7 Formal Power Series

Commutative rings: Set R
Operations $+, -, \cdot$
special elements $0, 1$
satisfying the usual properties

ex $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_n, \mathcal{F}(\mathbb{R}, \mathbb{R})$

non-ex $\mathbb{N}$, vector spaces

Inverses: If $ab = 1$, we say a and b are <u>invertible</u>, b is the <u>inverse</u> of a, written $a^{-1} = b$.

Zero divisors: If $a, b \neq 0$ and $ab = 0$, we say that a and b are <u>zero-divisors</u>.

A commutative ring with no zero-divisors is called an <u>integral domain</u>.

ex $\mathbb{Z}$ is an i.d., $\mathbb{Z}_{15}$ is not because $3 \cdot 5 = 0$
$\mathcal{F}(\mathbb{R}, \mathbb{R})$ is not, because $(x \mapsto \frac{1}{2}(|x| + x)) \cdot (x \mapsto \frac{1}{2}(|x| - x)) = 0$
$\mathbb{Z}_n$ is an integral domain if and only if n is a prime (or $n = 1$) (or $n = 0$)

A field is a ring in which every non-zero element is invertible.

ex $\mathbb{R}, \mathbb{Q}, \mathbb{C}$

Every field is an integral domain (easy).

Integral domain is what we need to make sense of division. If R is an integral domain we write, for $b \neq 0$,

$$\frac{a}{b} = c \quad \text{when} \quad bc = a$$

This behaves in the expected ways. This will go wrong if R isn't an i.d.

Constructions:

Let R be a commutative ring.

1. The polynomial ring

$$R[x] = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n ; n \in \mathbb{N}, a_0, \ldots, a_n \in R\}$$

Here x is an indeterminate.

$+, \cdot, -$ defined as you would expect

$R[x]$ is always a commutative ring.

$R[x]$ is an integral domain if R is an integral domain.

$R[x]$ is never a field.

- constants ← represents a definite value
- unknowns
- variables ← represents a range of definite values
- indeterminates ← pretending to be a variable; does not represent anything but can be manipulated according to same rules as variables

ex $\mathbb{Q}[x][y]$

2. The ring of rational functions

$$R(x) = \left\{ \frac{f(x)}{g(x)} ; f(x), g(x) \in R[x], g \neq 0, g \text{ not a zero divisor} \right\}$$

with usual $+, -, \cdot$ (and simplification)

If R is an integral domain then $R(x)$ is a field. The inverse of

$$\frac{f(x)}{g(x)} \text{ is } \frac{g(x)}{f(x)}$$

3. The ring of formal power series (FPS)

$$R[[x]] = \{ f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots ; a_n \in R \text{ for } n \in \mathbb{N}\}$$

$$f(x) = \sum_{n \geq 0} a_n x^n \qquad g(x) = \sum_{n \geq 0} b_n x^n$$

$$f(x) + g(x) = \sum_{n \geq 0} (a_n + b_n) x^n$$

$$f(x)g(x) = \left(\sum_{i \geq 0} a_i x^i\right)\left(\sum_{j \geq 0} b_j x^j\right) = \sum_{n \geq 0} \left(\sum_{i=0}^{n} a_i b_{n-i}\right) x^n$$

special elements $0 = 0 + 0x + 0x^2 + 0x^3 + \cdots$

$1 = 1 + 0x + 0x^2 + 0x^3 + \cdots$

$R[[x]]$ is a commutative ring

$R[[x]]$ is an integral domain if R is an integral domain. (Exercise.)

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What about inverses?

Suppose $f(x)g(x)=1$

Means

$$\sum_{n\geq 0}\left(\sum_{i=0}^{n} a_i b_{n-i}\right)x^n = 1 + 0x + 0x^2 + \cdots$$

Comparing coefficients.

$f(x)$ is invertible if and only if a_0 is invertible

$R[[x]]$ is never a field.

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Combining constructions

- $\mathbb{Z}[x,y] = \mathbb{Z}[x][y] = \mathbb{Z}[y][x]$
  if S is a finite set, $\omega: S\to\mathbb{N}^2$ a weight function,
  then $\Phi_S^\omega(x,y)\in\mathbb{Z}[x,y]$

- $\mathbb{Z}[[x,y]] = \mathbb{Z}[[x]][[y]] = \mathbb{Z}[[y]][[x]]$
  if S is a set and $\omega: S\to\mathbb{N}^2$ is a weight function,
  then $\Phi_S^\omega(x,y)\in\mathbb{Z}[[x,y]]$.

