Class Based Lecture Notes

CO 255: Introduction to Optimization (Advanced Level)

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Preface and Notation

This PDF document includes lecture notes for CO 255 - Introduction to Optimization (Advanced Level) taught by Chaitanya Swamy in Winter 2019.

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Notation

Throughout the course and the notes, unless otherwise is explicitly stated, we adopt the following conventions and notations.

- The university logo is used as a place holder.
- The animation works as expected on latest Adobe PDF reader but it does not work on primitive PDF readers.
- For a finite set S with size n, when we say $\mathbf{x} \in \mathbb{R}^S$ we mean $\mathbb{R}^{|S|} \ni \mathbf{x} = (x_1, \ldots, x_n)$.

Calvin KENT

Start of Lecture 1

Chapter 1 – Introduction to Optimization: Lectures 1–2

Definition 1.0.1: An ***optimization problem*** is the problem $\mathrm{P}_{\min}/\mathrm{P}_{\max}$ of finding the best solution (finding min or maximize) to a function $f(\mathbf{x})$ subject to $\mathbf{x} \in S$. Where

- $f : S \to \mathbb{R}$ is the ***objective function***,
- S is the ***feasible region***,
- $\mathbf{x}$ are the ***variables***.

We denote the point $\overline{\mathbf{x}} \in S$ as the ***feasible solution***. $\triangleleft$

Remark 1.0.2: $\max f(\mathbf{x})$ subject to $\mathbf{x} \in S = \min -f(\mathbf{x})$ subject to $\mathbf{x} \in S$. $\triangleleft$

1.1 Outcomes of an Optimization Problem

We have the following definitions for various outcomes.

1. If $S = \varnothing$ we say the problem is ***infeasible***.
2. If $S \neq \varnothing$ and $\exists\, \mathbf{x}^* \in S$ such that $\forall\, \overline{\mathbf{x}} \in S$, $f(\mathbf{x}^*) \leq f(\overline{\mathbf{x}}) \quad \forall\, \overline{\mathbf{x}} \in S$, then we say $\mathbf{x}^*$ is an ***optimal solution*** and $f(\mathbf{x}^*)$ is an ***optimal value***. For a program (P), we denote the optimal value of (P) as $\mathrm{OPT}_{(\mathrm{P})}$.
3. If $S \neq \varnothing$ but there are feasible solutions of arbitrarily small objective values, we say the problem is ***unbounded***.

1.2 Classes of Optimization Problems

Historically these problems are referred as *programs*. We classify the optimization problems as follows.

1. **Linear Programs:** $f(\mathbf{x}) = \mathbf{c}^\top \mathbf{x} = \sum_j c_j x_j$ where,

   $$S = \{\mathbf{x} \in \mathbb{R}^n \ \mid \ A\mathbf{x} \leq \mathbf{b} \text{ where } A \in M_{m\times n}(\mathbb{R}), \mathbf{b} \in \mathbb{R}^m\} = \{\mathbf{x} \in \mathbb{R}^n \ \mid \ a_i^\top \mathbf{x} \leq b_i \quad \forall\, i \in [m]\}.$$

   Note that for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $\mathbf{a} \leq \mathbf{b} \iff a_i \leq b_i \quad \forall\, i = 1, \ldots, n$.
2. **Integer (Linear) Programs:** Linear programs where $x \in \mathbb{Z}^n$.
3. **Convex Programs:**

   Definition: A set $S \subseteq \mathbb{R}^n$ is called a ***convex set*** if $\forall\, \mathbf{x}, \mathbf{y} \in S, \forall\, \lambda \in [0, 1], \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in S$. In other words, convex sets are the sets that contain the line which connects any of its two elements. If S is not convex then it is said to be a ***concave set***.

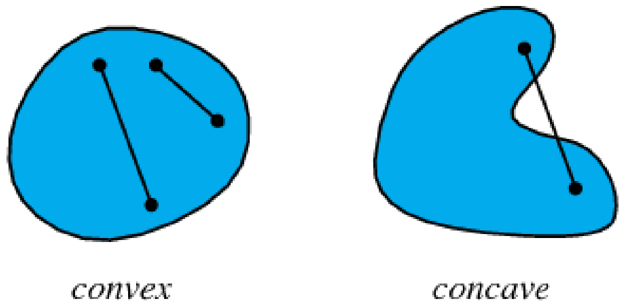


Figure 1.2.1: Convex and concave sets.

We say $f : S \to \mathbb{R}$ is a ***convex function*** if S is convex and $\forall\ \mathbf{x}, \mathbf{y} \in S, \lambda \in [0, 1]$ we have $f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y})$. In other words, we say $f : S \to \mathbb{R}$ is convex if $\forall\ \mathbf{x}, \mathbf{y} \in S$, the line segment joining $f(\mathbf{x}), f(\mathbf{y})$ lies above the graph of f. We say a function f is a ***concave function*** if $-f$ is convex.

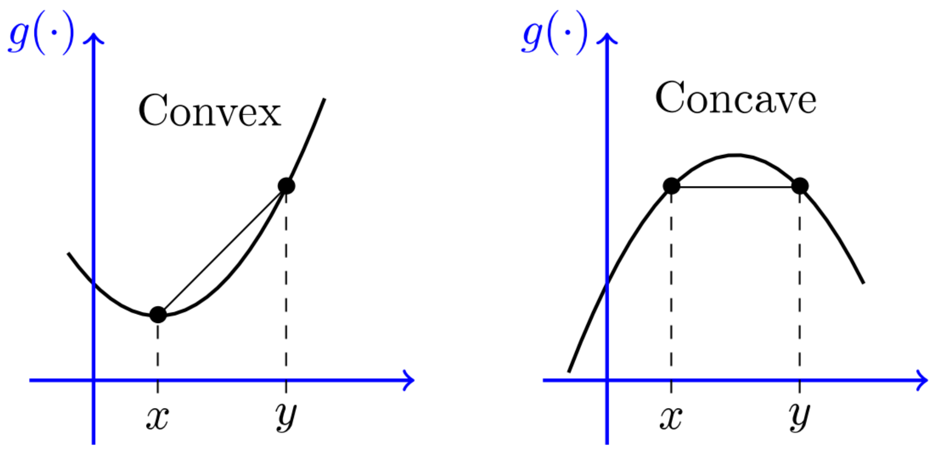


Figure 1.2.2: Convex function and concave functions.

