ex Compositions again²

The set of all compositions is $(\mathbb{Z}_{\geq 1})^*$

Define weight function $\omega : \mathbb{Z}_{\geq 1} \to \mathbb{N}^2$ by
$$\omega(c) = (c, 1).$$
Then
$$\omega^*((c_1, \dots, c_k)) = (c_1 + \cdots + c_k, k).$$
By finite string lemma,
$$\Phi^{\omega^*}_{(\mathbb{Z}_{\geq 1})^*}(x,y) = \frac{1}{1 - \Phi^{\omega}_{\mathbb{Z}_{\geq 1}}(x,y)} \overset{\text{see last time}}{=} \frac{1}{1 - \left(\frac{xy}{1-x}\right)} = \frac{1-x}{1-x-xy}$$
Therefore number of comp. of n with k parts is
$$[x^n y^k] \frac{1}{1 - \frac{xy}{1-x}}.$$

5 q-binomial Theorem

Recall that $\mathcal{P}(X)$ is the set of all subsets of X.

Binomial Theorem: Let $\omega : \mathcal{P}(N_n) \to \mathbb{N}$ be the weight function
$$\omega(S) = \# S.$$
We compute $\Phi^{\omega}_{\mathcal{P}(N_n)}(x)$ in two ways.

Method 1: Since
$$[x^k]\Phi^{\omega}_{\mathcal{P}(N_n)}(x) = \# \omega^{-1}(k) = \# \mathcal{B}(n,k) = \binom{n}{k}$$
for all $k \in \{0, \dots, n\}$,
$$\Phi^{\omega}_{\mathcal{P}(N_n)}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k.$$
Method 2: We have a bijection $f : \mathcal{P}(N_n) \to \{0,1\}^n$ defined by
$$f(S) = (a_1, \dots, a_n),$$
where
$$a_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$
Define a weight function $\eta : \{0,1\}^n \to \mathbb{N}$ by $\eta((a_1, \dots, a_n)) = a_1 + \cdots + a_n$. Then f is a weight preserving bijection. If $S \in \mathcal{P}(N_n)$ then $\omega(S) = \# S$,

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$$\nu(f(S)) = a_1 + \cdots + a_n$$

where

$$a_i := \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

so

$$\nu(f(S)) = \sum_{i \in S} 1 + \sum_{i \notin S} 0 = \#S.$$

Therefore

$$\Phi^{\omega}_{\mathcal{P}(N_n)}(x) = \Phi^{\nu}_{\{0,1\}^n}(x) = (1+x)^n$$

by the product lemma (skipped steps here). Therefore

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

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q-Binomial Theorem:

$$\omega : \mathcal{P}(N_n) \to \mathbb{N}^2$$
$$\omega(S) = (\#S, \mathrm{sum}(S)).$$

Two methods for computing $\Phi^{\omega}_{\mathcal{P}(N_n)}(x,q)$.

- Direct Method
- Product Lemma

Start with method 2. Bijection $f : \mathcal{P}(N_n) \to \{0,1\}^n$

$$f(S) = (a_1, \ldots, a_n), \quad a_i := \begin{cases} 1 & i \in S, \\ 0 & i \notin S. \end{cases}$$

What weight function on $\{0,1\}^n$ would make f a weight preserving bijection?

Define $\nu : \{0,1\}^n \to \mathbb{N}^2$ by

$$\nu((a_1, \ldots, a_n)) = \omega(f^{-1}((a_1, \ldots, a_n)))$$
$$= \left(\textstyle\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} i a_i\right).$$

By construction of ν, f is weight preserving. Hence

$$\Phi^{\omega}_{\mathcal{P}(N_n)}(x,q) = \Phi^{\nu}_{\{0,1\}^n}(x,q).$$

To compute the RHS, define weight functions $\mu_i : \{0,1\} \to \mathbb{N}^2$,
$\mu_i(a) = (a, ia)$.

