11 Exponential Generating Functions

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Def A class of structures (or Species) $\mathcal{A}$ is a rule that associates to each finite set X another finite set $\mathcal{A}_X$, called the set of $\mathcal{A}$-structures on X, satisfying:

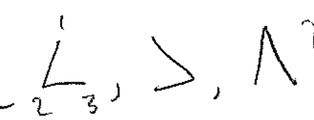
() if $X \neq Y$ then $\mathcal{A}_X \cap \mathcal{A}_Y = \phi$;
(i) if $X \rightleftarrows Y$ then $\mathcal{A}_X \rightleftarrows \mathcal{A}_Y$; ← just think of this
(ii) technical stuff, see chapter 12

ex Graphs: The species of graphs $\mathcal{G}$ assigns to X the set $\mathcal{G}_X$ of all graphs with vertex set X.

ex $\mathcal{G}_{\{1,2,3\}} = \{$  $\}$

Think of $\mathcal{A}_X$ as "the set of all ways to turn X into a ____", or as "the set of all ways to make a ___ out of X".

ex Trees: $\mathcal{T}_X$ is the set of all trees with vertex set X.

ex $\mathcal{T}_{\{1,2,3\}} = \{$  $\}$

Notation: Write $\mathcal{A}_n := \mathcal{A}_{N_n}$.
Note: If $\#X = n$ then $X \rightleftarrows N_n$ so $\mathcal{A}_X \rightleftarrows \mathcal{A}_n$. So $\#\mathcal{A}_X = \#\mathcal{A}_n$.
Problem. Determine $\#\mathcal{A}_n$ for $n \in \mathbb{N}$.

The exponential generating function (EGF) of $\mathcal{A}$ is the Formal Power Series

$$A(x) = \sum_{n=0}^{\infty} \#\mathcal{A}_n \frac{x^n}{n!}$$

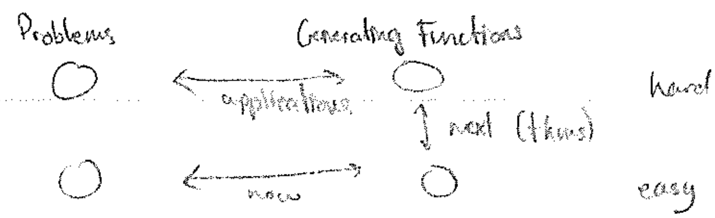
If we know the EGF for $\mathcal{A}$ then we can answer our question:

$$\#\mathcal{A}_n = n! [x^n] A(x).$$

The basic examples

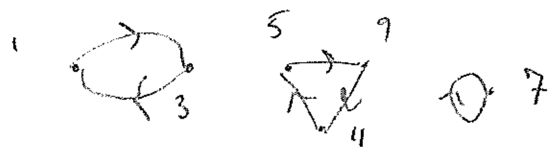
- Permutations ($\mathcal{S}$): For a finite set X, $\mathcal{S}_X$ is the set of all permutations of X. We know that if $\#X = n$ then $\#\mathcal{S}_X = \#\mathcal{S}_n = n!$, so our main condition is satisfied. Therefore the EFG is

$$S(x) = \sum_{n\geq 0} \#\mathcal{S}_n \frac{x^n}{n!} = \sum_{n\geq 0} n! \frac{x^n}{n!} = \sum_{n\geq 0} x^n = \frac{1}{1-x}$$



A permutation can be visualized by a diagram.

ex $X = \{1,3,5,7,9,11\}$, $\sigma(1)=3$, $\sigma(3)=1$, $\sigma(5)=9$, $\sigma(7)=7$, $\sigma(9)=11$, $\sigma(11)=5$.



Another way to think about $\mathcal{A}_X$ is as "all ways to sprinkle labels from X onto a diagram representing an $\mathcal{A}$-structure."

Warning: Different looking sprinklings of a diagram may represent the same mathematical object.

A permutation is cyclic if its diagram has one component.