A problem of the form $\min f(\mathbf{x})$ subject to $\mathbf{x} \in S$ where S is convex and f is a convex or a concave function is called a ***convex program***. ◁

Example 1.2.1:

For example, this is a convex program

$$\min f(\mathbf{x}) \quad \text{subject to } \mathbf{x} \in S \quad \text{where} \quad f \quad \text{is convex} \quad \text{and } S \text{ is convex},$$

and its equivalent is

$$\max -f(\mathbf{x}) \text{ subject to } \mathbf{x} \in S \quad \text{where} \quad -f \quad \text{is concave} \quad \text{and } S \text{ is convex}.$$

Each of these two programs are equivalents. ◁

1.3 Examples of Optimization Problems

1.3.1 Transportation Problem

Consider the scenario where a company has a set of F centers and C clients.

Each center $i \in F$ can supply u_i units and each client $j \in C$ demands d_j units.

Shipping from center i to client j costs \$$c_{ij}$ per unit. We want to find a minimum cost solution. We assign variables $x_{ij}\ \forall\, i, j \in F, C$ (units sent from $i \to j$). We want to minimize the cost so our problem is

$$\underbrace{\min \sum_i \sum_j c_{ij} x_{ij}}_{\text{minimize cost}} \text{ subject to } \underbrace{\sum_{j \in C} x_{ij} \le u_i \quad \forall\, i}_{\text{cannot exceed supply}} \text{ and } \underbrace{\sum_{i \in F} x_{ij} \le d_i \quad \forall\, i}_{\text{must meet demand}}.$$

End of Lecture 1

Start of Lecture 2

1.4 Examples of Optimization Problems (continued)

1.4.1 2-player Game

Consider a game with 2-players Rose (R) and Colin (C) with a known matrix $A \in \mathbb{R}^{m\times n} = (a_{ij})$ where $i = 1, \dots, m$ and $j = 1, \dots, n$.

- R's strategy is to choose a row $i \in \{1, \dots, m\}$.
- C's strategy is to choose a column $j \in \{1, \dots, n\}$.
- If R chooses i, and C chooses j then Rose pays Colin amount a_{ij}.

Example 1.4.1: Let $A = \begin{bmatrix} 5 & -2 \\ 1 & 6 \end{bmatrix}$.

If C chooses $j = 1$, we can guarantee a payoff of 1 =min-entry in column j. No matter what R chooses.

If C chooses $j = 2$, we can guarantee a payoff of $= -2$, no matter what R chooses.

If C chooses $j = 1$ with probability $\frac{1}{2}$ (and $j = 2$ with same probability), then C would see the following expected payoffs under R's choices.

$$\mathbf{v} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \implies A\mathbf{v} = \begin{bmatrix} \frac{1}{2}\cdot 5 + \frac{1}{2}(-2) = 1.5 \\ \\ \frac{1}{2}\cdot 1 + \frac{1}{2}(\ \ 6) = 3.5 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 3.5 \end{bmatrix}.$$

Now C guarantees a payoff of 1.5 no matter what R does. In general, what is Colin's best randomized strategy?

C wants to choose some probabilities $p_1, \dots, p_n \geq 0$ such that $\sum p_j = 1$. C wants to maximize

$$\min_{i=1\dots m} \left(\underbrace{\sum_{j=1}^{n} a_{ij} p_j}_{(Ap)_i} \right) \quad \text{(payoff if } R \text{ chooses row } i).$$

We can think C's strategy as "Colin wants to maximize whatever he can get". C's problem becomes

$$\max \min_{i=1\dots m} \left(\sum_{j=1}^{n} a_{ij} p_j \right) \text{ such that } \sum p_j = 1 \text{ where } p = (p_1, \dots, p_n)^\top \geq 0.$$

We can think this problem as maximizing a variable v as follows

$$C\text{'s linear problem } (C - LP) \equiv \max v \text{ where } v \leq \sum_{j=1}^{n} a_{ij} p_j.$$

R's problem is to choose a randomized strategy, in other words, if we denote the probabilities of R's as $q_1, \dots, q_m \geq 0$ such that $\sum q_j = 1$, then R's problem is to minimize

$$\max_{j=1\dots n} \left(\underbrace{\sum_{i=1}^{n} a_{ij} q_i}_{(q^\top A)_j} \right) \quad \text{(payoff if } C \text{ chooses column } j).$$

We can think of R's strategy as "Rose wants to minimize her loses". Similarly, we can think this problem as minimizing the variable w as

$$R\text{'s linear problem } (R - LP) \equiv \min w \text{ where } w \geq \sum_{i=1}^{n} a_{ij} q_i. \qquad \triangleleft$$

Remark 1.4.2:

① $(C - LP)$ is feasible and unbounded, in fact $(C - LP)$ has an optimal solution (same holds for $(R - LP)$).

② Suppose (v, p) is a feasible solution to $(C - LP)$ (similarly with (w, q) for $(R - LP)$).

$$w \geq \max_{j=1\dots n} \left(\sum_{i=1}^{m} a_{ij} q_i \right) \geq \min_{i=1\dots m} \left(\sum_{j=1}^{n} a_{ij} p_j \right) \geq v.$$

This is similar to relationship $\inf \leq \sup$.

③ Optimal values of $C - LP$ and $R - LP$ are equal, that is $v = w$.

