4 Generating Functions

You have 2 (ordinary 6-sided) dice. How many ways to roll a total of 9? Dice are distinguishable. (Imagine one is red, one is blue.)
Mathematically, a dice roll is represented by an element of N_6.

$$\text{two dice} \longleftrightarrow N_6 \times N_6.$$

Determine $\#\{(a,b) \in N_6 \times N_6 ; a+b=9\}$. To solve this, consider

$$\sum_{(a,b)\in N_6 \times N_6} x^{a+b}.$$

We get x^9 every time $a+b=9$. Answer is coefficient of x^9 in this expression.

$$\sum_{(a,b)\in N_6\times N_6} x^{a+b} = \sum_{a\in N_6}\sum_{b\in N_6} x^a x^b = \left(\sum_{a\in N_6} x^a\right)\left(\sum_{b\in N_6} x^b\right)$$

$$= (x+x^2+x^3+x^4+x^5+x^6)(x+x^2+x^3+x^4+x^5+x^6)$$

$$= \left(\frac{x-x^7}{1-x}\right)\left(\frac{x-x^7}{1-x}\right) = \left(\frac{x-x^7}{1-x}\right)^2.$$

ex A composition of n with k parts is a k-tuple $(c_1,\dots,c_k) \in (\mathbb{Z}_{\ge 1})^k$ where

$$c_1+\dots+c_k = n.$$

How many? Determine $\#\{(c_1,\dots,c_n)\in \mathbb{Z}_{\ge 1}^k ; c_1+\dots+c_k=n\}$. Consider

$$\sum_{(c_1,\dots,c_k)\in(\mathbb{Z}_{\ge1})^k} x^{c_1+\dots+c_k}$$

We get a x^n when $c_1+\dots+c_k=n$. Answer is $[x^n](\ )$. This equals

$$\sum_{c_1\in\mathbb{Z}_{\ge1}} \cdots \sum_{c_k\in\mathbb{Z}_{\ge1}} x^{c_1}\cdots x^{c_k} = \left(\sum_{c_1\in\mathbb{Z}_{\ge1}} x^{c_1}\right)\cdots\left(\sum_{c_k\in\mathbb{Z}_{\ge1}} x^{c_k}\right)$$

$$= (x+x^2+\dots)^k$$

$$= \left(\frac{1}{1-x}\right)^k.$$

From last time, consider

$$\sum_{(c_1,\dots,c_k)\in \mathbb{Z}_{\geq 1}^k} x^{c_1+\cdots+c_k}.$$

- $\mathbb{Z}_{\geq 1}^k$ is the set of objects
- $c_1+\cdots+c_k$ is the weight

This problem has two parameters, n and k. But they get treated very differently:
- n is the weight: disappears from the problem for a while, only shows up at the end when we extract $[x^n]$;
- k is part of the set of objects: present throughout the solution.

Makes it hard to get the solution, unless you already know how.

In this course, we consider vector valued weight functions.

MORE UNDERSTANDING
LESS THINKING

Theory: Let $\alpha=(\alpha_1,\dots,\alpha_r)\in\mathbb{N}^r$. If $x=(x_1,\dots,x_r)$ is a sequence of (pairwise commuting) indeterminates, we use the notation

$$x^\alpha := x_1^{\alpha_1}\cdots x_r^{\alpha_r}.$$

If $\alpha=(\alpha_1,\dots,\alpha_r)$, $\beta=(\beta_1,\dots,\beta_r)$,

$$\alpha+\beta=(\alpha_1+\beta_1,\dots,\alpha_r+\beta_r),$$
$$x^{\alpha+\beta}=x^\alpha x^\beta.$$

If $\alpha_i\leq\beta_i$ for all i, we write $\alpha\leq\beta$. Hence x^α divides $x^\beta \iff \alpha\leq\beta$.

<u>Def</u> If S is a set (may or may not be finite) and $\omega: S\to\mathbb{N}^r$ has the property that $\omega^{-1}(\alpha)$ is finite for all $\alpha\in\mathbb{N}^r$, we say ω is a <u>weight function</u>.