- Cyclic Permutations ($\mathcal{C}$): For a finite set X, $\mathcal{C}_X$ is the set of all cyclic permutations of X.

Claim: If $\#X = n$ then

$$\#\mathcal{C}_X = \begin{cases} (n-1)! & \text{if } n\geq 1, \\ 0 & \text{if } n = 0. \end{cases}$$

Informal Argument: For each n, there is one diagram for a cyclic permutation, which is a directed cycle of size n. There are $n!$ ways to sprinkle labels on any diagram. Two sprinklings represent the same mathematical object if one is a rotation of the other. So we over-counted by a factor of n.

So we have

$$C(x) = \sum_{n\geq 0} \# \mathcal{C}_n \frac{x^n}{n!} = \sum_{n\geq 1} (n-1)! \frac{x^n}{n!} = \sum_{n\geq 1} \frac{x^n}{n} = \log\left(\frac{1}{1-x}\right) = \log(S(x))$$

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- Sets ($\mathcal{E}$): For any finite set X, $\mathcal{E}_X = \{X\}$. Think: $\mathcal{E}_X$ is the set of all ways to turn X into a set. Since X is a set, there is only one way to this. Note

$$E(x) = \sum_{n\geq 0} \mathcal{E}_n \frac{x^n}{n!} = \sum_{n\geq 1} \frac{x^n}{n!} = \exp(x).$$

Remark: Informally, a <u>permutation</u> is a <u>set</u> of <u>cycles</u> This is mirrored in

$$S(x) = E(C(x)).$$

Goal: Understand this in a formal way. For relationships among these classes, which we'll write as

$$\mathcal{S} \equiv \mathcal{E}[\mathcal{C}].$$

<u>Def</u>] If $\mathcal{A}$ and $\mathcal{B}$ are species, we say that $\mathcal{A}$ and $\mathcal{B}$ are <u>equivalent</u> if $\mathcal{A}_X \rightleftarrows \mathcal{B}_X$ for all finite sets X. We write $\mathcal{A} \equiv \mathcal{B}$.

Aside: There are two versions of this:
- numerical equivalence: $\#\mathcal{A}_X = \#\mathcal{B}_X$
- natural equivalence: see chapter 12

<u>Operations</u>

- Sum of classes If $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \dots$ is a (finite or infinite) list of species and $\mathcal{A}^{(i)}_X \cap \mathcal{A}^{(j)}_X = \emptyset$ for all $i \neq j$ and for every finite set X, then define

$$\mathcal{B} = \mathcal{A}^{(1)} \oplus \mathcal{A}^{(2)} \oplus \cdots$$

to be the species in which

$$\mathcal{B}_X = \mathcal{A}^{(1)}_X \cup \mathcal{A}^{(2)}_X \cup \cdots$$

Informally, $\oplus \longleftrightarrow$ 'or'.

If $\mathcal{A}^{(1)}_X, \mathcal{A}^{(2)}_X, \dots$ are not pairwise disjoint we make them disjoint by replacing

$$\mathcal{A}^{(i)}_X \longleftrightarrow \{i\} \times \mathcal{A}^{(i)}_X.$$

ex k-set ($\mathcal{E}_k$):

$$(\mathcal{E}_k)_X = \begin{cases} \{X\} & \text{if } \#X = k \\ \emptyset & \text{if } \#X \neq k \end{cases}$$

Interpretation. $\mathcal{E}_k$ is all ways to turn _____ into a k-set.

$$\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \cdots = \bigoplus_{k \geq 0} \mathcal{E}_k$$

"A set is either a 0-set or a 1-set or ..."