Note that ② and ③ are consequences of $LP-$Duality. $\triangleleft$

1.4.2 General 2-Person Game

Consider a game with 2 matrices $A, B \in \mathbb{R}^{m \times n}$ giving the payoffs for C and R respectively.

If R plays i, and C plays j then C gets payoff a_{ij} and R gets payoff b_{ij}. . Recall, previously we had $B = -A$ (zero sum game).

How should R, C choose their randomized strategies $\mathbf{q} \in \mathbb{R}^m, \mathbf{p} \in \mathbb{R}^n$?

Definition 1.4.3: We say $(\mathbf{p}, \mathbf{q})$ is ***an equilibrium*** if no player has an incentive to deviate even if other player's strategy is revealed. Equilibrium always exists as a consequence of John Nash's theorem. $\triangleleft$

For R's case, given $\mathbf{p}$, R sees the expected payoffs $\begin{bmatrix} (B\mathbf{p})_1 \\ (B\mathbf{p})_2 \\ \vdots \\ (B\mathbf{p})_m \end{bmatrix}$. So for R to not deviate from $\mathbf{q}$, R's expected payoff under $\mathbf{q}$ should be equal to

$$\begin{bmatrix} (B\mathbf{p})_1 \\ (B\mathbf{p})_2 \\ \vdots \\ (B\mathbf{p})_m \end{bmatrix} = \max_{i=1\ldots m} (B\mathbf{p})_i.$$

In other words,

$$\mathbf{q}^\top B\mathbf{p} = \max_{i\ldots m}(B\mathbf{p})_i \equiv \mathbf{q}^\top B\mathbf{p} \geq (B\mathbf{p})_i \quad \forall\, i = 1, \ldots, m \tag{1}$$

For C's case, given $\mathbf{q}$, C sees the expected payoffs $((\mathbf{q}^\top A)_1, \ldots, (\mathbf{q}^\top A)_n)$. So for C to not deviate from $\mathbf{p}$, C's expected payoff under $\mathbf{p}$ must be equal to

$$\mathbf{q}^\top A\mathbf{p} = \max_{j=1\ldots n}(\mathbf{q}^\top A)_j \equiv \mathbf{q}^\top A\mathbf{p} \geq (\mathbf{q}^\top A)_j \quad \forall\, j = 1, \ldots, n \tag{2}$$

Finding an Equilibrium: Feasible solution always exists (1),(2)

$$\sum_{j=1}^{n} p_j = 1, \quad \mathbf{p} \geq \mathbf{0}, \tag{3}$$

$$\sum_{j=1}^{m} q_i = 1, \quad \mathbf{q} \geq \mathbf{0} \tag{4}$$

We want to find an equilibrium that maximizes the payout to both players. i.e. we want an equilibrium that makes total payoff to the players

$$\max \mathbf{p}^\top (A+B)\mathbf{q} \quad \text{subject to constraints (1)-(4).}$$

Note that $\mathbf{p}^\top(A+B)\mathbf{q}$ is not a concave function of $\mathbf{p}$ and $\mathbf{q}$.

1.4.3 Fair Division

Consider a game with n players $i = 1, \ldots, n$ and m items $j = 1, \ldots m$. 1 unit of each item j, divisible, assigning $x-$fraction of j to i gives player i utility $= u_{ij} \cdot x$. We want a fair assignment of items to players.

Fair assignment: We want to find an assignment maximizing the product of player utilities, that is $\prod$(utility of i). We assign variables x_{ij} for the fraction of item j given to player i.

$$\max \prod_{i=1}^{n} \left(\sum_{j=1}^{m} u_{ij} x_{ij} \right) \text{ subject to } \sum_{i=1}^{n} x_{ij} \leq 1 \text{ where } x \geq 0.$$

To get a convex program, we write our program as

$$\max \sum_{i=1}^{n} \underbrace{\ln \left(\sum_{j=1}^{m} u_{ij} x_{ij} \right)}_{\text{concave function}} \text{ subject to } \sum_{i=1}^{n} x_{ij} \leq 1 \text{ where } x \geq 0.$$

1.4.4 Job Assignment Problem

Consider n workers and n jobs with following conditions:

- Assigning job j to worker i costs \$$c_{ij}$.
- Each job must be assigned to one worker.
- Each worker can be assigned only 1 job.

We want to find a min-cost assignment. We assign binary variables

$$x_{ij} = \delta_j^i \quad \forall\, i, j = 1, \ldots, n$$

and we try to solve for

$$\begin{aligned} \min \sum_{i,j} c_{ij} x_{ij} \text{ such that } & \sum_{j=1}^{n} x_{ij} = 1 \quad \forall\, i = 1, \ldots, n \\ & \sum_{i=1}^{n} x_{ij} = 1 \quad \forall\, j = 1, \ldots, n \text{ where } x_{ij} = 0 \text{ or } 1. \end{aligned}$$

Remark: There exists an optimal solution $\mathbf{x}^*$ to LP obtained by dropping "x_{ij} integer $\forall\ i, j$" constraints that satisfies x_{ij}^* $\forall\, i, j$. ◁

End of Lecture 2

Start of Lecture 3

Chapter 2 – Linear Programming (LP): Lectures 3–6

Recall 2.0.1: A ***linear program*** (LP) is a problem of the form:

$$\max / \min \mathbf{c}^\top \mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b}, \text{ where}$$

- $\mathbf{x} = (x_1, \ldots, x_n)^\top$ and $A \in M_{m\times n}$ and,
- we have a linear objective function and,
- we have a finite number of linear constraints.

