CO 330 - Combinatorial Enumeration

2014 09 08

Philosophy: Given a finite set, <u>how many</u> elements does it have?

1 Sets and Bijections

ex $\mathbb{N} = \{0,1,2,3,\dots\}$
ex For $n \in \mathbb{N}$, $N_n = \{1,\dots,n\}$
ex If X, Y are sets, $X \times Y = \{(x,y);\ x \in X,\ y \in Y\}$
ex $\mathcal{F}(X,Y)$ is the set of all functions with domain X and codomain Y
ex The set of full binary trees.
ex The set of perfect matchings in K_n

ex Set of all sets doesn't exist.
ex Set of all graphs.. doesn't exist. (or finite sets)
↳ to fix this: species (combinatorial structures)

$f: S \to T$ is...
- an injection if for every $s, s' \in S$, if $f(s) = f(s')$ then $s = s'$
- a surjection if for every $t \in T$, there is an $s \in S$ such that $f(s) = t$
- a bijection if it is both an injection and a surjection

In this course, you'll need to
(1) state bijections and
(2) prove it's correct.
(1) State domain and codomain, and state a rule that defines the function.
(2) codomain is correct, well-defined, injection, surjection

If there is a bijection $f: S \to T$, there is an <u>inverse</u> map $g: T \to S$ satisfying (1) $g(f(s)) = s\ \forall s \in S$, (2) $f(g(t)) = t\ \forall t \in T$. 2014 09 10
If f has an inverse then f is ~~bijective~~ a bijection because (1) $\Rightarrow$ f is surjective, and (2) $\Rightarrow$ f is injective.
This gives us another way to prove f is a bijection: Find the inverse and check (1) and (2).
We write f^{-1} for the inverse map.

If there is a bijection $f: S \to T$, we write $S \rightleftarrows T$, and say S is <u>equicardinal</u> with T.
If $S \rightleftarrows N_n$, we say S is a finite set and the <u>cardinality</u> of S is n, written
$$\#S = n \quad \text{or} \quad |S| = n.$$

Theorem: (1) A finite set can only have one cardinality.
(2) Two finite sets have the same cardinality if and only if they are equicardinal.
Proof: (1) This will be a homework exercise. (2) is proposition 1.1 in the notes. ∎

We can also say $\#S = \sum_{s \in S} 1$.
This is very useful.

If $f: S \to T$ any function (S, T finite sets), define
$$f^{-1}(\{t\}) = \{s \in S;\ f(s) = t\}, \quad t \in T.$$
Common practise: drop curly brackets when they're annoying and confusion caused is minimal.

Proposition: If S, T are finite sets and $f: S \to T$ is any function then
$$\#S = \sum_{t \in T} \# f^{-1}(t).$$
Proof: $$\#S = \sum_{s \in S} 1 = \sum_{t \in T} \left(\sum_{s \in f^{-1}(t)} 1 \right) = \sum_{t \in T} \# f^{-1}(t).$$ ∎

<u>Products</u>

Proposition: If S, T are finite sets then $\#(S \times T) = (\#S)(\#T)$.
Proof: Define a function $f: S \times T \to T$ by $f((s,t)) = t$. The previous proposition gives us
$$\#(S \times T) = \sum_{t \in T} \# f^{-1}(t) = \sum_{t \in T} \#S = \#S \left(\sum_{t \in T} 1 \right) = \#S \, \#T$$

More generally, if $S_1, \dots, S_m$ are finite sets then
$$\#(S_1 \times \cdots \times S_m) = (\#S_1) \cdots (\#S_m).$$
If S is a set, $S^m = \underbrace{S \times \cdots \times S}_{m \text{ times}}$.

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Functions:

Proposition: If X, Y are finite sets,
$$\#\mathcal{F}(X,Y) = (\#Y)^{(\#X)}$$
Proof: First half. Prove it in the special case where $X = N_m$. I want to show $\#\mathcal{F}(N_m, Y) = (\#Y)^m$. To do this, we will construct a bijection between $\mathcal{F}(N_m, Y)$ and Y^m. This will show that $\#\mathcal{F}(N_m,Y) = \#(Y^m) = (\#Y)^m$.
Define $\phi: \mathcal{F}(N_m, Y) \to Y^m$ as follows. For $f \in \mathcal{F}(N_m, Y)$, define
$$\phi(f) = (f(1), \dots, f(m)) \in Y^m.$$
This is clearly well-defined and lands in the codomain. As an exercise, show it is injective and surjective.
Second half: Prove that if $\#X = m$ then
$$\mathcal{F}(X,Y) \rightleftarrows \mathcal{F}(N_m, Y).$$
See the notes for details. This proves
$$\#\mathcal{F}(X,Y) = \#\mathcal{F}(N_m, Y) = (\#Y)^m = (\#Y)^{(\#X)}. \quad \blacksquare$$