· Difference of Classes: If $\mathcal{A}$ and $\mathcal{B}$ are classes and $\mathcal{A}_X \subseteq \mathcal{B}_X$ for every X we say that $\mathcal{A}$ is a subclass of $\mathcal{B}$ and define $(\mathcal{B} \setminus \mathcal{A})$ to be the species

$$(\mathcal{B} \setminus \mathcal{A})_X = \mathcal{B}_X \setminus \mathcal{A}_X.$$

ex $\mathcal{E} \setminus \mathcal{E}_0$ is the species of non-empty sets

The EGF for $\mathcal{A}^{(1)} \oplus \mathcal{A}^{(2)} \oplus \cdots$ is $A^{(1)}(x) + A^{(2)}(x) + \cdots$
The EGF for $\mathcal{B} \setminus \mathcal{A}$ is $B(x) - A(x)$.

· Superposition of Classes:

$$(\mathcal{A} \,\&\, \mathcal{B})_X = \mathcal{A}_X \times \mathcal{B}_X$$

$$\#(\mathcal{A} \,\&\, \mathcal{B})_X = (\#\mathcal{A}_X)(\#\mathcal{B}_X)$$

(Does not lead to a nice EGF formula.)

· Product of Classes:

$$(\mathcal{A} * \mathcal{B})_X = \bigcup_{S \subseteq X} \mathcal{A}_S \times \mathcal{B}_{X \setminus S}$$

To form an $\mathcal{A} * \mathcal{B}$-structure on X, take a subset $S \subseteq X$, form an $\mathcal{A}$-structure on S, and form a $\mathcal{B}$-structure on $X \setminus S$.
If $\#X = n$ then

$$\begin{aligned} \#(\mathcal{A} * \mathcal{B})_X &= \sum_{S \subseteq X} (\#\mathcal{A}_S)(\#\mathcal{B}_{X \setminus S}) \\ &= \sum_{k=0}^{n} \sum_{S \in \mathcal{B}(X,k)} (\#\mathcal{A}_S)(\#\mathcal{B}_{X \setminus S}) \\ &= \sum_{k=0}^{n} \binom{n}{k} (\#\mathcal{A}_k)(\#\mathcal{B}_{n-k}). \end{aligned}$$

So

$$\frac{1}{n!} \#(\mathcal{A} * \mathcal{B})_n = \sum_{k=0}^{n} \left(\frac{1}{k!} \#\mathcal{A}_k \right) \left(\frac{1}{(n-k)!} \#\mathcal{B}_{n-k} \right) = [x^n] A(x) B(x).$$

Therefore EGF for $\mathcal{A} * \mathcal{B}$ is $A(x) B(x)$.

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Elements are ordered pairs (α, β) where:

- α is an $\mathcal{A}$-structure on a subset of X
- β is a $\mathcal{B}$-structure on the complement of the subset of X

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ex $(\mathcal{E}*\mathcal{E})_X = \{(S, X\setminus S);\ S \subseteq X\}$

Equivalently, can view $\mathcal{E}*\mathcal{E}$ as ways to pick a subset.

EGF for $\mathcal{E}*\mathcal{E}$ is $E(x)E(x)$:

$$e^x e^x = e^{2x} = \sum_{n\geq 0} \frac{(2x)^n}{n!} = \sum_{n\geq 0} 2^n \frac{x^n}{n!}$$

as expected.

ex $\mathcal{E}*\mathcal{E}*\mathcal{E}$ is ways to partition a set into an ordered triple

EGF is

$$e^{3x}, \quad n![x^n]e^{3x} = 3^n$$

Note: we have a bijection between

$$(\mathcal{E}*\mathcal{E}*\mathcal{E})_X \rightleftarrows \mathcal{F}(X, N_3)$$

ex $k \in \mathbb{N}$, $\mathcal{E}_k * \mathcal{E}$

- Take input set X
  - partition X into two sets $(S, X\setminus S)$
  - put an $\mathcal{E}_k$-structure on S
  - do nothing to $X\setminus S$

Procedurally, think of $\mathcal{E}_k$ as a filter "make sure set has k elements"