<u>Def</u> If S is a set and $\omega: S\to\mathbb{N}^r$ is a weight function, then the generating function for S with respect to ω is

$$\Phi_S^\omega(x)=\sum_{\sigma\in S} x^{\omega(\sigma)}$$

ex For the set $\mathbb{N}$, define the weight function $\omega:\mathbb{N}\to\mathbb{N}$, $\omega(n)=n$. Then

$$\Phi_{\mathbb{N}}^\omega(x)=\sum_{n\in\mathbb{N}} x^{\omega(n)}=\sum_{n\in\mathbb{N}} x^n = 1+x+x^2+\cdots = \frac{1}{1-x}.$$

Main Principle: one coefficient at a time, no limits.

ex Again for $\mathbb{N}$, let $\omega: \mathbb{N} \to \mathbb{N}$, $\omega(n) = 1$. Then ω is not a valid weight function, because $\omega^{-1}(1) = \mathbb{N}$ is not finite.

ex Combining the last two, $\omega: \mathbb{N} \to \mathbb{N}^2$, $\omega(n) = (n, 1)$. Then

$$\Phi^{\omega}_{\mathbb{N}}(x,y) = \sum_{n \in \mathbb{N}} x^n y^1 = y + xy + x^2 y + \cdots = \frac{y}{1-x}.$$

ex Let $S = N_6^3$ be the set of possible outcomes from rolling three dice. Define $\omega: S \to \mathbb{N}^3$, $(a,b,c) \mapsto (a,b,c)$. Then

$$\begin{aligned}
\Phi^{\omega}_{S}(x,y,z) &= \sum_{(a,b,c) \in N_6^3} x^a y^b z^c \\
&= \sum_{a \in N_6} \sum_{b \in N_6} \sum_{c \in N_6} x^a y^b z^c \\
&= \left(\sum_{a \in N_6} x^a\right)\left(\sum_{b \in N_6} y^b\right)\left(\sum_{c \in N_6} z^c\right) \\
&= D(x) D(y) D(z)
\end{aligned}$$

where $D(t) = t + t^2 + t^3 + t^4 + t^5 + t^6$.

Proposition: Let S be a set with weight function $\omega: S \to \mathbb{N}^r$. Then

$$\Phi^{\omega}_{S}(x) = \sum_{\alpha \in \mathbb{N}^r} \left(\# \omega^{-1}(\alpha)\right) x^{\alpha}.$$

In other words, the coefficient of x^α in the generating function is the answer to the question 'how many elements of S have weight α?'.

Proof:
$$\Phi^{\omega}_{S}(x) = \sum_{s \in S} x^{\omega(s)} = \sum_{\alpha \in \mathbb{N}^r} \sum_{s \in \omega^{-1}(\alpha)} x^{\omega(s)} = \sum_{\alpha \in \mathbb{N}^r} \sum_{s \in \omega^{-1}(\alpha)} x^{\alpha}$$
$$= \sum_{\alpha \in \mathbb{N}^r} x^{\alpha} \sum_{s \in \omega^{-1}(\alpha)} 1 = \sum_{\alpha \in \mathbb{N}^r} x^{\alpha} \left(\# \omega^{-1}(\alpha)\right). \quad \blacksquare$$

~~Midterm coverage ends here~~

Think of $\Phi_S^\omega(x)$ as a generalization of cardinality.

generating functions	sets
What is $\Phi_S^\omega(x)$?	What is $\#S$?
weight pres. bij. $\Leftrightarrow \Phi_S^\omega(x) = \Phi_T^\nu(x)$	$S \rightleftarrows T \Leftrightarrow \#S = \#T$
sum lemma	$\#(\cup_i S_i) = \sum_i \#S_i$ if $S_i \cap S_j = \emptyset$
product lemma	$\#S \times T = \#S \#T$

Def] Let S, T be sets with weight functions $\omega : S \to \mathbb{N}^r$ and $\nu : T \to \mathbb{N}^r$. A function $f : S \to T$ is called a <u>weight preserving bijection</u> if:

① f is a bijection;

② $\omega(s) = \nu(f(s))$ $\forall s \in S$

Proposition: Let S, T be sets with weight functions $\omega : S \to \mathbb{N}^r$ and $\nu : T \to \mathbb{N}^r$. There is a weight preserving bijection $f : S \to T$ if and only if $\Phi_S^\omega(x) = \Phi_T^\nu(x)$.

