } L = \emptyset. \end{cases}$$

Rewrite the bijection

$$\mathcal{T} \rightleftarrows \left(\{\odot\} \times \mathcal{T} \times (\mathcal{T} \cup \{\emptyset\})\right) \cup \left(\{\odot\} \times \{\emptyset\} \times (\mathcal{T} \cup \{\emptyset\})\right).$$

Define weight of $\odot$ differently for the two sets. For the first,

$$\mu_1(\odot) = (1, 0),$$

and for the second,

$$\mu_2(\odot) = (1, 1).$$

Then we have a weight preserving bijection, so

$$\Phi^{\omega}_{\mathcal{T}}(x,y) = \Phi^{\mu_1}_{\{\odot\}}(x,y)\, \Phi^{\omega}_{\mathcal{T}}(x,y) \left(\Phi^{\omega}_{\mathcal{T}}(x,y) + 1\right) + \Phi^{\mu_2}_{\{\odot\}}(x,y) \cdot 1 \cdot \left(\Phi^{\omega}_{\mathcal{T}}(x,y) + 1\right).$$

Let

$$A = A(x,y) = \Phi^{\omega}_{\mathcal{T}}(x,y).$$

Then

$$A = xA(A+1) + xy(A+1)$$
$$= x\left((A+y)(A+1)\right).$$

This is in the form for LIFT, with $\mathbb{K} = \mathbb{Q}[y]$, $G(u) = (u+y)(u+1)$.

Therefore

$$[x^n]A(x,y) = \frac{1}{n}[u^{n-1}]G(u)^n$$

$$= \frac{1}{n}[u^{n-1}]\left((u+y)(u+1)\right)^n.$$

We want $[x^n y^k] A(x,y)$, which is

$$\frac{1}{n}[u^{n-1}y^k](u+y)^n(u+1)^n = \frac{1}{n}[u^{n-1}]\binom{n}{k}u^{n-k}(u+1)^n$$

$$= \frac{1}{n}[u^{k-1}]\binom{n}{k}(u+1)^n$$

$$= \frac{1}{n}\binom{n}{k}\binom{n}{k-1} \quad (\text{for } n \geq 1)$$

Formal Laurent Series (FLS)

For a commutative ring R,
$$R((x)) = \{a_r x^r + a_{r-1} x^{r-1} + \cdots ; r \in \mathbb{Z}, a_r, a_{r+1}, \ldots \in R\}.$$

ex $\frac{1}{x} + 1 + x + x^2 + x^3 + x^4 + \cdots \in \mathbb{Z}((x))$.

Note that there are only finitely many negative powers of x.
Every FLS can be written in the form
$$\frac{f(x)}{x^r}$$
where $f(x) \in R[[x]]$, $r \in \mathbb{Z}$.
$$\frac{f(x)}{x^r} + \frac{g(x)}{x^s} = \frac{x^s f(x) + x^r g(x)}{x^{r+s}}$$
$$\frac{f(x)}{x^r}\frac{g(x)}{x^s} = \frac{f(x)g(x)}{x^{r+s}}$$
Define $I(f)$ to be the smallest number n such that $[x^n]f(x) \neq 0$.

ex $I(\frac{1}{x} + 1 + x + \cdots) = -1$.

A FLS $f(x) \in R((x))$ is invertible if and only if $[x^{I(f)}]f(x)$ is invertible in R.
Note that
$$\frac{f(x)}{x^{I(f)}}$$
is a FPS, with $[x^0]\uparrow = [x^{I(f)}]f(x)$. Not hard to see why from here.

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Formal Derivatives and Integrals
$$f(x) = \sum_{n \geq I(f)} a_n x^n \in R((x))$$
Assume $\mathbb{Q} \subseteq R$.

1. Formal derivative
$$f'(x) = \sum_{n \geq I(f)} n a_n x^n = \sum_{m \geq I(f)-1} (m+1) a_{m+1} x^m$$

I don't want to re-explain that. If you didn't understand, please ignore every word I just said

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2. Formal Integral

assume $[x^{-1}]f(x)=0$:

$$\int f(x)\,dx = \sum_{n\geq I(f),\, n\neq -1} \frac{a_n}{n+1} x^{n+1}$$

In particular, if $f(x)\in R[[x]]$, $\int f(x)dx$ is defined.

3. Formal Residue

$$[x^{-1}]f(x)\in R$$

Properties: All usual rules for derivatives:

- sum rule
- product rule
- quotient rule
- chain rule

All usual rules for integrals:

- fundamental theorem of calculus
$$\int f'(x)dx = f(x)+c$$
- integration by parts
$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx + c$$
- change of variables
$$\int f(g(x))g'(x)dx = F(g(x)) + c$$
where $F(x) = \int f(x)dx$.