So $\mathcal{E}_k * \mathcal{E}$ is ways to pick a k-element subset. EGF is

$$\frac{x^k}{k!} e^x = \frac{x^k}{k!} \sum_{n\geq 0} \frac{x^n}{n!} = \sum_{n\geq 0} \frac{x^{n+k}}{k!\,n!} = \sum_{m\geq k} \frac{x^m}{(m-k)!\,k!}$$

So

$$n![x^n]\frac{x^k}{k!}e^x = \left\{\begin{matrix} \frac{n!}{(n-k)!\,k!} & \text{if } n\geq k \\ 0 & \text{if } n<k \end{matrix}\right\} = \binom{n}{k}.$$

Define $\mathcal{X} = \mathcal{E}_1$ "singletons"

ex For any class $\mathcal{A}$, $\mathcal{X}*\mathcal{A}$:

- input set X
- partition X into $(S, X\setminus S)$
- make sure S has 1 element
- put an $\mathcal{A}$-structure on $X\setminus S$

In other "words"

$$(\mathcal{X} * \mathcal{A})_X = \{(x,\alpha) ;\ x \in X,\ \alpha \in \mathcal{A}_{X \setminus \{x\}}\}$$

(Pick out one element to be special and put an $\mathcal{A}$-structure on everything else.)

<u>Powers</u> (k-tuples of $\mathcal{A}$-structures)

For any species $\mathcal{A}$,

$$\mathcal{A}^k = \begin{cases} \mathcal{A} * \cdots * \mathcal{A} & \text{if } k \geq 1 \\ \mathcal{E}_0 & \text{if } k = 0 \end{cases}$$

ex $\mathcal{E}^k \equiv \mathcal{F}(\cdot, N_k)$

Means: for any set X

$$(\mathcal{E}^k)_X \rightleftarrows \mathcal{F}(X, N_k)$$

<u>Def</u> A class $\mathcal{A}$ is <u>connected</u> if $\mathcal{A}_\emptyset = \emptyset$.

If $\mathcal{A}$ is connected, define

$$\mathcal{A}^* = \mathcal{A}^0 \oplus \mathcal{A}^1 \oplus \mathcal{A}^2 \oplus \cdots$$

EGF:

$$A(x)^0 + A(x)^1 + \cdots = \frac{1}{1-A(x)}$$

ex $\mathcal{X}^*$: An $\mathcal{X}^k$ structure is a k-tuple $(x_1, \dots, x_k)$ where $X = \{x_1, \dots, x_k\}$ and $x_i \neq x_j\ \forall i \neq j$

$(\mathcal{X}^*)_X = \{(x_1, \dots, x_k) ;\ X = \{x_1, \dots, x_k\},\ x_i \neq x_j\ \forall i \neq j,\ k = \#X\}$

$\mathcal{X}^*$ is called the class of linear order

elements of $(\mathcal{X}^*)_X$ are ways to order elements of X

EGF: $\dfrac{1}{1-x}$ $\qquad n!\,[x^n]\dfrac{1}{1-x} = n! \leftarrow$ # ways to order N_n

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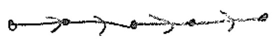
$\mathcal{X}^*$ linear orderings $\#(\mathcal{X}^*)_n = n!$

$\mathcal{S}$ permutations $\#(\mathcal{S})_n = n!$

two subtle differences:

- pictures  vs 
- see chapter 12

Back to sum operation:

If we have an infinite sum

$$\mathcal{B} = \mathcal{A}^{(1)} \oplus \mathcal{A}^{(2)} \oplus \cdots$$

$$\mathcal{B}_X = \mathcal{A}^{(1)}_X \cup \mathcal{A}^{(2)}_X \cup \cdots \quad \leftarrow \text{need this to be finite}$$

When this is true for all finite sets X, we say that the sequence $(\mathcal{A}^{(1)}, \ldots)$ is locally finite.