Then

$$\nu((a_1, \ldots, a_n)) = \mu_1(a_1) + \cdots + \mu_n(a_n).$$

Thus by the product lemma

$$\Phi^{\nu}_{\{0,1\}^n}(x,q) = \Phi^{\mu_1}_{\{0,1\}}(x,q)\cdot \cdots \cdot \Phi^{\mu_n}_{\{0,1\}}(x,q)$$
$$= (x^0q^0 + x^1q^1)\cdot \cdots \cdot (x^0q^0 + x^1q^n)$$
$$= (1+xq)\cdots(1+xq^n)$$
$$= \prod_{i=1}^{n}(1+xq^i).$$

To generalize method 1, observe that we can collect powers of x:

$$\Phi^{\omega}_{P(N_n)}(x,q) = \sum_{k=0}^{n} B_{n,k}(q)\, x^k,$$

where $B_{n,k}(q)$ is some polynomial in q. Since

$$\Phi^{\omega}_{P(N_n)}(x) = \sum_{A\in P(N_n)} q^{\mathrm{sum}(A)} x^{\#A}$$
$$= \sum_{k=0}^{n}\sum_{A\in B(n,k)} q^{\mathrm{sum}(A)} x^{\#A}$$
$$= \sum_{k=0}^{n}\left(\sum_{A\in B(n,k)} q^{\mathrm{sum}(A)}\right) x^{k}.$$

Therefore

$$B_{n,k}(q) = \sum_{A\in B(n,k)} q^{\mathrm{sum}(A)} = \Phi^{\mathrm{sum}}_{B(n,k)}(q).$$

Our goal is to compute RHS by generalizing results of chapter 2.
We start with S_n.

Def) Let $\sigma = a_1\cdots a_n \in S_n$ be a permutation of length n. An inversion of σ is a pair (i,j) where $1\le i<j\le n$ and $a_i > a_j$. Let $\mathrm{inv}(\sigma)$ denote the number of inversions of σ.

ex The inversions of $\sigma = 514263$ are $(1,2),(1,3),(1,4),(1,6),(3,4),(3,5),(5,6)$ so $\mathrm{inv}(\sigma) = 7$.

ex $\mathrm{inv}(1\cdots n) = 0$, $\mathrm{inv}(n\cdots 1) = \binom{n}{2}$ are the extremes

Theorem: $\Phi^{\mathrm{inv}}_{S_n}(q) = [n]!_q$

where $[k]_q = 1 + \cdots + q^{k-1}$ and $[n]!_q = [n]_q \cdots [1]_q$, which are called q-analogues of k and of $n!$ respectively. This "means" that if we put $q=1$ then $[k]_1 = k$ and $[n]!_1 = n!$.

Note: We showed that

$$B_{n,k}(q) = \sum_{A \in B(n,k)} q^{\mathrm{sum}(A)}.$$

$\Phi^{\mathrm{sum}}_{B(n,k)}(q)$

Putting $q=1$,

$$B_{n,k}(1) = \sum_{A \in B(n,k)} 1^{\mathrm{sum}(A)} = \# B(n,k) = \binom{n}{k}.$$

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Proof: Recall that we have a bijection

$$I_n : S_n \to N_n \times N_{n-1} \times \cdots \times N_1 =: C_n.$$

Suppose $I_n(\sigma) = (r_1+1, \ldots, r_n+1)$. I claim that $\mathrm{inv}(\sigma) = r_1 + \cdots + r_n$. This is true because r_k is the number of inversions of σ that are of the form $(i, -)$. Therefore we define $\mu_k : N_k \to \mathbb{N}$ by $\mu_k(c) = c-1$, and define $\upsilon : N_n \times \cdots \times N_1 \to \mathbb{N}$ by

$$\upsilon((c_1, \ldots, c_n)) = \mu_n(c_1) + \cdots + \mu_1(c_n) = (c_1 - 1) + \cdots + (c_n - 1).$$

Then I_n is a weight preserving bijection. Therefore

$$\Phi^{\mathrm{inv}}_{S_n}(q) = \Phi^{\upsilon}_{C_n}(q).$$

By the product lemma,

$$\Phi^{\upsilon}_{C_n}(q) = \Phi^{\mu_n}_{N_n}(q) \cdots \Phi^{\mu_1}_{1}(q) = [n]_q \cdots [1]_q = [n]!_q. \blacksquare$$

Theorem. $B_{n,k}(q) = q^{\frac{k(k+1)}{2}} \dfrac{[n]!_q}{[k]!_q [n-k]!_q}.$

We define

$$\binom{n}{k}_q := \frac{[n]!_q}{[k]!_q [n-k]!_q}.$$

Proof: Recall we have the bijection

$$\Psi : S_n \to \mathcal{B}(n,k) \times S_k \times S_{n-k}$$

Claim: If $\Psi_{n,k}(\sigma) = (A, \beta, \gamma)$, then

$$\mathrm{inv}(\sigma) = \left(\mathrm{sum}(A) - \frac{k(k+1)}{2}\right) + \mathrm{inv}(\beta) + \mathrm{inv}(\gamma).$$

