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Binomial Theorem

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \quad \forall \alpha \in \mathbb{C}\ \forall x \in \mathbb{C},\ |x|<1.$$

We can also use this if x is an indeterminate (more on this later).
Two special cases that come up in chapter 6 are $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$:

$$\sqrt{1-4x} = 1-2\sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^{n+1}$$

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$$

Proof: See notes.

6 Recursive Structures

Def] A binary rooted tree (BRT) is a tree with a designated node ⊙ called the root, drawn in the plane, such that each node has at most two children, one to the left and one to the right. If a node only has one child, it is still designated either left or right.

ex.  or 

(smallest possible)

How many BRTs are there with n nodes?
Solution: Let $\mathcal{T}$ be the set of all BRTs. Let $n: \mathcal{T} \to \mathbb{N}$ be the weight function $n(T) = \#$ of nodes of T.
Given a BRT $T \in \mathcal{T}$, we can ~~obtain~~ remove the root node to ~~get a~~ obtain a pair of subtrees (L, R), which are either empty or BRTs themselves.

ex. 

L is rooted at the left child of ~~the~~ ⊙ if it exists, otherwise $L = \emptyset$.
Similarly with R. The construction gives a bijection
$$\mathcal{T} \rightleftarrows \{⊙\} \times (\mathcal{T} \cup \{\emptyset\}) \times (\mathcal{T} \cup \{\emptyset\}).$$

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If

$$T \leftrightarrow (\odot, L, R)$$

then

$$n(T) = n(\odot) + n(L) + n(R)$$

Therefore, by the product lemma,

$$(*) \quad \Phi^n_{\mathcal{T}}(x) = \Phi^n_{\{\odot\}}(x)\, \Phi^n_{\mathcal{T}\cup\{\emptyset\}}(x)\, \Phi^n_{\mathcal{T}\cup\{\emptyset\}}(x)$$

Let

$$A(x) = \Phi^n_{\mathcal{T}}(x).$$

Then

$$\Phi^n_{\{\odot\}}(x) = x$$

$$\Phi^n_{\mathcal{T}\cup\{\emptyset\}}(x) = \Phi^n_{\mathcal{T}}(x) + \Phi^n_{\{\emptyset\}}(x) = A(x) + 1.$$

Thus $(*)$ becomes

$$A(x) = x\,(A(x)+1)^2$$

or just

$$A = x(A+1)^2.$$

Expanding yields

$$xA^2 + (2x-1)A + x = 0.$$

By the quadratic formula,

$$A = \frac{(1-2x) \pm \sqrt{(2x-1)^2 - 4x^2}}{2x} = \frac{1}{2x} - 1 \pm \frac{1}{2x}\sqrt{1-4x}$$

$$= \frac{1}{2x} - 1 \pm \frac{1}{2x}\left(1 - 2\sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^{n+1}\right).$$

Since we need the $\frac{1}{2x}$ to cancel, we have to choose the "$-$" solution. Thus

$$A(x) = \frac{1}{2x} - 1 - \frac{1}{2x}\left(1 - 2\sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}x^{n+1}\right)$$

$$= -1 + \frac{1}{\cancel{2x}}\cancel{2}\sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}x^{n\cancel{+1}}$$

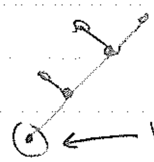
$$= \sum_{n\geq 1}\frac{1}{n+1}\binom{2n}{n}x^n.$$

For $n \geq 1$, the number of BRTs with n nodes is

$$[x^n]A(x) = \frac{1}{n+1}\binom{2n}{n}.$$

A <u>terminal</u> in a BRT is a node with no children.
A terminal is not quite the same as a leaf.

ex



$\odot \leftarrow$ leaf, but not terminal $\qquad\qquad \odot \leftarrow$ terminal, not a leaf

What is the average number of terminals among all BRTs with n nodes.
Set of objects will be $\mathcal{T}$. For $T \in \mathcal{T}$, let $\tau(T)$ be the number of terminals in T. Define the weight function

$$\omega : \mathcal{T} \to \mathbb{N}^2,$$
$$\omega(T) = (n(T), \tau(T)).$$

We have the same bijection

$$\mathcal{T} \rightleftarrows \{\odot\} \times (\mathcal{T} \cup \{\emptyset\}) \times (\mathcal{T} \cup \{\emptyset\}),$$
$$T \mapsto (\odot, L, R).$$

