Quantum Physics and the Representation Theory of $SU(2)$

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Abstract. Over the past several decades, developments in Quantum Physics have provided motivation for research into representation theory. In this paper, we will explain why representation theory occurs in Quantum Physics and provide a classification of the irreducible, unitary, finite-dimensional representations of $SU(2)$, conditional on some key theorems from the representation theory of compact groups. We will then explain of what consequence this classification is to physics, and shed some light on the relationship between the two subjects.

1 Introduction

In Quantum Physics, states of physical systems are elements of a complex vector space. These vector spaces come with particular choices of orthonormal bases corresponding to measurable quantities, linear transformation between the bases, and self-adjoint operators which are diagonal in these bases. In most introductions to Quantum Mechanics, students are introduced to many such vector spaces example by example, told how each corresponds to particular physical systems, and learn how to make predictions by transforming states represented in one basis into another and computing the amplitude of the coefficients.

This approach allows them to successfully make predictions for physical systems whose quantum theory is described by the vector space constructions they are familiar with, but leaves open as to how one obtains the vector spaces, the corresponding bases and operators, and the transformation laws. This question is altogether more difficult, and there are many important considerations. One answer is that in situations when quantum effects are only relevant for small-scale behaviour, the quantum theory should give predictions which are indistinguishable from that of the classical theory if one cannot measure very precisely. In this case, one often says that the quantum theory should "converge" to the classical theory in the "limit" when an important physical constant $\hbar$ goes to 0. Ways of producing quantum theories from existing classical ones where such notions can be made precise are often called methods of "quantizing" a classical theory.

However, not all quantum theories for systems found in nature can be obtained in this way, and in such cases other considerations are needed to obtain the right quantum theory. An important one is the role of physical symmetries. Physical symmetries are transformations of the physical states which preserve the underlying physics. For instance, physical laws are typically invariant under orthogonal transformations of Euclidean space, since it doesn't matter where one puts or how one orients the coordinate axes. These symmetries take the form a group action by a "symmetry" group on the set of states of the system. In quantum physics, the set of states form a vector space, and so one is naturally led to consider group actions on vector spaces, i.e., group representations.

In this paper, we will specifically look at the group $SU(2)$. This is an infinite group, not a finite one, so we will begin with a discussion (without proof) of which key results from finite-dimensional representation theory hold in the infinite case. We will then prove a classification result to classify the physically important representations of $SU(2)$, and conclude by connecting this result to Physics.

2 Infinite Dimensional Representation Theory

Our study of finite-dimensional representation theory made heavy use of the theory of characters. In particular, we were able to define an inner product $\langle \cdot, \cdot \rangle$ on the space of complex-valued class functions, and found that the characters of the irreducible representations (ρ, V), defined by $\chi_\rho(g) = \text{Tr}[\rho(g)]$, formed an orthonormal basis for this space of class functions. This allowed us to check if a representation was irreducible simply by deciding whether $\langle \chi_\rho, \chi_\rho \rangle = 1$, and to check if (ρ, V) was the same as another irreducible representation (τ, W) by deciding whether $\langle \chi_\rho, \chi_\tau \rangle = 0$.

In the infinite case, we would much like to be able to replicate these results, but we are immediately faced with a problem. In the finite case, we defined for a group G

$$\langle \alpha, \beta \rangle := \frac{1}{|G|} \sum_{g \in G} \alpha(g) \overline{\beta}(g).$$

In the infinite case, $|G|$ is not finite, and we cannot sum over the group elements. The usual solutions for replacing finite summations when dealing with infinite objects are to use either convergent sums of countably infinitely many elements, or to use integrals. If we choose countably infinite summations, we cannot assign a non-zero value to $|G|$ without weighing the elements of $|G|$ non-uniformly. A similar problem occurs for uncountably infinite groups which are topologically non-compact. For compact groups however, we can define an appropriate integration measure, called the *Haar measure*, which allows for a suitable definition of $\langle \cdot, \cdot \rangle$.

Another condition that's needed in the infinite case is the requirement that the map $\rho : G \to GL(V)$ be continuous. In some ways this is not a restriction, since in the finite case our maps ρ were continuous with respect to the discrete topology. In physics, one can always assume that any continuous function is smooth, since every continuous function is arbitrarily-well approximated (in some natural sense) by a smooth function, and so non-differentiability can have no relevance in experiments which only ever determine quantities to within some experimental uncertainty. We will make the same assumption here. The assumption that all relevant maps be smooth puts us in the context of the theory of differentiable manifolds, which we will not use explicitly, and makes the groups under consideration *Lie groups*.