Formal Residue also behaves like an integral:

- FTC: $[x^{-1}]f'(x)=0$

Proof: $$[x^{-1}]f'(x) = [x^{-1}]\sum_{n\geq I(f)} n a_n x^{n-1} = 0\cdot a_0 = 0. \qquad \square$$

- integration by parts
$$[x^{-1}]f(x)g'(x) = -[x^{-1}]f'(x)g(x)$$

Proof: By FTC, $[x^{-1}]\frac{d}{dx}(f(x)g(x))=0$.
$[x^{-1}](f'(x)g(x)+f(x)g'(x))=0$, expand. $\square$

- change of variables

If $f(y)\in R((y))$, $g\in R[[x]]$ with $[x^0]g(x)=0$ and $[x^{I(g)}]g(x)$ invertible, then

$$[x^{-1}]f(g(x))g'(x) = I(g)[y^{-1}]f(y)$$

Proof: Write $f(y) = a_{-1}y^{-1} + h(y)$, where $[y^{-1}]h(y) = 0$, $a_{-1} = [y^{-1}]f(y)$.
Let $H(y) = \int h(y)\,dy$. Then $H'(y) = h(y)$, $f(y) = a_{-1}y^{-1} + H'(y)$.
RHS $= I(g)[y^{-1}]f'(y) = I(g)a_{-1}$.

$$\begin{aligned}
\text{LHS} &= [x^{-1}]f(g(x))g'(x) \\
&= [x^{-1}](a_{-1}g(x)^{-1} + H'(g(x)))g'(x) \\
&= [x^{-1}]a_{-1}g(x)^{-1}g'(x) + [x^{-1}]H'(g(x))g'(x) \\
&= a_{-1}[x^{-1}]\frac{g'(x)}{g(x)} + [x^{-1}]\cancel{\frac{d}{dx}(H(g(x)))} \to 0
\end{aligned}$$

Write $g(x) = x^{I(g)}k(x)$. So

$$\begin{aligned}
&= a_{-1}[x^{-1}]\frac{I(g)x^{I(g)-1}k(x) + x^{I(g)}k'(x)}{x^{I(g)}k(x)} \\
&= a_{-1}[x^{-1}]\left(I(g)x^{-1} + \frac{k'(x)}{k(x)}\right) \quad \text{FPS} \to 0 \\
&= a_{-1}I(g).
\end{aligned}$$

$\square$

Compositional Inverse

Lemma: Let $A(x), B(x) \in R[[x]]$, where R is an integral domain. Suppose $[x^0]A(x) = [x^0]B(x) = 0$. If $A(B(x)) = 0$ then either $A(x) = 0$ or $B(x) = 0$.
Proof: Exercise.

Theorem: Let $f(x) \in R[[x]]$, $g(u) \in R[[u]]$. Suppose $[x^0]f(x) = 0$, so that $g(f(x))$ is defined. If $g(f(x)) = x$ then the following are true:
(i) $[u^0]g(u) = 0$, $[u^1]g(u)[x^1]f(x) = 1$
(ii) $f(g(u)) = u$
(iii) By (i), (ii), $G(u) = \frac{u}{g(u)} \in R[[u]]$ is defined. Then $f(x)$ is the solution to the LIFT equation $f = xG(f)$.
If $f(g(x)) = x$, we say that f is the <u>compositional inverse of g</u>.
Proof: (i) For any FPS, $[x^0]A(x) = A(0)$, $[x^1]A'(x) = A'(0)$. So
$g(f(0)) = 0 \Rightarrow g(0) = 0 \Rightarrow [u^0]g(u) = 0$
$\frac{d}{dx}g(f(x))$

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(ii): Let $A(u) = f(g(u)) - u$. Then

$$A(f(x)) = f(g(f(x))) - f(x) = f(x) - f(x) = 0.$$

Since $f(x) \neq 0$ (as $[x]f(x)$ is invertible by (i)), $A(u) = 0$.

So $f(g(u)) - u = 0$

(iii) $RHS = xG(f(x)) = x \frac{f(x)}{g(f(x))} = x \frac{f(x)}{x} = f(x) = LHS.$

Corollary: If $G(u)$ is invertible then the LIFT equation

$$W = xG(W)$$

is equivalent to

$$W\left(\frac{u}{G(u)}\right) = u.$$

Indeed, (iii) says the first is the compositional inverse of $u/G(u)$.

<u>General Statement of LIFT:</u>

- $\mathbb{K}$ a commutative ring containing $\mathbb{Q}$ or an integral domain containing $\mathbb{Z}$

$G(u) \in \mathbb{K}[[u]]$

(i) There is a unique FPS $W(x) \in \mathbb{K}[[x]]$ such that

$$W(x) = x\,G(W(x)).$$

(ii) $[x^0]W(x) = 0$,

$$[x^n]W(x) = \frac{1}{n}[u^{n-1}]G(u)^n, \quad n \geq 1$$

(iii) For any $F(u) \in \mathbb{K}[[u]]$, $[x^0]F(W(x)) = F(0)$,

$$[x^n]F(W(x)) = \frac{1}{n}[u^{n-1}]F'(u)\,G(u)^n$$

Note: (ii) is a special case of (iii) where $F(u) = u$.