Proof: See notes

Sum lemma: Let S be a set with weight function $\omega : S \to \mathbb{N}^r$. If $S = S_1 \cup S_2 \cup \cdots$ is a decomposition of S as a union of finite or infinitely many pairwise disjoint sets, then

$$\Phi_S^\omega(x) = \sum_i \Phi_{S_i}^\omega(x).$$

Product lemma: Suppose S and T are sets with weight functions $\omega : S \to \mathbb{N}^r$ and $\nu : T \to \mathbb{N}^r$. Define a weight function $\varphi : S \times T \to \mathbb{N}^r$ by

$$\varphi((s,t)) = \omega(s) + \nu(t).$$

Then

$$\Phi_{S \times T}^\varphi(x) = \Phi_S^\omega(x)\, \Phi_T^\nu(x).$$

Note: Need to verify that φ is a weight function. (Exercise in chapter 4.)

ex. Compositions, again!

Let S be the set of all compositions. Define the weight function $\phi : S \to \mathbb{N}^2$ by

$$\phi((c_1, \ldots, c_k)) = (c_1 + \cdots + c_k, k).$$

Now

$$S = \coprod_{k\geq 0} (\mathbb{Z}_{\geq 1})^k = (\mathbb{Z}_{\geq 1})^0 \cup (\mathbb{Z}_{\geq 1}^1) \cup \cdots .$$

Mantra: The weight of a composite object must be the sum of the weights of its components.

Define a weight function of $\mathbb{Z}_{\geq 1}$

$$\omega : \mathbb{Z}_{\geq 1} \to \mathbb{N}^2,$$
$$\omega(c) = (c, 1).$$

Note that $\varphi((c_1, \ldots, c_k)) = \omega(c_1) + \cdots + \omega(c_k)$. By sum and product lemma,

$$\Phi_S^{\varphi}(x) = \sum_{k\geq 0} \left(\Phi_{\mathbb{Z}_{\geq 1}}^{\omega}(x)\right)^k$$
$$= \sum_{k\geq 0} \left(xy + x^2y + x^3y + \cdots\right)^k$$
$$= \sum_{k\geq 0} \left(\frac{xy}{1-x}\right)^k$$
$$= \frac{1}{1 - \frac{xy}{1-x}}.$$

So the number of compositions of n with k parts is

$$[x^n y^k] \frac{1}{1-\frac{xy}{1-x}} = \cdots \text{algebra}.$$

Def| Let S be a set with weight function $\omega : S \to \mathbb{N}^r$. The set of all finite strings on S is ~~the~~ defined as

$$S^* = \bigcup_{k=0}^{\infty} S^k.$$

If $\sigma \in S^*$ then $\sigma \in S^k$ for some unique k, called the length of σ.
Note $S^0 = \{\varepsilon\}$ where ε is the "empty string" of length 0.
We define

$$\omega^* : S^* \to \mathbb{N}^r,$$
$$\omega^*((s_1, \ldots, s_k)) = \omega(s_1) + \cdots + \omega(s_k).$$

Lemma: Let S be a set with weight function $\omega : S \to \mathbb{N}^r$. Then $\omega^* : S^* \to \mathbb{N}^r$ is a weight function if and only if $\omega^{-1}(0) = \emptyset$

Proof: ($\Rightarrow$): Suppose there exists an element $z \in \omega^{-1}(0)$. Consider the elements

$$z_k = \underbrace{(z, \dots, z)}_{k \text{ times}} \in S^*,$$

for all $k \in \mathbb{N}$. Then

$$\omega^*(z_k) = \underbrace{\omega(z) + \cdots + \omega(z)}_{k \text{ times}} = k\omega(z) = k \cdot 0 = 0.$$

So $z^k \in \omega^{*-1}(0)$ for all $k \in \mathbb{N}$. That is, $\omega^{*-1}(0)$ is infinite. Therefore ω^* is not a weight function

($\Leftarrow$): If $\beta = (\beta_1, \dots, \beta_r) \in \mathbb{N}^r$ write $|\beta| = \beta_1 + \cdots + \beta_r$.