A last assumption that's needed is that our representations be unitary. For finite groups this is automatic, but for the infinite dimensional case it needs to be imposed. To justify this we again look to physics. We've mentioned that quantum theories on a vector space $\mathcal{H}$ frequently come with special orthonormal bases corresponding to measurable quantities. Suppose $\mathcal{B}$ is one such basis. This means that a physical state $\psi \in \mathcal{H}$ can be decomposed as $\psi = \sum_{b \in \mathcal{B}} \alpha_b b$ for coefficients $\alpha_b \in \mathbb{C}$. It is a postulate of such theories that $|\alpha_b|^2$ is the probability of observing the state b when the state ψ is measured. This tells us that $\|\psi\| = \sum_{b \in \mathcal{B}} |\alpha_b|^2 = 1$, since probabilities must sum to 1. Furthermore, if $\rho(g)$ is a linear transformation corresponding to a representation $(\rho, \mathcal{H})$ of some symmetry group G, the map $\rho(g)$ must preserve the norm, i.e., we must have $\|\rho(g)\psi\| = 1$, since the probabilities must still sum to 1 in any valid state.

The above discussion tells us that our representations must be *norm preserving*, a condition that is equivalent to $(\rho, \mathcal{H})$ being a unitary representation. Taking all these assumptions at once, we now get the following theorem.

Theorem: We can define an inner-product $\langle \cdot, \cdot \rangle$ on the space of complex-valued class functions such that for irreducible unitary representations (ρ, V) and (ρ', V') of a compact Lie Group G, we have

$$\langle \chi_\rho, \chi_{\rho'} \rangle = \delta_{\rho,\rho'}.$$

For a proof, see [2].

The requirement that our representations be unitary also has an additional benefit, which comes from a general result called the Peter-Weyl Theorem. We have

Peter-Weyl Theorem (Part II): If $(\rho, \mathcal{H})$ is a unitary representation of a compact group G, then $(\rho, \mathcal{H})$ is a direct sum of irreducible finite-dimensional unitary representations.

For a source, see [6].

With all this in mind, we can follow the same programme for studying representation theory of $SU(2)$ as for the finite-dimensional case — classify the finite-dimensional irreducible representations, and form the others as all possible direct sums of these. We pursue this programme in the next section.

3 The Representation Theory of SU(2)

3.1 The Group SU(2)

In general, the group $SU(n)$ is the group of $n \times n$ unitary complex matrices with determinant 1. Let's consider a matrix $U := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ for our case $n = 2$. The unitarity condition tells us that $U^* = U^{-1}$. The standard formula for the inverse of a 2×2 matrix and the condition $\det(U) = 1$ gives us that

$$\begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{pmatrix} = U^* = U^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} .$$

This tells us that $\delta = \overline{\alpha}$, and that $\gamma = -\overline{\beta}$, so $SU(2)$ takes the form

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathbb{M}_2(\mathbb{C}) : |\alpha|^2 + |\beta|^2 = 1 \right\}$$

This formulation immediately suggests an identification of $SU(2)$ with S^3, since if $\alpha = x_1 + ix_2$ and $\beta = x_3 + ix_4$ for $\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, we have $|\alpha|^2 + |\beta|^2 = 1$ if and only if $\|\boldsymbol{x}\| = 1$. Because the standard topology on $SU(2)$ corresponds to the standard topology on S^3 inherited from $\mathbb{R}^4$, this shows that $SU(2)$ is a compact group, which is a fact we will need in the next section.

We would also like to classify the conjugacy classes of $SU(2)$, as this information will be important for understanding its character theory. Since $A \in SU(2)$ is unitary, we have $AA^* = I = A^*A$, and so A is normal (i.e., $AA^* = A^*A$). The spectral theorem then tells us that A must be diagonalizable by a unitary matrix U, so $A = UDU^{-1}$ for some diagonal matrix D. The condition that $1 = \det(A) = \det(D)$ tells us that $D = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ for some $\theta \in [0, 2\pi)$. In general, we may have that $U \notin SU(2)$. But since U is unitary, we know $\det(U) = e^{i\alpha}$ for some $\alpha \in [0, 2\pi)$. This lets us choose $U' = e^{-i\alpha/2}U$ and get that

$$U'(U')^* = (e^{-i\alpha/2}U)(e^{i\alpha/2}U^*) = UU^* = I \qquad U'DU'^{-1} = UDU^{-1} = A.$$

We also then have that $\det(U') = 1$, so we see that every $A \in SU(2)$ is conjugate to a matrix of the form D. Since conjugate matrices must have the same eigenvalues, we conclude that each conjugacy class in $SU(2)$ is specified by a parameter $\theta \in [0, 2\pi)$, and generated by a matrix $D_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$.