Whenever we take an infinite product, we should check that the sequence is locally finite.

ex $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \cdots$

locally finite because

$$(\mathcal{E}_k)_X = \begin{cases} \{X\} & \text{if } \#X = k, \\ \emptyset & \text{otherwise.} \end{cases}$$

so for any X, one is finite and the rest are empty

Exercise: Verify that if $\mathcal{A}$ is connected,

$$\mathcal{A}^* = \mathcal{A}^0 \oplus \mathcal{A}^1 \oplus \cdots$$

is locally finite.

For all X all but finitely many $\mathcal{A}^{(i)}_X$ are empty.

<u>Rooted Structures</u>

If $\mathcal{A}$ is any class, define $\mathcal{A}^\circ$ to be the class of 'rooted $\mathcal{A}$-structures', defined by

$$\mathcal{A}^\circ_X = \mathcal{A}_X \times X.$$

To form a rooted $\mathcal{A}$-structure on X:

- put an $\mathcal{A}$-structure on X
- pick an element of X to call the root

ex rooted trees $\mathcal{T}^\circ$

$\in \mathcal{T}_3$

tree

$\in \mathcal{T}^\circ_3$

rooted tree

Notes: $\mathcal{A}^\circ$ is not the same as $\mathcal{X} * \mathcal{A}$.

build structure first then pick special element

pick special element, build structure without using it

$\mathcal{A}^\circ$ is always connected

$$\#\mathcal{A}^\circ_X = (\#X)(\#\mathcal{A}_X) \Rightarrow \#\mathcal{A}^\circ_n = n\#\mathcal{A}_n.$$

Therefore

$$A^\circ(x) = \sum_{n\geq 0} n\#\mathcal{A}_n \frac{x^n}{n!} = x\frac{d}{dx}A(x)$$

From now on I will assume that if $X \neq Y$ then $\mathcal{A}_X \cap \mathcal{A}_Y = \emptyset$.

"elements of $\mathcal{A}_X$ remember X"

"$\mathcal{A}$-structures on X use all elements of X"

k-Sets of Structures

If $\mathcal{A}$ is a connected class, define $\mathcal{E}_k[\mathcal{A}]$ to be the class of k-sets of $\mathcal{A}$-structures:

$$(\mathcal{E}_k[\mathcal{A}])_X = \{\{\alpha_1, \ldots, \alpha_k\} ; (\alpha_1, \ldots, \alpha_k) \in \mathcal{A}^k_X\}$$

- take a k-tuple of $\mathcal{A}$-structures ($\mathcal{A}^k$-structure) and forget the order
- take a set partition of X with k subsets $\{S_1, \ldots, S_k\}$ and put an $\mathcal{A}$-structure on each S_i

ex $\mathcal{E}_k[\mathcal{X}] \equiv \mathcal{E}_k$ because $\mathcal{X}^k$ lists elements as long as there are exactly k

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Recall EGFs for $\mathcal{A}^k$ and $\mathcal{E}_k[\mathcal{A}]$ are $A(x)^k$ and $A(x)^k/k!$ respectively, for $\mathcal{A}$ a connected ~~species~~ class.

ex $\mathcal{E}_2[\mathcal{E}_2]$ "2-sets of 2-sets"

$\mathcal{E}_2[\mathcal{E}_2]_4 = \{\{\{1,2\},\{3,4\}\},\{\{1,3\},\{2,4\}\},\{\{1,4\},\{2,3\}\}\}$

EGF is
$$\frac{\left(\frac{x^2}{2!}\right)^2}{2!} = \frac{x^4}{8}, \quad 4![x^4] = 3 \text{ as expected.}$$

$\mathcal{E}_2[\mathcal{E}_2]_X = \emptyset$ where $\#X \neq 4$.