What is the relationship between the two sides for weight functions?

$$n(T) = n(\odot) + n(L) + n(R)$$
$$\tau(T) = \begin{cases} 0 + \tau(L) + \tau(R) & \text{if } T \neq \odot \\ 1 & \text{if } T = \odot \end{cases}$$

Define

$$\mu : \{\odot\} \to \mathbb{N}^2$$
$$\mu(\odot) = (1, 0)$$

Then

$$\omega(T) = \mu(\odot) + \omega(L) + \omega(R),$$

unless $T = \odot$. Therefore we have a weight preserving bijection

$$\mathcal{T} \setminus \{\odot\} \rightleftarrows \{\odot\} \times (\mathcal{T} \cup \{\emptyset\}) \times (\mathcal{T} \cup \{\emptyset\}) \setminus \{(\odot, \emptyset, \emptyset)\}.$$

Thus

$$\Phi^{\omega}_{\mathcal{T}}(x,y) - \Phi^{\omega}_{\{\odot\}}(x,y) = \Phi^{\mu}_{\{\odot\}}(x,y)\, \Phi^{\omega}_{\mathcal{T} \cup \{\emptyset\}}(x,y)^2 - \Phi^{\mu+\omega+\omega}_{\{(\odot,\emptyset,\emptyset)\}}(x,y)$$

Let

$$A = A(x,y) = \Phi^{\omega}_{\mathcal{T}}(x,y)$$

Then
$$A - xy = x(A+1)^2 - x.$$
So
$$xA^2 + (2x-1)A + xy = 0$$
and
$$A(x,y) = \frac{1-2x \pm \sqrt{(1-2x)^2 - 4x^2y}}{2x}.$$
We want the average number of terminals among all BRTs with n nodes:
$$\frac{\sum_{T \in \mathcal{T};\, n(T)=n} \tau(T)}{\#\{T \in \mathcal{T};\, n(T) = n\}} \longleftarrow = \frac{1}{n+1}\binom{2n}{n}$$
Note that
$$[x^n] \frac{\partial}{\partial y} A(x,y)\Big|_{y=1} = \sum_{T \in \mathcal{T};\, n(T) = n} \tau(T),$$
as on A3.

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If we plug in $y=1$, we get $\Phi_{\mathcal{T}}^{n}(x)$. So we must have the "$-$" solution.

Well,
$$\frac{\partial}{\partial y} A(x,y) = -\frac{1}{2x} \cdot \frac{1}{2}\left((1-2x)^2 - 4x^2y\right)^{-1/2} \cdot (-4x^2)$$
$$\frac{\partial}{\partial y} A(x,y)\Big|_{y=1} = -\frac{1}{2x} \cdot \frac{1}{2}\left((1-2x)^2 - 4x^2\right)^{-1/2} \cdot (-4x^2) = \frac{x}{\sqrt{1-4x}}$$
$$= x\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \sum_{n=0}^{\infty} \binom{2n}{n} x^{n+1} = \sum_{m=1}^{\infty} \binom{2m-2}{m-1} x^m.$$
So
$$[x^n] \frac{\partial}{\partial y} A(x,y)\Big|_{y=1} = \binom{2n-2}{n-1}.$$
The answer is
$$\frac{\binom{2n-2}{n-1}}{\frac{1}{n+1}\binom{2n}{n}} = \cdots = \frac{n(n+1)}{4n-2}$$

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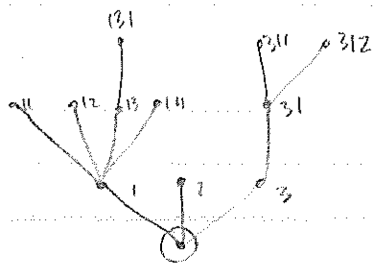
ex Plane Planted Trees

<u>Def</u>] A plane planted tree (PPT) is a tree with a designated node $\odot$, called the root, drawn in the plane such that each node has its children ordered from left to right. A node can have any number of children.

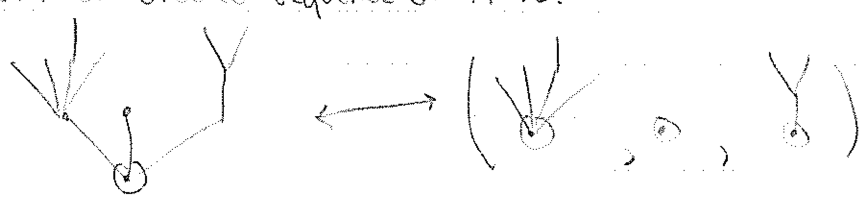
ex



How many PPTs are there with n nodes?