To see this, split up the inversions of σ into:

$$E_1 = \{\text{inversions } (i,j) \text{ where } i<j \le k\},$$
$$E_2 = \{\quad \text{"} \quad k < i < j\},$$
$$E_3 = \{\quad \text{"} \quad i \le k < j\}.$$

Then $\mathrm{inv}(\sigma) = \#E_1 + \#E_2 + \#E_3$. Then $\#E_1 = \mathrm{inv}(\beta)$ and $\#E_2 = \mathrm{inv}(\gamma)$ (see notes for details).

Finally, we show $\#E_3 = \mathrm{sum}(A) - k(k+1)/2$. Write $\sigma = a_1 \cdots a_n$. Then $(i,j) \in E_3$ if and only if $(a_i, a_j) \in N_n \times N_n$ is a pair such that $a_i \in A$, $a_j \notin A$ and $a_i > a_j$.

Therefore $\#E_3 = \#\{(a,z) \in N_n \times N_n ; a \in A, z \notin A, a > z\}$.

Sort the elements of A as $s_1 < s_2 < \cdots < s_k$. For each i, there are i elements of A that are less than or equal to s_k. Therefore there are $s_i - i$ elements of $N_n \setminus A$ that are less than s_i. ~~Therefore the total number of~~

Therefore # taken over all $i \in \{1, \ldots, k\}$ is

$$(s_1 - 1) + \cdots + (s_k - k)$$
$$= \mathrm{sum}(A) - (1 + \cdots + k)$$
$$= \mathrm{sum}(A) + \tfrac{1}{2}k(k+1)$$

This tells us that

$$\Phi^{\mathrm{inv}}_{S_n}(q) = \left(\Phi^{\mathrm{sum}}_{\mathcal{B}(n,k)}(q)\, q^{-\frac{k(k+1)}{2}}\right) \cdot \left(\Phi^{\mathrm{inv}}_{S_k}(q)\right) \cdot \left(\Phi^{\mathrm{inv}}_{S_{n-k}}(q)\right).$$

Thus

$$[n]!_q = B_{n,k}(q) \cdot q^{-\frac{k(k+1)}{2}} [k]!_q [n-k]!_q. \quad \boxtimes$$

Recall: At the beginning of this chapter, we were trying to compute

$$\Phi^{(\#, \mathrm{sum})}_{P(N_n)}(x, q).$$

Method 2 gave $(1+xq) \cdot \ldots \cdot (1+xq^n)$.

Method 1 gave

$$\sum_{k=0}^{n} B_{n,k}(q)\, x^k.$$

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By what we just showed, we therefore get:

Theorem (q-binomial Theorem):

$$(1+xq)\cdot \dots \cdot (1+xq^n) = \sum_{k=0}^{n} q^{\frac{k(k+1)}{2}} \binom{n}{k}_q x^k.$$

Other interpretations of $\binom{n}{k}_q$ exist.

Theorem: Fix $(a,b) \in \mathbb{N}^2$. For $P \in \mathcal{L}(a,b)$, let $\text{area}(P)$ denote the area enclosed by P, the y-axis, and the line $y=b$. Then

$$\Phi^{\text{area}}_{\mathcal{L}(a,b)}(q) = \binom{a+b}{b}_q.$$

Proof: Homework

Theorem: Fix $0 \le k \le n$, and let $q = p^c$ be a prime power. Let $\mathbb{F}_q$ be the field with q elements. The number of k-dimensional subspaces of the vector space $\mathbb{F}_q^n$ over $\mathbb{F}_q$ is $\binom{n}{k}_q$.

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ex Prove the identity

$$\binom{2n}{n}_q = \sum_{k=0}^{n} q^{k^2} \binom{n}{k}_q^2.$$

Observe that if $q=1$ then this becomes

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Once upon a time, we found a bijection

$$\Phi : \bigcup_{k=0}^{j} \mathcal{L}(k,n-k) \times \mathcal{L}(n-k,k) \longrightarrow \mathcal{L}(n,n),$$