More generally,
$$\mathcal{E}_k[\mathcal{E}_2]_X = \begin{cases} \text{perfect matchings in } K_X & \text{if } \#X = 2k \\ \emptyset & \text{otherwise} \end{cases}$$

How many matchings in K_{2k}?
$$(2k)!\,[x^{2k}]\,\frac{\left(\frac{x^2}{2!}\right)^k}{k!} = \frac{(2k)!}{(2!)^k\,k!}$$

Recall $\mathcal{E}\setminus\mathcal{E}_0$ is "non-empty sets", $\mathcal{E}_k[\mathcal{E}\setminus\mathcal{E}_0]$ is k-sets of non-empty sets, or set partitions of size k. EGF:

EGF for $\mathcal{E}\setminus\mathcal{E}_0$ is $e^x - 1$
$$\frac{(e^x-1)^k}{k!}$$

How many set partitions of N_n into k subsets?
$$n!\,[x^n]\,\frac{(e^x-1)^k}{k!} = \frac{n!}{k!}[x^n](e^x-1)^k$$

Sets of Structures

If $\mathcal{A}$ is a connected class,
$$\mathcal{E}[\mathcal{A}] = \mathcal{E}_0[\mathcal{A}] \oplus \mathcal{E}_1[\mathcal{A}] \oplus \mathcal{E}_2[\mathcal{A}] \oplus \cdots$$

- partition X into any number of non-empty subsets and put an $\mathcal{A}$-structure on each subset

EGF is
$$\sum_{k\geq 0} \frac{A(x)^k}{k!} = \exp(A(x)) = e^{A(x)}$$

ex $\mathcal{E}[\mathcal{E}\setminus\mathcal{E}_0]$ is the class of set partitions (of any size)
How many set partitions of N_n?

$$n!\,[x^n]\exp(e^x-1)$$

- If $\mathcal{T}$ is the class of trees then $\mathcal{E}[\mathcal{T}]$ is the class of forests.
- $\mathcal{E}[\mathcal{C}] = \mathcal{S}$

$$\Rightarrow \exp\left(\log\left(\frac{1}{1-x}\right)\right) = \frac{1}{1-x}$$

- Recall $\mathcal{T}^\circ$ is the class of rooted trees

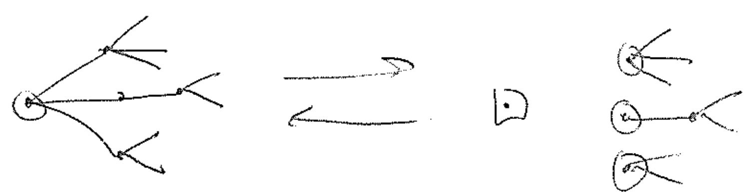
$\mathcal{E}[\mathcal{T}^\circ]$ = forest in which each tree has a "root"

Claim $\mathcal{T}^\circ \equiv \mathcal{X} * \mathcal{E}[\mathcal{T}^\circ]$

pick a special vertex to be outside

partition into non-empty sets

make a rooted tree out of each subset



Since we have an equivalence, EGF's are equal

$$T^\circ(x) = x\exp(T^\circ(x))$$

Solve using LIFT in $\mathbb{Q}[[x]]$ with $G(u) = e^u$:

$$\begin{aligned}
[x^n]T^\circ(x) &= \frac{1}{n}[u^{n-1}]G(u)^n \\
&= \frac{1}{n}[u^{n-1}]\,e^{un} \\
&= \frac{1}{n}[u^{n-1}]\sum_{k\geq 0}\frac{(un)^k}{k!} \\
&= \frac{1}{n}[u^{n-1}]\sum_{k\geq 0}\frac{\cancel{u^k}\,n^k}{k!}u^k \\
&= \frac{1}{n}\,\frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{n!}
\end{aligned}$$

Therefore the number of rooted trees on n vertices is

$$n!\,[x^n]T^\circ(x) = n^{n-1}.$$

How many trees are there on n vertices?