3.2 Some Irreducible Representations of $SU(2)$

To begin with, we will give an explicit construction of some irreducible representations of $SU(2)$. We will eventually see that, up to isomorphism, such representations exhaust our search.

Take V_n for $n \in \mathbb{Z}_{\geq 0}$ to be the $n+1$ dimensional vector space of homogeneous polynomials in two variables, z_1 and z_2, over $\mathbb{C}$. That is, each $f \in V_n$ is of the form

$$f(z_1, z_2) = a_0 z_1^n + a_1 z_1^{n-1} z_2 + \cdots + a_{n-1} z_1 z_2^{n-1} + a_n z_2^n,$$

where $a_j \in \mathbb{C}$ for $0 \leq j \leq n$. We then define the representation (π_n, V_n) of $SU(2)$ on $A = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(2)$ by

$$(\pi_n(A) f)(z_1, z_2) = f\left(A^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)^T = f(\overline{\alpha} z_1 - \beta z_2, \overline{\beta} z_1 + \alpha z_2)$$

If $A, A' \in SU(2)$, we easily see that $\pi_n(AA')f = \pi_n(A)(\pi_n(A')f)$. The fact that each $\pi_n(A)$ is linear is also easily checked. We also need to check that the result is again a homogenous polynomial in z_1 and z_2, but this is a consequence of the fact that when we expand $(\overline{\alpha} z_1 - \beta z_2)^k (\overline{\beta} z_1 + \alpha z_2)^{n-k}$ we will only ever get terms of total degree n. Thus, we have a well-defined representation.

Suppose we have some $SU(2)$-invariant subspace W of V_n which contains a non-zero f. Then $g_\alpha(z_1, z_2) = f(\alpha z_1, \alpha^{-1} z_2)$ for $|\alpha| = 1$ is also in W, as are linear combinations of elements of this form. Each g_α is equal to f but with each term $a_k z_1^k z_2^{n-k} \mapsto a_k \alpha^{2k-n} z_1^k z_2^{n-k}$. Choosing the α such that $\alpha^{2k-n} = -1$, we can find g_α such that $f + g_\alpha$ has one less term than f. Inductively, we can find an element of W which has but a single term, which by choosing α to cancel the coefficient we can suppose is a monomial of the form $z_1^k z_2^{n-k}$.

Now consider $A = \begin{pmatrix} \gamma & i\delta \\ i\delta & \gamma \end{pmatrix}$ where $\gamma, \delta \in \mathbb{R}$ and $\gamma^2 + \delta^2 = 1$. We have that $A \in SU(2)$, and acting on $z_1^k z_2^{n-k}$ we get

$$\begin{aligned}(\gamma z_1 - i\delta z_2)^k (-i\delta z_1 + \gamma z_2)^{n-k} &= \left(\sum_{i=0}^{k} \gamma^i (-i\delta)^{k-i} z_1^i z_2^{k-i}\right)\left(\sum_{j=0}^{n-k} (-i\delta)^j \gamma^{n-k-j} z_1^j z_2^{n-k-j}\right) \\ &= \sum_{l=0}^{n}\left(\sum_{i+j=l} (i\delta)^{k+j-i} \gamma^{n-k-j+i}\right) z_1^l z_2^{n-l} \\ &= \sum_{l=0}^{n} (i\delta)^k \gamma^{n-k}\left(\sum_{i=0}^{l} (i\delta)^{l-2i} \gamma^{2i-l}\right) z_1^l z_2^{n-l}\end{aligned}$$

Consider the situation when $\gamma \to 1$. The coefficient of the $z_1^l z_2^{n-l}$ term approaches

$$(i\delta)^k \sum_{i=0}^{l} (i\delta)^{l-2i} = (i\delta)^{k+l} \sum_{i=0}^{l} (i\delta)^{-2i} = (i\delta)^{k+l} \frac{(i\delta)^{-2} - (i\delta)^{-2(l+1)}}{1 - (i\delta)^{-2}}.$$

This is non-zero for $\delta \neq 0$ and away from ± 1, so there is some value of (γ, δ) in a neighbourhood of $(1, 0) \in S^1 \subset \mathbb{R}^2$ such that none of the coefficients of $\pi_n(A)(z_1^k z_2^{n-k})$ are zero. This shows that there is an element of W all of whose terms are non-zero. By applying the cancellation procedure we used to turn f into a monomial, we can thus obtain any monomial in W. This shows that W contains all the monomials, and thus we must have $W = V_n$. Since W was an arbitrary $SU(2)$-invariant subspace of V_n, we have proved:

Proposition: The representation (π_n, V_n) of $SU(2)$ is irreducible for all $n \in \mathbb{Z}_{\geq 0}$.