Suppose $\omega(s) \neq 0$ $\forall s \in S$. Then $|\omega(s)| \geq 1$ $\forall s \in S$.

Suppose $\sigma = (s_1, \dots, s_k) \in S^*$. Then

$$\omega^*(\sigma) = \omega(s_1) + \cdots + \omega(s_k)$$

$$\therefore \quad |\omega^*(\sigma)| = |\omega(s_1)| + \cdots + |\omega(s_k)| \geq 1 + \cdots + 1 = k$$

That is to say, $|\omega^*(\sigma)| \geq \text{length}(\sigma)$, for any $\sigma \in S^*$.

Now if $\sigma \in \omega^{*-1}(\alpha)$ then we must have $\omega(s_i) \leq \alpha$ for all $i \in [k]$. That is,

$$s_i \in \bigcup_{\beta \leq \alpha} \overset{?}{\omega^{-1}}(\beta) =: U_\alpha. \leftarrow \text{finite set}$$

This U_α is a finite set because each set $\omega^{-1}(\beta)$ is finite and there are finitely many $\beta \leq \alpha$. We have shown that every string in $\omega^{*-1}(\alpha)$ has length at most $|\alpha|$ and each entry of the string belongs to U_α. In other words, every string in $\omega^{*-1}(\alpha)$ belongs to

$$\{\varepsilon\} \cup U_\alpha \cup U_\alpha^2 \cup \cdots \cup U_\alpha^{|\alpha|},$$

which is a finite set. Thus $\omega^{*-1}(\alpha)$ is a subset of a finite set which implies $\omega^{*-1}(\alpha)$ is a finite set. This proves ω^* is a weight function. ☐

Theorem [Finite String Lemma]: Let S be a set with weight function $\omega: S \to \mathbb{N}^r$ such that $\omega^{-1}(0) = \emptyset$. Then $\Phi^{\omega^*}_{S^*}(x) = (1 - \Phi^{\omega}_{S}(x))^{-1}$

Intuitive idea of above proof:

If $\sigma \in \omega^{-1}(\alpha)$ then length(σ) can't be too large, and there are only finitely many things with bounded length ex Compositions again^2

The set of all compositions is $(\mathbb{Z}_{\geq 1})^*$

Define weight function $\omega: \mathbb{Z}_{\geq 1} \to \mathbb{N}^2$ by

$$\omega(c) = (c, 1).$$

Then

$$\omega^*((c_1, \ldots, c_k)) = (c_1 + \cdots + c_k, k).$$

By finite string lemma,

see last time

$$\Phi^{\omega^*}_{(\mathbb{Z}_{\geq 1})^*}(x,y) = \frac{1}{1 - \Phi^{\omega}_{\mathbb{Z}_{\geq 1}}(x,y)} = \frac{1}{1 - \left(\frac{xy}{1-x}\right)} = \frac{1-x}{1-x-xy}$$

Therefore number of comp. of n with k parts is

$$[x^n y^k] \frac{1}{1 - \frac{xy}{1-x}}.$$

5 q-binomial Theorem

Recall that $\mathcal{P}(X)$ is the set of all subsets of X.

Binomial Theorem: Let $\omega: \mathcal{P}(N_n) \to \mathbb{N}$ be the weight function

$$\omega(S) = \#S.$$

We compute $\Phi^{\omega}_{\mathcal{P}(N_n)}(x)$ in two ways.

Method 1: Since

$$[x^k]\Phi^{\omega}_{\mathcal{P}(N_n)}(x) = \#\omega^{-1}(k) = \#\mathcal{B}(n,k) = \binom{n}{k}_q$$

for all $k \in \{0, \ldots, n\}$,

$$\Phi^{\omega}_{\mathcal{P}(N_n)}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Method 2: We have a bijection $f: \mathcal{P}(N_n) \to \{0,1\}^n$ defined by

$$f(S) = (a_1, \ldots, a_n),$$

where

$$a_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

Define a weight function $\eta: \{0,1\}^n \to \mathbb{N}$ by $\eta((a_1, \ldots, a_n)) = a_1 + \cdots + a_n$. Then f is a weight preserving bijection. If $S \in \mathcal{P}(N_n)$ then $\omega(S) = \#S$,