$$\#\mathcal{T}_n = \frac{1}{n}\#\mathcal{T}^\circ_n = \frac{1}{n}(n^{n-1}) = n^{n-2}.$$

ex Let $\mathcal{J}$ be the class of 2-regular graphs. Compute EGF.
Let $\mathcal{H}$ be the class of connected 2-reg graphs,
ie $\mathcal{H}$ is the class of graphs that are cycles.
So $\#\mathcal{H}_0 = \#\mathcal{H}_1 = \#\mathcal{H}_2 = 0$

$$\#\mathcal{H}_n = \frac{(n-1)!}{2} \quad \text{if } n \geq 3 \qquad \text{(easy to see)}$$

$$H(x) = \sum_{n\geq 3} \frac{(n-1)!}{2} \frac{x^n}{n!} = \frac{1}{2} \sum_{n \geq 3} \frac{x^n}{n} = \frac{1}{2}\left(\log\left(\frac{1}{1-x}\right) - x - \frac{x^2}{2}\right)$$

Since $\mathcal{J} \equiv \mathcal{E}[\mathcal{H}]$,

$$J(x) = \exp(H(x)) = \exp\left(\frac{1}{2}\left(\log\left(\frac{1}{1-x}\right) - x - \frac{x^2}{2}\right)\right)$$

$$= \frac{e^{-\frac{x}{2} - \frac{x^2}{4}}}{\sqrt{1-x}}$$

ex We know that there are $\binom{n}{2}$ graphs with n vertices,

$$G(x) = \sum_{n\geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!}$$

For connected graphs we have $\mathcal{E}[\tilde{\mathcal{G}}] \equiv \mathcal{G}$, so

$$\exp(\tilde{G}(x)) = G(x)$$

$$\Rightarrow \tilde{G}(x) = \log(G(x)) = \log\left(\sum_{n \geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!}\right).$$

Composition of Classes

Let $\mathcal{A}$ be a connected class, let $\mathcal{B}$ be any class. Define $\mathcal{B}[\mathcal{A}]$, "$\mathcal{B}$-structures of $\mathcal{A}$-structures", by

$$\mathcal{B}[\mathcal{A}]_X = \bigcup_{\xi \in \mathcal{E}[\mathcal{A}]_X} \{\xi\} \times \mathcal{B}_\xi$$

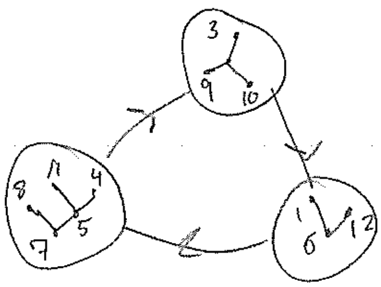
A $\mathcal{B}[\mathcal{A}]$-structure on X is a pair (ξ, β) where $\xi = \{\alpha_1, \ldots, \alpha_k\}$ is an $\mathcal{E}[\mathcal{A}]$-structure on X and β is a $\mathcal{B}$-structure on ξ.

Note: This is consist with defs of $\mathcal{E}[\mathcal{A}]$ and $\mathcal{E}_k[\mathcal{A}]$.

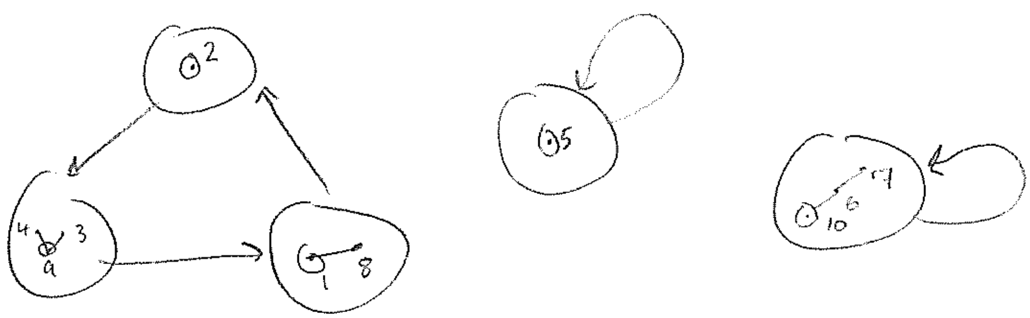
How to draw a $\mathcal{B}[\mathcal{A}]$-structure.