We make one final observation about the representations (π_n, V_n). We showed earlier that the conjugacy classes of $SU(2)$ are indexed by $\theta \in [0, 2\pi)$ and generated by elements of the form $D_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$. Taking the monomials $z_1^k z_2^{n-k}$ as our basis for V_n, we can see that $\pi_n(D_\theta)(z_1^k z_2^{n-k}) = e^{i\theta(2k-n)} z_1^k z_2^{n-k}$, and so the eigenvalues of $\pi_n(D_\theta)$ are $\{e^{i\theta(2k-n)}\}_{k=0}^n$, and the character for π_n takes the form

$$\chi_n(A) = \chi_n(\theta(A)) = \sum_{k=0}^{n} e^{i\theta(2k-n)}. \tag{1}$$

where $\theta(A)$ is the associated θ-value for the conjugacy class of A.

3.3 The Classification Theorem

We now prove our main theorem.

Theorem: Every finite-dimensional irreducible unitary representation of $SU(2)$ is isomorphic to (π_n, V_n) for some $n \in \mathbb{Z}_{\geq 0}$.

Proof: Suppose that (ρ, V) is a finite $(n+1)$-dimensional irreducible unitary representation of $SU(2)$. Consider the group $H := \{D_\theta : \theta \in [0, 2\pi)\} \cong U(1) \subset SU(2)$. We can restrict (ρ, V) to a representation of H, which we can denote by $(\rho|_H, V)$. We know that H is abelian, so its irreducible representations are 1-dimensional. This lets us write

$$(\rho|_H, V) = (\rho_1|_H, \mathbb{C}) \oplus \cdots \oplus (\rho_{n+1}|_H, \mathbb{C}),$$

where each $v \in V$ is written as $v = v_1 \oplus \cdots \oplus v_{n+1}$ and for each $D_\theta \in H$ we know that $(\rho_j|_H(D_\theta))v_j = e^{ik_j\theta} v_j$ for some constant k_j. Because we require our representations to be continuous, we must have $e^{ik_j\theta} \to 1$ when $\theta \to 2\pi$, which tells us that $k_j \in \mathbb{Z}$.

We can then extend $\rho_{k_j}|_H$ to act on all of $SU(2)$ by simply defining it to agree with ρ on v_j and act as the zero map on each v_i for $i \neq j$. We then get that $\rho = \rho_1 \oplus \cdots \oplus \rho_{n+1}$. Because the character of a representation is defined on each conjugacy class of the group, it suffices to consider $(\rho|_H, V)$ to determine χ_ρ. This tells us that $\chi_\rho = \chi_1 + \cdots + \chi_{n+1}$, where χ_j is the character for $(\rho_j|_H, \mathbb{C})$.

As the characters are just the sum of the eigenvalues, we have for $A \in SU(2)$ that

$$\chi_\rho(\theta) := \chi_\rho(D_{\theta(A)}) = \sum_{j=1}^{n+1} e^{ik_j\theta} = \sum_{j=1}^{n+1} (e^{i\theta})^{k_j}. \tag{2}$$

Since χ_ρ acts on conjugacy classes, its action should be invariant under conjugation by $P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We have that

$$PD_\theta P^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} = D_{-\theta},$$

which tells us that $\chi_\rho(\theta)$ is symmetric under the interchange $\theta \mapsto -\theta$.

Our goal now is to show that χ_ρ is not orthogonal to all of the characters of the form given in (1). To do this, it suffices to show that χ_ρ is a linear combination of such characters. Both the expressions (1) and (2) can be regarded as Laurent polynomials in $\omega := e^{i\theta}$. Furthermore, looking at (1) and considering the symmetry of (2) about θ, we can see that they are both *symmetric* Laurent polynomials which are unchanged under the map $\omega \mapsto \omega^{-1}$. Call the degree of such a polynomial the magnitude of the largest (or smallest) exponent d. The set of symmetric Laurent polynomials of degree at most d forms a vector space under the usual operations we can label L_d. We have that

$$\mathbb{C} = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_d \subset L_{d+1} \subset \cdots$$

The dimension of L_d is $d+1$. In Section 3.2, we exhibited the irreducible representations $(V_0, \pi_0), (V_1, \pi_1), \cdots, (V_d, \pi_d)$ which have characters which are polynomials in L_d. The fact that these are distinct irreducible representations shows that the set of these polynomials is orthonormal, and so those characters form a basis for L_d. Then since $\chi_\rho(\theta)$ is also of this form, it cannot be orthogonal to the characters of the representations exhibited in Section 3.2. This shows that (V, π) is not distinct from the representations we have already seen, and completes the proof.