We can incorporate $\leq$ (less than or equal to) or $\geq$ (greater than or equal to) constraints but we do not allow strict inequalities. e.g. $\mathbf{x} < \mathbf{0}$. This is because in strict inequality case $\mathbf{x}$ can be very close to the limit (arbitrarily close to 0) but can never get to the limit. ◁

Notation 2.0.2: For a matrix $A \in M_{m\times n}$ we denote the rows ($a_i^\top$) and columns (A_j) of A as follows.

$$A = \begin{bmatrix} — a_1^\top — \\ — a_2^\top — \\ \vdots \\ — a_m^\top — \end{bmatrix} = \begin{bmatrix} A_1 & | & A_2 & | & \cdots & | & A_n \end{bmatrix}.$$

◁

Remark 2.0.3: Every LP can be converted into an equivalent LP of the following form:

$$\max \mathbf{c}^\top \mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}.$$

This form is called the ***standard inequality form*** (SIF). Note that this conversion may introduce extra variables and constraints. ◁

2.1 Feasibility of a Linear System

Our goal is to characterize when a system of linear inequalities ($A\mathbf{x} \leq \mathbf{b}$) is infeasible. We consider the equality case as special case.

Special Case: Suppose we have a system

$$A_{m\times n}\mathbf{x} = \mathbf{b} \text{ where } \mathbf{x} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^m.$$

From linear algebra we know that

$$\begin{aligned} \mathbf{b} \notin \mathrm{Col}(A) &\iff \exists\, \mathbf{y} \text{ such that } \mathbf{y}^\top A_j = 0 \;\forall\, j = 1, \ldots, n \text{ and } \mathbf{y}^\top \mathbf{b} \neq 0, \\ &\iff \exists\, \mathbf{y} \text{ such that } \mathbf{y}^\top A = 0, \mathbf{y}^\top \mathbf{b} < 0. \end{aligned}$$

2.1.1 Fourier-Motzkin Elimination

The idea is to repeatedly eliminate a variable to determine feasibility of a linear system $A\mathbf{x} \leq \mathbf{b}$.

Example 2.1.1: Consider the following system

$$\begin{aligned}
2x_1 + x_2 + x_3 &\leq 5, && (1)\\
-x_1 + 3x_2 + 2x_3 &\leq 6, && (2)\\
3x_1 - x_2 &\leq 0, && (3)\\
x_1 - 2x_2 - x_3 &\leq -2, && (4)
\end{aligned}$$

We want to eliminate x_3 and get a new equivalent system. We have the relation

$$\begin{aligned}
(1),(2) &\implies x_3 \leq \min\left\{5 - 2x_1, x_2, \frac{6 + x_1 - 3x_2}{2}\right\},\\
(4) &\implies x_3 \geq x_1 - 2x_2 + 2.
\end{aligned}$$

For $(1) - (4)$ to be feasible we need

$$x_1 - 2x_2 + 2 \leq x_3 \leq \min\left\{5 - 2x_1, x_2, \frac{6 + x_1 - 3x_2}{2}\right\} \text{ and } 3x_1 - x_2 \leq 0.$$

In other words, for $(1) - (4)$ to be feasible, we need x_1, x_2 to satisfy

$$\begin{aligned}
x_1 - 2x_2 + 2 &\leq 5 - 2x_1 - x_2,\\
x_1 - 2x_2 + 2 &\leq \frac{6 + x_1 - 3x_2}{2},\\
3x_1 - x_2 &\leq 0.
\end{aligned}$$

Note that the system is obtained from the non-negative linear combinations of the system in $(1) - (4)$. ◁

Formally, given a system $A\mathbf{x} \leq \mathbf{b}$ with $A \in M_{m\times n}$, let

$$\begin{aligned}
I_+ &= \{i \in \{1, \ldots, m\} \mid a_{in} > 0\},\\
I_- &= \{i \in \{1, \ldots, m\} \mid a_{in} < 0\},\\
I_0 &= \{i \in \{1, \ldots, m\} \mid a_{in} = 0\}.
\end{aligned}$$

For all $k \in I_+$ and $\ell \in I_-$, consider the inequality and scale it by $\frac{1}{a_{kn}}$ and $\frac{1}{|a_{\ell n}|}$.

$$\begin{matrix} a_k^\top \mathbf{x} \leq b_k, \\ a_\ell^\top \mathbf{x} \leq b_\ell. \end{matrix} \quad \longrightarrow \quad \underbrace{\begin{matrix} \left[a_k^\top \mathbf{x} \leq b_k\right] \cdot \dfrac{1}{a_{kn}}, \\ \left[a_\ell^\top \mathbf{x} \leq b_\ell\right] \cdot \dfrac{1}{|a_{\ell n}|}. \end{matrix}}_{(k-\ell)-\text{inequality}}$$

We denote the inequality on the right as $(k - \ell) -$ inequality.

Remark 2.1.2: $(k - \ell) -$ inequality does not involve any x_n. ◁

Form the new system $A'\mathbf{x} \leq \mathbf{b}'$ consisting of

- $a_i^\top \leq b_i \quad \forall\, i \in I_0$,
- all $(k-\ell)$ – inequalities.

By construction x_n does not appear in $A'\mathbf{x} \leq \mathbf{b}'$. Successively applying this method we can eliminate all n variables which gives us the system $A^0\mathbf{x} \leq \mathbf{b}^0$ where

$$A^0 = \mathbf{0}_{m\times n} \text{ is feasible} \iff \mathbf{b}^0 \geq \mathbf{0}.$$

2.1.1.1 Properties of Fourier-Motzkin Elimination

Suppose we go from $A\mathbf{x} \leq \mathbf{b}$ to $A'\mathbf{x} \leq \mathbf{b}'$ by eliminating x_n as shown. Then we have the following properties:

P1 If $I_+ = \varnothing$ or if $I_- = \varnothing$ then there are no $(k-\ell)$ inequalities.

P2 Every inequality of $A'\mathbf{x} \leq \mathbf{b}'$ is a non-negative linear combination of inequalities of $A \leq \mathbf{b}$.