- draw an unlabelled $\mathcal{B}$-structure with big ~~[illegible]~~ circles in places of objects that would normally get labels
- draw an unlabelled $\mathcal{A}$-structure in each circle
- assign labels distributed among the $\mathcal{A}$-structures

ex Draw a $\mathcal{C}[\mathcal{T}]$-structure on $\mathbb{N}_{12}$

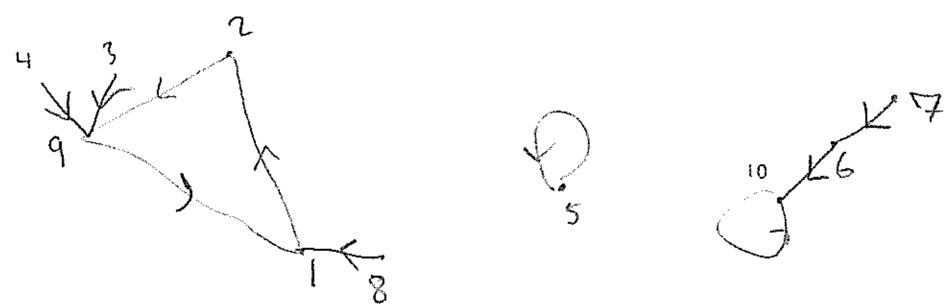


ex $\mathcal{S}[\mathcal{T}^\bullet]$



collapse

and direct edges to root



$f : \mathbb{N}_{10} \to \mathbb{N}_{11}$

We have an equivalence

$$\mathcal{S}[\mathcal{T}^\bullet] \equiv \mathcal{F}$$

where $\mathcal{F}$ is the class of endofunctions, $\mathcal{F}_X = \mathcal{F}(X;X)$.

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Theorem: The EGF for $\mathcal{B}[\mathcal{a}]$ is $B(A(x))$.

Proof: Since a $\mathcal{B}[\mathcal{a}]$-structure consists of an $\mathcal{E}_k[\mathcal{a}]$-structure and a $\mathcal{B}$-structure on a k-element set where $k \in \mathbb{N}$. So

$$\#\mathcal{B}[\mathcal{a}]_n = \sum_{k \geq 0} \underset{\substack{\uparrow \\ \text{\# ways to put} \\ \mathcal{B}\text{-struct. on k-set}}}{\#\mathcal{B}_k} \cdot \underset{\substack{\uparrow \\ \text{ways to put } \mathcal{E}_k[\mathcal{a}]\text{-struct} \\ \text{on an n-set}}}{\#\mathcal{E}_k[\mathcal{a}]_n}$$

So EGF for $\mathcal{B}[\mathcal{a}]$ is

$$\sum_{n \geq 0} \left(\sum_{k \geq 0} \#\mathcal{B}_k \#\mathcal{E}_k[\mathcal{a}]_n \right) \frac{x^n}{n!}$$

$$= \sum_{k \geq 0} \#\mathcal{B}_k \left(\sum_{n \geq 0} \#\mathcal{E}_k[\mathcal{a}]_n \frac{x^n}{n!} \right)$$

EGF for $\mathcal{E}_k[\mathcal{a}]$

$$= \sum_{k \geq 0} \#\mathcal{B}_k \frac{A(x)^k}{k!} = B(A(x))$$

ex Last time, $\mathcal{F} \equiv \mathcal{S}[\mathcal{T}^\circ]$

$$F(x) = \sum_{n \geq 0} n^n \frac{x^n}{n!} \qquad S(x) = \frac{1}{1-x} \qquad T^\circ(x) = \sum_{n \geq 1} n^{n-1} \frac{x^n}{n!}$$

$$F(x) = S(T^\circ(x))$$

$$\sum_{n \geq 0} n^n \frac{x^n}{n!} = \frac{1}{1 - \sum_{n \geq 1} n^{n-1} \frac{x^n}{n!}}$$

inverse!