4 $SU(2)$ in Quantum Physics

The fact that the representation theory of $SU(2)$ is of importance in Physics may seem surprising. We argued in Section 1 that we can expect that the representation theory of groups which represent symmetries of physical systems to occur in Quantum Physics, but what kind of physical situation is symmetric under action by $SU(2)$? One can easily imagine that physical systems could be symmetric under the action of $SO(n)$ for $n = 2$ or $n = 3$; there are many examples of rotationally symmetric systems in physics. But while $SU(2)$ does correspond to a kind of norm-preserving rotation, the matrix elements are complex-valued, making it difficult to imagine how it could play a role in physical theories where measurable quantities are required to be real.

The key observation is that $SU(2)$ is isomorphic to another group, commonly called Spin(3). Spin(3) is what's known as a "double cover" of $SO(3)$ – a larger group where each element of $SO(3)$ is associated to two distinct elements of Spin(3) in a way that behaves smoothly with the differential structure on both. If one imagines moving through Spin(3) embedded in some external space, then the structure of the group dictates that you have to "go around twice" to get back to where you started, whereas one full rotation suffices with $SO(3)$.

The map $\kappa : \text{Spin}(3) \to SO(3)$ which projects back onto $SO(3)$ is a group homomorphism, called a *covering map*. If we have a representation (ρ, V) of $SO(3)$, the composition $\rho \circ \kappa$ gives us a representation of $\text{Spin}(3) \cong SU(2)$. So by classifying representations of $SU(2)$, we have in fact also given a classification of representations of $SO(3)$. In particular, it turns out that the representations (π_n, V_n) for even values of n are also irreducible representations of $SO(3)$.

But if the goal is representations of $SO(3)$, why bother with $SU(2)$? Well, remarkably, the representations (π_n, V_n) for odd n, which are "genuine" representations only of $SU(2) \cong \text{Spin}(3)$, are also physically realized. Physicists typically define a parameter $s := \frac{n}{2}$, called the "spin" of the representation, which gives the $SO(3)$ representations an integer value and gives the Spin(3) representations "half-integer" values. This quantity is also associated to all particles in nature. Protons, neutrons and electrons have spin $s = \frac{1}{2}$; photons, particles of light, carry $s = 1$; the Higgs particle, recently discovered at the LHC, has spin 0; the hypothesized graviton particle is believed to have $s = 2$, and composite particles can have arbitrarily large values of s, in principle. The value of s defines the theory of angular momentum of the particle under consideration — if a particle has spin s, its vector space of angular momentum states will be isomorphic to the representation (π_{2s}, V_{2s}). As distinct observable quantities in quantum physics are typically associated to orthonormal basis elements, the dimensionality of the space dictates the number of spin values we can measure. For the electron, $\dim V_{2s} = 2s + 1 = 2$, so the spin of an electron can take two distinct values, commonly called "up" and "down". For large macroscopic objects, the value of s is enormous, which is said to explain why macroscopic objects appear to have a continuous range of possible angular momenta despite the constituent particles only taking on angular momenta from a finite set of discrete values.

One of the most remarkable consequences of the relationship between representations of $SU(2) \cong \text{Spin}(3)$ and angular momentum is that for half-integer values of s the difference between the two "halves" of Spin(3) are physically realized. Since the state spaces for half-integer spin particles are not representations of $SO(3)$, acting by a full 2π-rotation does *not* return the system to its original state. Rather, it requires a 4π-rotation, two ordinary full rotations, to return the system back to its starting point. In some sense, electrons, protons and neutrons must be rotated by 720 degrees to return to their original form – 360 degrees is not enough.

Far from a mathematical oddity, this effect has experimental consequences. In two celebrated 1975 experiments, researchers directed two beams of of identically prepared neutrons through an apparatus which created an interference pattern.

A magnetic field was then used to rotate just one of the beams by a full 2π rotation, which changed the result [4][5].

5 Conclusion

We have seen that the representation theory of $SU(2)$ plays an important role in Physics. We also saw how the tools developed in the representation theory of finite groups have generalizations to compact groups, in particular a class of compact groups known as compact Lie groups. Although the theory in these cases is relatively well understood, there are many open problems in the representation theory of non-compact Lie groups, and of infinite groups in general. Moreover, many of the symmetries observed in physics, such those described by the Lorentz and Poincare groups, correspond to non-compact infinite groups. Their representations are also important in physics, particularly in Quantum Field Theory. Thus, we can expect that Physics will continue to motivate research in representation theory in the years to come.

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