P3 The $(k-\ell)$ inequalities can be equivalently viewed as follows:

- Every $k \in I_+$ gives the upper bound (UB)

$$x_n \leq \frac{b_k - \sum_{j=1}^{n-1} a_{kj}x_j}{a_{kn}}.$$

- Every $\ell \in I_-$ gives the lower bound (LB)

$$x_n \geq \frac{b_\ell - \sum_{j=1}^{n-1} a_{\ell j}x_j}{a_{\ell n}}.$$

So we must have

$$\underbrace{\max_{\ell\in I_-}\left(\frac{b_\ell - \sum_{j=1}^{n-1} a_{\ell j}x_j}{a_{\ell n}}\right) \leq \min_{k\in I_+}\left(\frac{b_k - \sum_{j=1}^{n-1} a_{kj}x_j}{a_{kn}}\right)}_{(\star)} \equiv_{\substack{\forall\, k\in I_+,\\ \forall\, \ell\in I_-,}} \frac{b_\ell - \sum_{j=1}^{n-1} a_{\ell j}x_j}{a_{\ell n}} \leq \frac{b_k - \sum_{j=1}^{n-1} a_{kj}x_j}{a_{kn}},$$

where $(\star)$ refers to all $(k-\ell)$ inequalities together.

P4 $A'\mathbf{x} \leq \mathbf{b}'$ involves $(n-1)$ variables but the number of constraints could be as large as $\frac{m^2}{4}$. So after k rounds, we can have about a total of $\frac{m^{2k}}{\text{constraints}}$ of inequalities.

Lemma 2.1.3: The system $A\mathbf{x} \leq \mathbf{b}$ is feasible $\iff$ $A'\mathbf{x} \leq \mathbf{b}'$ is feasible.

Exercise 2.1.4: Prove the $\Longrightarrow$ direction. *Hint:* It follows from **P2**. ◁

Proof: $\Longleftarrow$: Suppose $x_1, \ldots, x_{n-1}$ satisfy $A'\mathbf{x} \leq \mathbf{b}'$. Consider the easy cases where any I_+, I_- is empty.

$$\left.\begin{aligned} I_+ = \varnothing &\longrightarrow \text{ can choose } x_n \text{ large enough,}\\ I_- = \varnothing &\longrightarrow \text{ can choose } x_n \text{ small enough,}\end{aligned}\right\} \quad \text{so that} \quad (x_1, \ldots, x_{n-1}, x_n) \text{ satisfy } A\mathbf{x} \leq \mathbf{b}.$$

Now consider I_+, I_- are both non-empty. We know that $x_1, \ldots, x_{n-1}$ satisfy $(\star)$. In other words, the interval

$$J = \left[\max_{\ell \in I_-}\left(\frac{b_\ell - \sum_{j=1}^{n-1} a_{\ell j} x_j}{a_{\ell n}}\right), \min_{k \in I_+}\left(\frac{b_k - \sum_{j=1}^{n-1} a_{kj} x_j}{a_{kn}}\right)\right]$$

is non-empty. So we can choose $x_n \in J$ so that $(x_1, \ldots, x_n)$ satisfy $A\mathbf{x} \leq \mathbf{b}$. $\square$

Next lecture we will state and prove a variation of Farkas' lemma.

End of Lecture 3

Start of Lecture 4

Farkas' lemma has many forms. In today's lecture, we will prove the following.

Theorem 2.1.5 (Farkas' Lemma)**:** Let $A \in M_{m\times n}$.The system $A\mathbf{x}$ is infeasible if and only if $\exists\ \mathbf{y} \geq \mathbf{0} \in \mathbb{R}^m$ such that $\mathbf{y}^\top A = \mathbf{0}$ and $\mathbf{y}^\top \mathbf{b} < 0$.

Exercise 2.1.6: Prove the $\Longleftarrow$ direction. ◁

Proof: $\Longrightarrow$: Let $\mathbf{y} \geq \mathbf{0}$ where $\mathbb{R}^m = \mathbf{y} = (y_1, \ldots, y_m)$. Given $A\mathbf{x} \leq \mathbf{b}$, let $\alpha^\top \mathbf{x} \leq \beta$ be the inequality obtained by the linear combinations of the system

$$\begin{aligned} a_1^\top \mathbf{x} &\leq b_1 \ ] \cdot y_1 \\ a_2^\top \mathbf{x} &\leq b_2 \ ] \cdot y_2 \\ &\vdots \\ a_m^\top \mathbf{x} &\leq b_m \ ] \cdot y_m \end{aligned}$$

Running FME on $A\mathbf{x} \leq \mathbf{b}$, we will eventually derive an inequality $\mathbf{0}^\top \mathbf{x} \leq \beta$ where $\beta < 0$. Showing that $A\mathbf{x} \leq \mathbf{b}$ is infeasible. Every inequality we get by FME is a non-negative linear combination of inequalities of previous system and hence non-negative linear combination of inequalities of $A\mathbf{x} \leq \mathbf{b}$. Hence $\exists\ \mathbf{y} \geq \mathbf{0}$ such that $\mathbf{y}^\top A = \mathbf{0}$ and $\mathbf{y}^\top \mathbf{b} = \beta < 0$. □

Definition 2.1.7: The vector $\mathbf{y} \in \mathbb{R}^m$ that satisfies the condition in Farkas' lemma is called ***certificate of infeasibility***. ◁

Remark 2.1.8: The equivalent versions of Farkas' lemma are included below.

