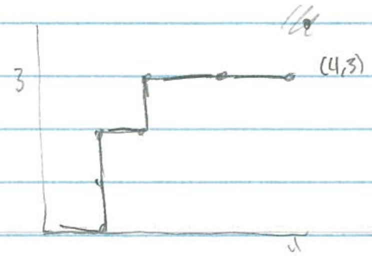
3 Lattice Paths and Polynomial Identities

For $a, b \in \mathbb{N}$, let $\mathcal{L}(a,b)$ be the set of lattice paths from $(0,0)$ to (a,b). These are paths that begin at $(0,0)$, end at (a,b), and consist of steps either east, $(x,y) \mapsto (x+1,y)$, or north $(x,y) \mapsto (x,y+1)$.

ex



$\in \mathcal{L}(4,3)$ Notation: ENNENEE.

What is $\#\mathcal{L}(a,b)$? Answer: $\binom{a+b}{a} = \binom{a+b}{b}$

Why? Each path is represented as a sequence $P = s_1 \cdots s_{a+b}$ where $s_i \in \{N, E\}$, and $P \in \mathcal{L}(a,b)$ if and only if a of its elements are E and b of its elements are N. This gives us a bijection

$$\mathcal{L}(a,b) \rightleftarrows \mathcal{B}(a+b, a)$$

$$s_1 \cdots s_{a+b} \longmapsto \{i \in N_{a+b} ; s_i = E\}$$

Therefore $\#\mathcal{L}(a,b) = \#\mathcal{B}(a+b,a) = \binom{a+b}{a}$.

ex Prove that for $a, b \in \mathbb{N}$,

$$\binom{a+1+b}{b} = \sum_{j=0}^{b} \binom{a+j}{j}$$

by using a bijection. Give a bijection

$$\mathcal{L}(a+1,b) \rightleftarrows \bigcup_{j=0}^{b} \mathcal{L}(a,j).$$

The bijection LHS $\to$ RHS truncate at the last E step.

Each path in $\mathcal{L}(a+1,b)$ consists of a path in $\mathcal{L}(a,j)$ followed by EN^{b-j}.

Define the bijection to be

$$s_1 \cdots s_{a+b-j} E N^{b-j} \mapsto s_1 \cdots s_{a+b-j}.$$

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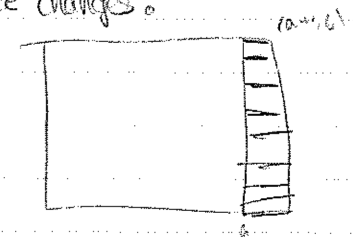
Three changes:

① 

- every path from $(0,0)$ to $(a+1,b)$ crosses <u>exactly one</u> green segment.
- decompose each path corresponding to which green segment is used

②
$$\mathcal{L}(a+1,b) \rightleftarrows \bigcup_{j=0}^{b} \mathcal{L}(a,j) \times \{EN^{b-j}\}$$

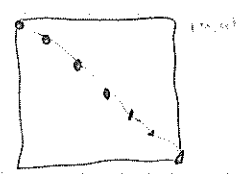
Optional: use symbol $\coprod$ instead of $\cup$ (means disjoint union)

③ State easy direction of the bijection first

$$\coprod_{j=0}^{b} \mathcal{L}(a,j) \times \{EN^{b-j}\} \to \mathcal{L}(a+1,b)$$
$$(\sigma,\tau) \mapsto \sigma\tau \qquad \text{(concatenation)}$$

ex For $n \in \mathbb{N}$, prove that
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Find a bijection
$$\coprod_{k=0}^{n} \mathcal{L}(k,n-k) \times \mathcal{L}(n-k,k) \rightleftarrows \mathcal{L}(n,n)$$



- every path crosses <u>exactly one</u> green dot
- decompose according to which green dot

ex Stirling Numbers of the Second Kind (aka Subset Stirling Numbers)

<u>Def</u> Let X be a finite set. A set partition of X is a finite set $\pi = \{B_1, \dots, B_k\}$ such that