Solution Let $\mathcal{U}$ be the set of all PPTs. Let $n : \mathcal{U} \to \mathbb{N}$ be such that $n(T)$ is the number of nodes of T, for any $T \in \mathcal{U}$.

To get a recursive decomposition of $\mathcal{U}$, remove the root, and we're left with an ordered sequence of PPTs.



In other words, we have a bijection

$$\mathcal{U} \rightleftarrows \{\odot\} \times \mathcal{U}^*,$$
$$T \leftrightarrow (\odot, T_1, \dots, T_k).$$

Note that

$$n(T) = n(\odot) + n(T_1) + \cdots + n(T_k).$$

Therefore

$$\Phi^n_{\mathcal{U}}(x) = \Phi^n_{\{\odot\}}(x) \cdot \Phi^{n^*}_{\mathcal{U}^*}(x) = x \frac{1}{1 - \Phi^n_{\mathcal{U}}(x)}.$$

Let $A = A(x) = \Phi^n_{\mathcal{U}}(x)$. Then $A^2 - A + x = 0$, so

$$A(x) = \frac{1 \pm \sqrt{1-4x}}{2} = \frac{1}{2} \pm \frac{1}{2}\left(1 - 2\sum_{n \geq 0} \frac{1}{n+1}\binom{2n}{n} x^{n+1}\right).$$

Since $[x^0]A(x) = 0$ (there are no PPTs with 0 nodes) so we must have the "$-$" solution be correct. Therefore

$$A(x) = \frac{1}{2} - \frac{1}{2}\left(1 - 2\sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^{n+1}\right)$$
$$= \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^{n+1}.$$

Thus the number of PPTs with $n+1$ nodes is

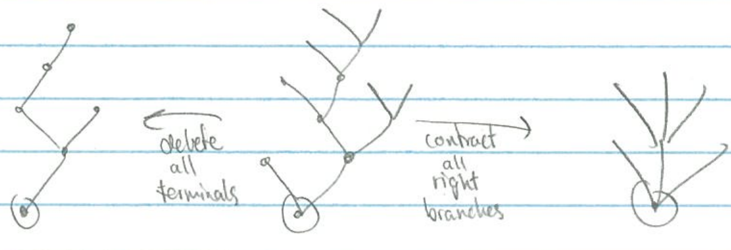
$$\frac{1}{n+1}\binom{2n}{n}.$$

These numbers are called the <u>Catalan numbers</u> and they come up in many different combinatorial settings.

Exercise: A full binary tree (FBT) is a BRT in which every node has 0 or 2 children.
Show that if $\mathbb{B}$ is the set of all FBTs then we have
$$\mathbb{B} \rightleftarrows \{\odot\} \cup (\{\odot\} \times \mathbb{B} \times \mathbb{B}).$$
Show that if $\tau(T)$ is the number of terminals of T, then
$$\Phi_{\mathbb{B}}^{\tau}(x) = x + \Phi_{\mathbb{B}}^{\tau}(x)^2.$$
Show that the number of FBTs with $n+1$ terminals is
$$\frac{1}{n+1}\binom{2n}{n}.$$

What is going on?

$$\begin{array}{c}\#\text{ of BRTs}\\ \text{with } n \text{ nodes}\end{array} = \begin{array}{c}\#\text{ of FBTs}\\ \text{with } n+1 \text{ terminals}\end{array} = \begin{array}{c}\#\text{PPTs with}\\ n+1 \text{ nodes}\end{array}$$



Dyck paths, or Catalan paths, are lattice paths in $\mathcal{L}(n,n)$ that don't go below the line $y = x$. The number of Dyck paths of length $2n$ is
$$\frac{1}{n+1}\binom{2n}{n}.$$

Let $\mathbb{D}$ be the set of all Dyck paths. Then (let ε be the empty path),
$$\{\varepsilon\} \cup \mathbb{D}^2 \rightleftarrows \mathbb{D} \rightleftarrows (\mathbb{D}\setminus\{\varepsilon\})^* \qquad \text{(see notes)}$$