The system		$A\mathbf{x} \leq \mathbf{b}$	$A\mathbf{x} = \mathbf{b}$
has no solution $\mathbf{x} \geq \mathbf{0}$	$\Longleftrightarrow$	$\exists\ \mathbf{y} \geq \mathbf{0} \in \mathbb{R}^m$ s.t $\mathbf{y}^\top A \geq \mathbf{0}$ and $\mathbf{y}^\top \mathbf{b} \leq 0$	$\exists\ \mathbf{y} \in \mathbb{R}^m$ s.t $\mathbf{y}^\top A^\top \geq \mathbf{0}$ and $\mathbf{y}^\top \mathbf{b} < 0$
has no solution $\mathbf{x} \in \mathbb{R}^n$	$\Longleftrightarrow$	$\exists\ \mathbf{y} \in \mathbb{R}^m \geq \mathbf{0}$ s.t $\mathbf{y}^\top A = \mathbf{0}$ and $\mathbf{y}^\top \mathbf{b} \leq 0$	$\exists\ \mathbf{y} \in \mathbb{R}^m$ s.t $\mathbf{y}^\top A^\top = \mathbf{0}$ and $\mathbf{y}^\top \mathbf{b} < 0$.

It is easy to prove these versions by using linear algebra. ◁

Definition 2.1.9: Let $S \subseteq \mathbb{R}^n$. Let $I \subseteq \{1, \ldots, n\}$. For $\mathbf{z} \in \mathbb{R}^n$, write $\mathbf{z} = (\mathbf{x}, \mathbf{y})$ where $\mathbf{x} = (\mathbf{z}_i)_{i\in I}$ and $\mathbf{y} = (\mathbf{z}_i)_{i\notin I}$. We define the ***projection*** of S on $\mathbf{x}$ (sometimes referred as projection of S onto the coordinates $\mathbf{x}$) as

$$\operatorname{proj}_{\mathbf{x}}(S)\{\mathbf{x} \in \mathbb{R}^{|I|} \mid \exists\ \mathbf{y} \in \mathbb{R}^{n-|I|} \text{ such that } \mathbf{z} = (\mathbf{x}, \mathbf{y}) \in S\}.$$

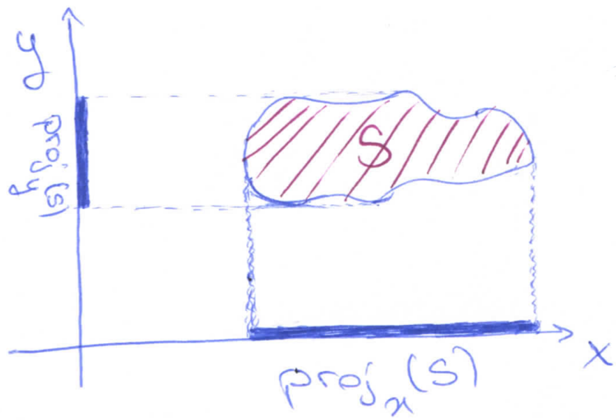


Figure 2.1.1: Projection of S on $\mathbf{x}$ and on $\mathbf{y}$.

◁

Definition 2.1.10: A ***polyhedron*** is a set of the form $\{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$ where $A \in M_{m\times n}(\mathbb{R})$. In other words, a polyhedron is the set corresponding to feasible region of an LP. The plural form of polyhedron is ***polyhedra***. Note that if $m = 1$ then we have a set

$$\{\mathbf{x} \in \mathbb{R}^n \mid \alpha^\top \mathbf{x} \leq \beta \text{ where } \alpha^\top \in \mathbb{R}^n \text{ and } \beta \in \mathbb{R}\}.$$

We call sets of this form a ***half-space***. ◁

Theorem 2.1.11: If $P \subseteq \mathbb{R}^n$ is a polyhedron and $I \subseteq \{1, \ldots, n\}$, then $\text{proj}_{(x_i)_{i\in I}}(\text{P})$ is also a polyhedron.

Proof: Exercise. *Hint:* Proof follows from FME. ◁

2.2 LP Duality

We want to know if we can we prove bounds on optimal value of an LP.

Example 2.2.1: Consider the LP

$$\begin{array}{lll} (\text{P}): \max & 3x_1 + x_2 + 2x_3 & \\ \quad \text{subject to} & 2x_1 - x_2 + x_3 \leq 4 & (1) \\ & x_1 + 2x_2 + x_3 \leq 5 & (2) \\ & x_1 + 2x_3 \leq 3 & (3) \\ \quad \text{with} & x_i \geq 0 \text{ for } i = 1, 2, 3. & \end{array}$$

We get

$$(2) \cdot 3 \implies 3x_1 + 6x_2 + 3x_3 \leq 15.$$

Since all $x_i \geq 0$, then 15 is an upper bound (UB) on the optimal value of (P), $\text{OPT}_{(\text{P})}$. We also have

$$(1) + (2) \implies 3x_1 + x_2 + 2x_3 \leq 9.$$

Hence 9 is an upper bound as well, in fact it's a better upper bound. In general, we can take some $y_i \geq 0$ and get the linear combination of the constraints as

$$y_1(1) + y_2(2) + y_3(3) = (2y_1 + y_2 + y_3)x_1 + (-y_1 + 2y_2)x_2 + (y_1 + y_2 + y_3)x_3 \leq 4y_1 + 5y_2 + 3y_2 \quad (\star)$$

For $(\star)$ to give an upper bound on $3x_1 + x_2 + 3x_3$, we need the constraints to be satisfied. That is,

$$\left.\begin{aligned} (x_1) &: 2y_1 + y_2 + y_3 \geq 3 \\ (x_2) &: -y_1 + 2y_2 \geq 1 \\ (x_3) &: y_1 + y_2 + 2y_3 \geq 2 \\ &\quad y_i \geq 0 \text{ for } i = 1, 2, 3. \end{aligned}\right\} (\dagger)$$

So to find tightest upper bound using this mechanism, we want to solve for

$$\min 4y_1 + 5y_2 + 3y_3 \text{ subject to constraints } (\dagger)$$

This problem is called the *dual of* (P). ◁

Definition 2.2.2: Let be (P) be an LP. The problem of finding the tightest bound on (P) by taking a suitable linear combination of inequalities of (P) is called the ***dual of*** (P). Generally, we denote the given LP as (P) and refer it as the ***primal LP*** (or just primal in short) and we denote the dual of (P) and (D) and call it the dual of (P) (or just dual in short). ◁