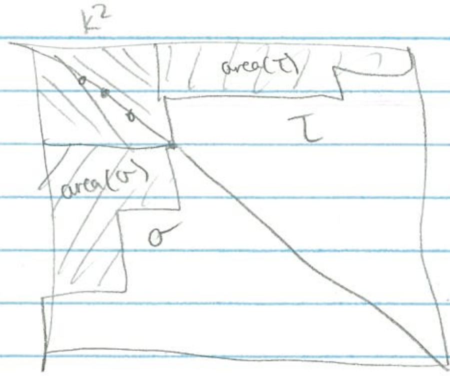
$$(\sigma,\tau) \longmapsto \sigma\tau$$

Small change:

$$\Phi : \bigcup_{k=0}^{j} \mathcal{L}(k,n-k) \times \mathcal{L}(n-k,k) \times \{k\} \longrightarrow \mathcal{L}(n,n)$$

$$(\sigma,\tau,k) \longmapsto \sigma\tau.$$

This is still a bijection. Compare both sides under the weight function area.



So $\operatorname{area}(\sigma\tau) = \operatorname{area}(\sigma) + \operatorname{area}(\tau) + k^2$. Therefore if we define

$$\omega((\sigma,\tau,k)) = \operatorname{area}(\sigma) + \operatorname{area}(\tau) + k^2$$

then Φ is a weight preserving bijection. Therefore

$$\binom{2n}{n}_q = \Phi^{\text{area}}_{\mathcal{L}(n,n)}(q) = \sum_{k=0}^{n} \Phi^{\omega}_{\mathcal{L}(k,n-k)\times\mathcal{L}(n-k,k)\times\{k\}}(q)$$

$$= \sum_{k=0}^{n} \binom{n}{k}_q \binom{n}{k}_q q^{k^2}.$$

Here we used the product lemma with the weight functions

$$\text{area}: \mathcal{L}(k,n-k) \to \mathbb{N},$$
$$\text{area}: \mathcal{L}(n-k,k) \to \mathbb{N},$$
$$k \mapsto k^2.$$

Method for solving enumeration problems with generating functions:

1. Identify the set S of objects under consideration (remove parameters from the problem.
   Define a weight function on S which reintroduces the parameters.
2. Find a bijection involving S or another description of S.
3. Define new weight functions if necessary and verify that they behave correctly:
   - Bijections are weight preserving
   - Weight of composite is sum of weights of components.
4. Use lemmas about generating functions to obtain a generating function equation involving Φs.
5. (Algebra) Solve this equation, extract coefficients, do whatever is necessary/possible to get answer from here.

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Binomial Theorem

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \quad \forall \alpha \in \mathbb{C}\ \forall x \in \mathbb{C},\ |x|<1.$$

We can also use this if x is an ~~indeterminate~~ (more on this later).
Two special cases that come up in chapter 6 are $\alpha \in \{-\frac{1}{2}, \frac{1}{2}\}$:

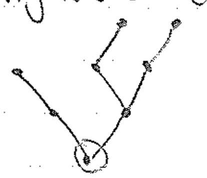
$$\sqrt{1-4x} = 1 - 2\sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^{n+1}$$

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$$

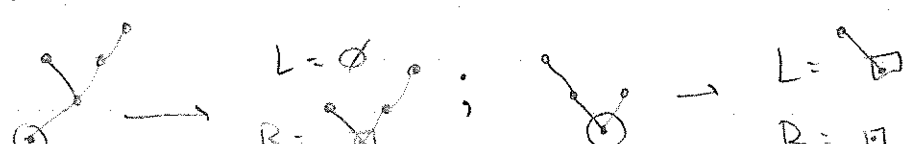
Proof: See notes.

6 Recursive Structures

Def] A binary rooted tree (BRT) is a tree with a designated node $\odot$ called the root, drawn in the plane, such that each node has at most two children, one to the left and one to the right. If a node only has one child, it is still designated either left or right.

ex  or $\odot$ (smallest possible)

How many BRTs are there with n nodes?
Solution: Let $\mathcal{T}$ be the set of all BRTs. Let $n: \mathcal{T} \to \mathbb{N}$ be the weight function $n(T) = \#$ of nodes of T.
Given a BRT $T \in \mathcal{T}$, we can ~~obtain~~ remove the root node to ~~get a~~ obtain a pair of subtrees (L, R), which are either empty or BRTs themselves.

ex. 

L is rooted at the left child of ~~the~~ $\odot$ if it exists, otherwise $L = \emptyset$.
Similarly with R. The construction gives a bijection

$$\mathcal{T} \rightleftarrows \{\odot\} \times (\mathcal{T} \cup \{\emptyset\}) \times (\mathcal{T} \cup \{\emptyset\}).$$