Jacobi Triple Product Formula

$$\sum_{h=-\infty}^{\infty} x^{h^2} y^h = \prod_{j=1}^{\infty} (1+x^{2j-1}y)(1+x^{2j-1}y^{-1})(1-x^{2j}).$$

Proof: Let $\mathcal{F}$ be the set of all subsets A of odd integers with the following properties:

- there are finitely many negative integers in A;
- there are finitely many positive odd integers not in A

We'll prove the thing by describing $\mathcal{F}$ in two ways.

Method 1: By two finite sets of odd integers:

For $A \in \mathcal{F}$ let $\alpha = \{j ;\ j \text{ odd},\ j>0,\ j \notin A\}$

$\beta = \{-j ;\ j \text{ odd},\ j<0,\ j \in A\}$

Then α, β "are" partitions with odd distinct parts

Define $\text{Energy}(\alpha,\beta) = \text{sum}(\alpha) + \text{sum}(\beta)$

$\text{Charge}(\alpha,\beta) = \#\alpha - \#\beta$

$$\sum_{(\alpha,\beta)} x^{\text{Energy}(\alpha,\beta)} y^{\text{charge}(\alpha,\beta)}$$

$$= \left(\sum_{\alpha} x^{\text{sum}(\alpha)} y^{\#\alpha}\right)\left(\sum_{\beta} x^{\text{sum}(\beta)} y^{-\#\beta}\right)$$

$$= \prod_{j=1}^{\infty} (1+x^{2j-1}y)(1+x^{2j-1}y^{-1})$$

Method 2. By a lattice path / (partition + number)

Start with a lattice path at $(0,k)$

where $2k+1$ is the smallest element of A. For $i \in \mathbb{N}$, let

$$\sigma_i = \begin{cases} E & 2(k+i)+1 \in A \\ N & \text{otherwise} \end{cases}$$

Note that this path ends with an infinite # of E steps on the line $y=h$.

If we consider the shape bounded by $y=h$, $x=0$, and the path we get the Ferrers diagram of a partition λ.

$$A \leftrightarrow (h,\lambda) \in \mathbb{Z} \times \mathcal{Y}$$

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Claim: If $A \leftrightarrow (\alpha, \beta)$ under method 1
$A \leftrightarrow (h, \lambda)$ " 2
then $\text{energy}(\alpha,\beta) = 2n(\lambda) + h^2$
$\text{charge}(\alpha,\beta) = h$

Therefore

$$\sum_{(\alpha,\beta)} x^{\text{energy}(\alpha,\beta)} y^{\text{charge}(\alpha,\beta)}$$

$$= \sum_{(h,\lambda) \in \mathbb{Z} \times \mathcal{Y}} x^{2n(\lambda)+h^2} y^h$$

$$= \sum_{h \in \mathbb{Z}} x^{h^2} y^h \sum_{y \in \mathcal{Y}} x^{2n(\lambda)} = \left(\sum_{h \in \mathbb{Z}} x^{h^2} y^h\right) \prod_{j=1}^{\infty} \frac{1}{1 - x^{2j}}. \quad \blacksquare$$