Remark 2.2.3:

1. If (P) is a max-LP then (D) is a min-LP.
2. Every primal constraint gives rise to a dual variable.
3. Every primal constraint gives rise to a dual constraint.
4. Dual of (D) is (P).
5. If (P) is primal with dual (D), then we refer (P), (D) as the ***primal-dual pair***. ◁

Example 2.2.4: content... ◁

... will be typed up later. Refer to https://tinyurl.com/y3j5fnwg

End of Lecture 4

Start of Lecture 5

LP Duality (continued)

Recall 2.2.5: Every LP (P) has a dual LP (D) that encodes the problem of finding *best bound* on $\text{OPT}_{(\text{P})}$ via taking a *suitable linear combination* of *constraints* of (P). $\lhd$

Theorem 2.2.6 (Weak Duality Theorem)**:** Let (P) be a max-LP with objective function $\max \mathbf{c}^\top \mathbf{x}$. Let (D) be the dual of (P) which is a min-LP, with objective function $\min \mathbf{b}^\top \mathbf{y}$. Let $\overline{\mathbf{x}}$ be the feasible solution to the primal (P) and $\overline{\mathbf{y}}$ be the feasible solution to (D). Then $\mathbf{c}^\top \mathbf{x} \leq \mathbf{b}^\top \mathbf{y}$.

Proof: We have seen this when constructing (D).

We will show this here when (P) is in standard-inequality-form (SIF):

Let (P) be the LP where

$$\begin{aligned} (\text{P}): \ \max \quad & \mathbf{c}^\top \mathbf{x}, \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b}, \\ \text{with} \quad & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

We multiply the constraints by $\mathbf{y}^\top \geq \mathbf{0}$. We get

$$\mathbf{c}^\top \mathbf{x} \leq (\mathbf{y}^\top A)\mathbf{x} \leq \mathbf{y}^\top \mathbf{b}.$$

We get the dual problem as

$$\begin{aligned} (\text{D}): \ \min \quad & \mathbf{y}^\top \mathbf{b}, \\ \text{subject to} \quad & \mathbf{c}^\top \leq \mathbf{y}^\top A \equiv A^\top \mathbf{y} \geq \mathbf{c}, \\ \text{with} \quad & \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

Since $\overline{\mathbf{x}}$ is feasible to (P) then $\overline{\mathbf{x}} \geq \mathbf{0}$ and $A\overline{\mathbf{x}} \leq \mathbf{b}$. Since $\overline{\mathbf{y}}$ is feasible to (D), then $\overline{\mathbf{y}} \geq \mathbf{0}$ and $A^\top \overline{\mathbf{y}} \geq \mathbf{c}$. Hence we have

$$\mathbf{c}^\top \overline{\mathbf{x}} \leq (A^\top \overline{\mathbf{y}})^\top \overline{\mathbf{x}} = \overline{\mathbf{y}}^\top A \overline{\mathbf{x}} \leq \overline{\mathbf{y}}^\top \mathbf{b} = \mathbf{b}^\top \overline{\mathbf{y}}. \qquad \square$$

Corollary 2.2.7: If (P) is unbounded, then (D) is infeasible. If (D) is unbounded, then (P) is infeasible.

Proof: Exercise. $\lhd$

Lemma 2.2.8: Let (P), (D) be primal-dual pair. If (P) is feasible and (D) is infeasible, then (P) is unbounded.

Proof: Exercise. $\lhd$

Corollary 2.2.9: (P) is unbounded $\iff$ (P) is feasible and (D) is infeasible.

Proof: Exercise. $\lhd$

Theorem 2.2.10 (Strong Duality Theorem)**:** Let (P), (D) be a primal-dual pair. Then the following are true:

1 If (P) has an optimal solution $\mathbf{x}^*$ then so does (D) and $\mathrm{OPT}_{(\mathrm{P})} = \mathrm{OPT}_{(\mathrm{D})}$.

2 If (P) and (D) are both feasible, then they both have opt. solutions and $\mathrm{OPT}_{(\mathrm{P})} = \mathrm{OPT}_{(\mathrm{D})}$.

Note that to prove this theorem we will first prove 2 then show 2 $\implies$ 1.

It is also easy to see 1 $\implies$ 2.

Exercise 2.2.11: Show 1 $\implies$ 2. ◁

Proof (of 2)**:** Suppose both (P) and (D) are feasible where

$$
\begin{array}{lll}
(\mathrm{P}): & \max & \mathbf{c}^\top \mathbf{x}, \\
& \text{subject to} & A\mathbf{x} \le \mathbf{b}, \\
& \text{with} & \mathbf{x} \ge \mathbf{0}.
\end{array}
\qquad
\begin{array}{lll}
(\mathrm{D}): & \min & \mathbf{b}^\top \mathbf{y}, \\
& \text{subject to} & A^\top \mathbf{y} \ge \mathbf{c}, \\
& \text{with} & \mathbf{y} \ge \mathbf{0}.
\end{array}
$$

By weak duality theorem, it suffices to show that the following system is infeasible.

$$
\begin{array}{r}
A\mathbf{x} \le \mathbf{b} \\
\mathbf{x} \ge \mathbf{0} \\
A^\top \mathbf{y} \ge \mathbf{c} \\
\mathbf{y} \ge \mathbf{0} \\
\mathbf{c}^\top \mathbf{x} \ge \mathbf{b}^\top \mathbf{y}
\end{array}
\quad \equiv \quad
\begin{array}{c}
\mathbf{x}, \mathbf{y} \ge \mathbf{0} \\
A\mathbf{x} \le \mathbf{b} \\
-A^\top \mathbf{y} \le -\mathbf{c} \\
-\mathbf{c}^\top \mathbf{x} + \mathbf{b}^\top \mathbf{y} \le \mathbf{0}
\end{array}
\tag{$\star$}
$$