- $\mathbb{Z}[y][[x]]$
  if S is a set, $\omega: S\to\mathbb{N}^2$ is a weight function and $\omega(\sigma)=(\alpha(\sigma),\beta(\sigma))$
  where $\alpha: S\to\mathbb{N}$ is also a weight function,
  then $\Phi_S^\omega(x,y)\in\mathbb{Z}[y][[x]]$

$$\Phi_S^\omega(x,y) = \sum_{\sigma\in S} x^{\alpha(\sigma)}y^{\beta(\sigma)} = \sum_{n\geq 0}\sum_{\sigma\in\alpha^{-1}(n)} x^{\alpha(\sigma)}y^{\beta(\sigma)}$$

$$= \sum_{n\geq 0}\underbrace{\left(\sum_{\sigma\in\alpha^{-1}(n)} y^{\beta(\sigma)}\right)}_{\in\mathbb{Z}[y]} x^n$$

Note: $\mathbb{Z}[[x]][y] \neq \mathbb{Z}[y][[x]]$. In fact, "$\subset$" holds.
less interesting

Composition (aka Substitution)

$f(x), g(x) \in R[x] \cup R(x) \cup R[[x]]$, different cases depending on what $f(x)$ is:

- if $f(x) \in R[x]$, then $f(g(x))$ is always defined
- if $f(x) \in R(x)$, then $f(g(x))$ defined if division makes sense
- if $f(x) \in R[[x]]$, ~~then~~ need $g(x) \in R[[x]]$ and $[x^0]g(x) = 0$

Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad , \quad g(x) = \sum_{n=1}^{\infty} b_n x^n$$

$$f(g(x)) = \sum_{n=0}^{\infty} a_n g(x)^n = a_0 + \sum_{n=1}^{\infty} a_n \left(\sum_{j_1=1}^{\infty} b_{j_1} x^{j_1} \right) \cdots \left(\sum_{j_n=1}^{\infty} b_{j_n} x^{j_n} \right)$$

$$= a_0 + \sum_{n=1}^{\infty} \left(\sum_{j_1, \dots, j_n \geq 1} a_n b_{j_1} \cdots b_{j_n} x^{j_1 + \cdots + j_n} \right)$$

$$= a_0 + \sum_{k=1}^{\infty} \left(\sum_{\substack{j_1, \dots, j_n \geq 1 \\ j_1 + \cdots + j_n = k}} a_n b_{j_1} \cdots b_{j_n} \right) x^k$$

definition of $f(g(x))$ — finite sum

In particular, we can't evaluate FPS at numbers other than 0. But if $f(x)$ is a polynomial we can plug in any number for x.

In particular, we can plug in numbers for y, but not x, in the ring $\mathbb{Z}[y][[x]]$.

This is why we can substitute $y=1$ in the calculation we did.

All identities you learned about Taylor series apply to FPS. Special FPS:

$$\exp(x) = \sum_{n \geq 0} \frac{1}{n!} x^n \in \mathbb{Q}[[x]]$$

$$\log\left(\frac{1}{1-x}\right) = \sum_{n \geq 1} \frac{1}{n} x^n \in \mathbb{Q}[[x]]$$

$$(1+x)^y = \sum_{n \geq 0} \binom{y}{n} x^n \in \mathbb{Q}[y][[x]] \quad , \quad \binom{y}{n} = \frac{y(y-1)\cdots(y-(n-1))}{n!}$$

Hence we can substitute $y = \alpha \in \mathbb{C}$ in $(1+x)^y$.

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These are inspired by Taylor series for functions but are technically different.
Doesn't make sense to compare FPS and functions directly.
But they satisfy the same identities.

ex. $\exp(x+y) = \exp(x)\exp(y)$
Verification:

$$\exp(x+y) = \sum_{n\geq 0}^{\infty} \frac{(x+y)^n}{n!}$$

$$= \sum_{n\geq 0} \sum_{k=0}^{n} \frac{1}{n!} \binom{n}{k} x^k y^{n-k}$$

$$= \sum_{n\geq 0} \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{l=0}^{\infty} \frac{y^l}{l!} = \exp(x)\exp(y)$$

Many others, we could prove them all using methods like this.
But won't.

Chapter 2?

Lagrange's Implicit Function Theorem

Let $\mathbb{K}$ be a commutative ring containing $\mathbb{Q}$, or an integral domain containing $\mathbb{Z}$. Let $G(u) \in \mathbb{K}[[u]]$.

(a) There is a unique FPS $W(x) \in \mathbb{K}[[x]]$ such that

$$W(x) = xG(W(x)).$$

(b) $[x^0]W(x) = 0$,

$$[x^n]W(x) = \frac{[u^{n-1}]G(u)^n}{n} \quad \text{for } n \geq 1.$$

ex. In the first BRT example, we had A satisfying the equation

$$A = x(1+A)^2.$$

Notice that if $G(u) = (1+u)^2$, then A is the solution to

$$A = xG(A).$$

Therefore we can apply LIFT.