Power Set

2014 09 12

For a set X, $\mathcal{P}(X)$ is the set of all subsets of X,
$$\mathcal{P}(X) = \{A ; A \subseteq X\}.$$
If X is a finite set what is $\#\mathcal{P}(X)$? $2^{\#X}$

ex $\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

To prove this, we will define a bijection $\mathcal{P}(X) \rightleftarrows \mathcal{F}(X, \{0,1\})$.
Characteristic function of a subset: If $A \subseteq X$ (ie $A \in \mathcal{P}(X)$) then
$$f_A : X \to \{0,1\},$$
$$f_A(x) = \begin{cases} 0 & \text{if } x \notin A, \\ 1 & \text{if } x \in A. \end{cases}$$

I claim $\psi : \mathcal{P}(X) \to \mathcal{F}(X, \{0,1\})$, $\psi(A) = f_A$, is a bijection. The inverse is $\rho: \mathcal{F}(X,\{0,1\}) \to \mathcal{P}(X)$, $\rho(f) = f^{-1}(1)$. Exercise: Check that ψ and ρ are mutually inverse functions.

implicitly assume all sets being #'d are finite

We've shown that $\mathcal{P}(X) \rightleftarrows \mathcal{F}(X, \{0,1\})$. Last time we showed that $\mathcal{F}(N_m, \{0,1\}) \rightleftarrows \{0,1\}^m$.

Intersections and Unions

If A, B are sets,

$$A \cup B = \{x ; x \in A \text{ or } x \in B\},$$
$$A \cap B = \{x ; x \in A \text{ and } x \in B\}.$$

Proposition: $\#(X \cup Y) = \#X + \#Y - \#(X \cap Y)$

$$\#(X \cup Y \cup Z) = \#X + \#Y + \#Z - \#(X \cap Y) - \#(X \cap Z) - \#(Y \cap Z) + \#(X \cap Y \cap Z)$$

To generalize this, let's introduce the following notation. Suppose $A_1, \ldots, A_m$ are finite sets. To each non-empty $S \subseteq N_m$, define

$$A_S = \bigcap_{i \in S} A_i$$

Theorem (Principle of Inclusion-Exclusion):

$$\#(A_1 \cup \cdots \cup A_m) = \sum_{\emptyset \subset S \subseteq N_m} (-1)^{\#S-1} \# A_S .$$

Special case: If $A_1, \ldots, A_m$ are pairwise disjoint (means $A_i \cap A_j = \emptyset$ for $i \neq j$):

$$\#(A_1 \cup \cdots \cup A_m) = \#A_1 + \cdots + \#A_m$$

For the proof of the theorem, see notes.

Note similarity to the following fact:

$$1 - (1-y_1)(1-y_2)(1-y_3) = y_1 + y_2 + y_3 - y_1y_2 - y_1y_3 - y_2y_3 + y_1y_2y_3$$

More generally, if $\emptyset \subset S \subseteq N_m$, $y_1, \ldots, y_m$ variables,

$$y^S = \prod_{i \in S} y_i .$$

Then

$$1 - \prod_{i=1}^{m} (1 - y_i) = \sum_{\emptyset \subset S \subseteq N_m} (-1)^{\#S-1} y^S$$

ex If $n \in \mathbb{N}$, the Euler totient of n is
$$\varphi(n) = \#\{k \in \mathbb{N}_n ; \gcd(k,n) = 1\}.$$

ex $\varphi(12) = 4$ because $1, 5, 7, 11$.

If $n = p_1^{c_1} \cdots p_m^{c_m}$ is the prime factorization, use inclusion-exclusion ~~to~~ to ~~looking at~~ compute $\varphi(n)$ by letting
$$A_i = \{k \in \mathbb{N}_n ; p_i \mid k\}.$$
Then
$$\varphi(n) = n - \#(A_1 \cup \cdots \cup A_m),$$
Since $A_1 \cup \cdots \cup A_m$ is the set of numbers in $\mathbb{N}_n$ which have at least one prime factor in common with n.
For $\emptyset \subset S \subseteq \mathbb{N}_m$,
$$b \in A_S \iff p_i \mid b \text{ for all } i \in S$$
$$\iff \left(\textstyle\prod_{i \in S} p_i\right) \mid b.$$
Therefore
$$\# A_S = \frac{n}{\prod_{i \in S} p_i} = n \prod_{i \in S} \frac{1}{p_i}$$
Let $y_i = \frac{1}{p_i}$. Can rewrite as $\# A_S = n \cdot y^S$. Now
$$\#(A_1 \cup \cdots \cup A_m) = \sum_{\emptyset \subset S \subseteq \mathbb{N}_m} (-1)^{\#S-1} \# A_S$$
$$= \sum (-1)^{\#S-1} (n \cdot y^S)$$
$$= n \sum (-1)^{\#S-1} y^S$$
$$= n \left(1 - \prod_{i=1}^{m} (1 - y_i) \right)$$
$$= n \left(1 - \prod_{i=1}^{m} \left(1 - \frac{1}{p_i}\right) \right).$$
So
$$\varphi(n) = n - \#(A_1 \cup \cdots \cup A_m) = n \prod_{i=1}^{m} \left(1 - \frac{1}{p_i}\right)$$

ex $n = 12 = 2^2 3 \Rightarrow \varphi(n) = 12 \cdot \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 4$