In the proof I will make one additional assumption $G(u)$ is invertible.

The theorem is still true even if $G(u)$ is not invertible.

Proof: Write

$$G(u) = \sum_{n=0}^{\infty} a_n u^n, \quad W(x) = \sum_{n \geq 0} b_n x^n.$$

$W(x) = xG(W(x)) \Rightarrow b_0 = W(0) = 0\,G(W(0)) = 0.$

Now compute coeffs on both sides:

$$b_n = [x^n]W(x) = [x^n]\,x\,G(W(x))$$

$$= [x^{n-1}]\,G(W(x))$$

$$= \sum_{k=1}^{n-1} \sum_{\substack{j_1,\dots,j_k \geq 1 \\ j_1+\dots+j_k = n-1}} a_k b_{j_1} \cdots b_{j_k}$$

↑ involves only $a_1, \dots, a_{n-1}$
$b_1, \dots, b_{n-1}$

Thus we can recursively compute $b_0, b_1, b_2, \dots \Rightarrow W(x)$ exists and is unique.

(iii) Saw $W(0)=0$, so $F(W(0))=0$ so $[x^0]F(W(x))=0$.

For $n \geq 1$ we calculate

$$[x^n]F(W(x)) = [x^{-1}]\,x^{-n-1}F(W(x))$$

Let $f(x) = x^{-n-1}F(W(x))$, $g(u) = u/G(u)$, $I(g)=1$, $W(u/G(u))=1$.

$$= I(g)\,[x^{-1}]f(x)$$

change of vars $\rightarrow$ $= [u^{-1}]\,f(g(u))\,g'(u)$

$$= [u^{-1}]\,g(u)^{-n-1}F(W(u/G(u)))\,g'(u)$$

$$= [u^{-1}]\,g(u)^{-n-1}F(u)\,g'(u).$$

$$= [u^{-1}]\,F(u)\cdot \tfrac{1}{-n}\tfrac{d}{du}\,g(u)^{-n}$$

$$= -[u^{-1}]\,F'(u)\cdot\tfrac{1}{-n}\,g(u)^{-n}$$

$$= \tfrac{1}{n}[u^{-1}]\,F'(u)\,(u/G(u))^{-n}$$

$$= \frac{1}{n}[u^{-1}]\,u^{-n}F'(u)G(u)^n$$

$$= \frac{1}{n}[u^{n-1}]\,F'(u)G(u)^n. \quad \blacksquare$$

ex Let $\mathcal{P}$ be the set of all PPTs. For $T \in \mathcal{P}$ let $\omega(T) = (n(T), d(T))$ where $d(T)$ is the degree of the root. We use the bijection,

$$\mathcal{P} \rightleftarrows \{\odot\} \times \mathcal{P}^*$$

$$T \leftrightarrow (\odot, C_1, \dots, C_k).$$

Then

$$n(T) = n(\odot) + \dots + n(C_k)$$

$$d(T) = k$$

Define the weight function $\nu : \mathcal{P} \to \mathbb{N}^2$ by

$$\nu(T) = (n(T), 1)$$

and $\nu(\odot) = (1, 0)$

Then we have a weight preserving bijection where ω ~~the~~ is used for $\mathcal{P}$ on the LHS, and ν is used for $\mathcal{P}$ on the RHS

ie $\omega(T) = \mu(0) + \nu(C_1) + \cdots + \nu(C_k)$.

Therefore

$$\Phi_{\mathcal{P}}^{\omega}(x,y) = \Phi_{\{0\}}^{\mu}(x,y)\,\Phi_{\mathcal{P}^*}^{\nu^*}(x,y)$$

$$= x \frac{1}{1-\Phi_{\mathcal{P}}^{\nu}(x,y)}$$

But

$$\Phi_{\mathcal{P}}^{\nu}(x,y) = \sum_{T\in\mathcal{P}} x^{n(T)} y^1$$

$$= y \sum_{T\in\mathcal{P}} x^{n(T)} = y\,\Phi_{\mathcal{P}}^{n}(x).$$

We know that if $A(x) = \Phi_{\mathcal{P}}^{n}(x)$ then

$$A = \frac{x}{1-A}.$$

And therefore

$$\Phi_{\mathcal{P}}^{\omega}(x,y) = x\frac{1}{1-yA}.$$

To solve this, let $\mathbb{K} = \mathbb{Q}[y]$ and let

$$G(u) = \frac{1}{1-u}$$

and let

$$F(u) = \frac{1}{1-yu}.$$

Then

$$[x^n]\,\Phi_{\mathcal{P}}^{\omega}(x,y) = [x^n]\,x F(A(x))$$

Note $A = xG(A)$. So

$$= [x^{n-1}]\,F(A(x))$$

$$= \frac{1}{n-1}[u^{n-2}]\,F'(u)\,G(u)^{n-1}$$

$$= \frac{1}{n-1}[u^{n-2}]\,y\left(\frac{1}{1-yu}\right)^2\left(\frac{1}{1-u}\right)^{n-1} \quad \text{etc.}$$