- B_i is a finite non-empty set $\forall i \in [k]$
- $B_i \cap B_j = \emptyset$ for $i \neq j$
- $B_1 \cup \dots \cup B_k = X$

ex. $\{\{1,4,6\},\{2\},\{3,5\},\{7\}\}$ is a set partition of $\mathbb{N}_7$

Let $\Pi(n,k)$ be the set of all set partitions of N_n of size k. The Stirling Numbers of the second kind are defined to be

$$S(n,k) = \#\Pi(n,k).$$

ex Compute $S(4,2)$ by listing all set partitions of N_4 of size 2:

no

Theorem: $S(0,0)=1$

$S(n,0)=0 \quad n\geq 1$

$S(0,k)=0 \quad k\geq 1$

$S(n,k)=S(n-1,k-1)+kS(n-1,k) \quad k,n\geq 1$

Note: if $k>n$ then $S(n,k)=0$.

Proof: ~~Want a bijection~~

$$\cancel{\Pi(n,k) \rightleftarrows \Pi(n-1,k-1)+?}$$

Let $\Pi'(n,k)=\{\pi\in\Pi(n,k);\ \{n\}\in\pi\}$ and $\Pi''(n,k)=\Pi(n,k)\setminus\Pi'(n,k)$. Then

$$\Pi(n,k)=\Pi'(n,k)\sqcup\Pi''(n,k).$$

To prove the recursion, we show that:

① $\#\Pi'(n,k)=\#\Pi(n-1,k-1)$;

② $\#\Pi''(n,k)=k\cdot\#\Pi(n-1,k)$.

For ① we have a bijection

$$\Pi'(n,k)\rightleftarrows\Pi(n-1,k-1),$$

$$\pi\mapsto\pi\setminus\{\{n\}\},$$

$$\mu\cup\{\{n\}\}\leftarrow\!\shortmid\ \mu.$$

We have a map

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$$\phi:\Pi''(n,k)\to\Pi(n-1,k)$$

$$\pi\mapsto\{B\setminus\{n\};\ B\in\pi\}$$

ex $\{\{1,3,5,7\},\{2,4,6\}\}\longmapsto\{\{1,3,5\},\{2,4,6\}\}$

For every $\mu\in\Pi(n-1,k)$, the preimage of μ has size k. Therefore

$$\#\Pi''(n,k)=\sum_{\mu\in\Pi(n-1,k)}\#\phi^{-1}(\mu)=\sum_{''}k=k\,\#\Pi(n-1,k)_{''}=k\cdot S(n-1,k).$$

In conclusion,

$$S(n,k)=\#\Pi(n,k)=\#\Pi'(n,k)+\#\Pi''(n,k)=S(n-1,k-1)+kS(n-1,k)$$

For an example where these are used see exercise 3.8.

Proving Polynomial Identities

Theorem: Let $p(y)$ and $q(y)$ be polynomials. If $p(n)=q(n)$ for infinitely many numbers $n\in\mathbb{N}$, then $p(y)\equiv q(y)$.

Proof: Let $f(y)=p(y)-q(y)$. $f(y)$ is a polynomial. If $f(y)\not\equiv 0$ then it has only finitely many roots. But by assumption $p(n)=q(n)$ for infinitely many $n\in\mathbb{N}$, so $f(n)=0$ for infinitely many $n\in\mathbb{N}$. Therefore $f(y)\equiv 0$ and so $p(y)=q(y)$. ∎

ex If y is a variable (or indeterminate) define

$$\binom{y}{k}=\frac{y(y-1)\cdots(y-k+1)}{k!}, \quad k\in\mathbb{N}.$$

It follows from Monday's class that

$$\binom{y+1+b}{b}=\sum_{j=0}^{b}\binom{y+j}{j} \quad \text{as polynomials.}$$

Justification: LHS and RHS are both polynomials in y. When $y=a\in\mathbb{N}$, we observed that the identity is true. Since it is true for infinitely many numbers, the identity holds.

Generalization. If $p(y_1,\dots,y_m)$ and $q(y_1,\dots,y_m)$ are polynomials and

$$p(n_1,\dots,n_m)=q(n_1,\dots,n_m)$$

for all $(n_1,\dots,n_m)\in S_1\times\cdots\times S_m$ where S_i is an infinite set then $p\equiv q$.

Remark: The identity

$$!! \quad \binom{2n}{n}=\sum_{k=0}^{n}\binom{n}{k}^2 \quad !!$$

does not have a corresponding polynomial identity. Here LHS and RHS are not polynomial functions of n.