Recall: By Farkas' lemma we have,

$$A'\mathbf{x}' \le \mathbf{b}', \mathbf{x}' \ge \mathbf{0} \text{ is infeasible} \iff \exists\, \mathbf{y}' \ge \mathbf{0} \text{ s.t } \mathbf{y}'^\top A' \ge \mathbf{0} \text{ and } \mathbf{y}'^\top \mathbf{b}' < 0. \quad ◁$$

If $(\star)$ is infeasible, then there exists non-negative $\mathbf{u} \in \mathbb{R}^m, \mathbf{v} \in \mathbb{R}^n, \lambda \in \mathbb{R}$ such that

$$
\begin{array}{c}
\mathbf{u}^\top A - \mathbf{c}^\top \lambda \ge 0 \\
-\mathbf{v}^\top A^\top + \lambda \mathbf{b}^\top \ge 0 \\
\mathbf{u}^\top \mathbf{b} - \mathbf{v}^\top \mathbf{c} < 0
\end{array}
\quad \equiv \quad
\begin{array}{cr}
A^\top \mathbf{u} \ge \lambda \mathbf{c} & (1) \\
A\mathbf{v} \le \lambda \mathbf{b} & (2) \\
\mathbf{c}^\top \mathbf{v} > \mathbf{b}^\top \mathbf{u}. & (3) \\
\mathbf{u}, \mathbf{v}, \lambda \ge \mathbf{0} &
\end{array}
$$

We have two cases: $\lambda > 0$ or $\lambda = 0$.

Case 1: $\lambda > 0$. We have

$$
\left.
\begin{array}{l}
\frac{\mathbf{u}}{\lambda} \ge \mathbf{0},\ A^\top\left(\frac{\mathbf{u}}{\lambda}\right) \ge \mathbf{c} \\
\frac{\mathbf{v}}{\lambda} \ge \mathbf{0},\ A\left(\frac{\mathbf{v}}{\lambda}\right) \le \mathbf{b}
\end{array}
\right\}
\begin{array}{l}
\text{so } \frac{\mathbf{v}}{\lambda}, \frac{\mathbf{u}}{\lambda} \text{ are feasible} \\
\text{solutions to (P), (D)}
\end{array}
$$

But by (3) we have

$$\mathbf{c}^\top\left(\frac{\mathbf{v}}{\lambda}\right) > \mathbf{b}^\top\left(\frac{\mathbf{u}}{\lambda}\right),$$

which contradicts weak duality.

Case 2: $\lambda = 0$. By hypothesis, we know that (P) has a feasible solution $\overline{\mathbf{x}}$ and (D) has a feasible solution $\overline{\mathbf{y}}$. Since $A^\top \overline{\mathbf{y}} \geq \mathbf{c}$ and $\mathbf{v} \geq \mathbf{0}$ then $\mathbf{c}^\top \mathbf{v} \leq (A^\top \overline{\mathbf{y}})^\top \mathbf{v}$. Since $\overline{\mathbf{y}} \geq \mathbf{0}$ and $A\mathbf{v} \leq \mathbf{0}$ (why?) then $\overline{\mathbf{y}}^\top (A\mathbf{v}) \leq 0$. Hence we have

$$\mathbf{c}^\top \mathbf{v} \leq (A^\top \overline{\mathbf{y}})^\top \mathbf{v} = \overline{\mathbf{y}}^\top (A\mathbf{v}) \leq 0.$$

Since $\mathbf{u} \geq \mathbf{0}$ and by inequality (1) and $\overline{\mathbf{x}} \geq \mathbf{0}$ we also have

$$\mathbf{b}^\top \mathbf{u} \geq (A\overline{\mathbf{x}})^\top \mathbf{u} = \overline{\mathbf{x}}^\top (A^\top \mathbf{u}) \geq 0.$$

But we have

$$\mathbf{c}^\top \mathbf{v} \leq 0 \leq \mathbf{b}^\top \mathbf{u},$$

which contradicts (3). Hence ($\star$) is infeasible. Hence, both (P) and (D) have same optimal solutions.

$\square$

Proof (2 $\implies$ 1)**:** Suppose 2 is true and suppose (P) has optimal solution. Then, by FTLP, (P) is feasible and not unbounded. Then, by Lemma 2.2.8, (D) is feasible. Since both primal and the dual are feasible, then by hypothesis $\text{OPT}_{(\text{P})} = \text{OPT}_{(\text{D})}$. $\square$

Theorem 2.2.12 (Fundamental Theorem of Linear Programming)**:** Let (P) be any LP. Then, (P) is either infeasible, unbounded or has an optimal solution.

Proof: Exercise. $\triangleleft$

End of Lecture 5

Start of Lecture 6

2.3 Applications and Interpretations of Duality

Definition 2.3.1: We say an inequality $\alpha^\top \mathbf{x} \le \beta$ where $\alpha, \mathbf{x} \in \mathbb{R}^n, \beta \in \mathbb{R}$ is ***valid*** for a set $S \subseteq \mathbb{R}^n$ if $\alpha^\top \overline{\mathbf{x}} \le \beta \quad \forall\, \overline{\mathbf{x}} \in S$. ◁

Consider a polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \le \mathbf{b}\}$. Then,

$$\begin{aligned}
\alpha^\top \mathbf{x} \le \beta \text{ is valid for } P &\iff \left(\max_{\mathbf{x}\in P} \alpha^\top \mathbf{x}\right) \le \beta \\
&\underset{(\star)}{\iff} \exists \text{ feasible dual and } \mathbf{y} \text{ such that } \mathbf{b}^\top \mathbf{y} \le \beta \\
&\iff \exists\, \mathbf{y} \ge 0, \mathbf{y}^\top A = \alpha^\top.
\end{aligned}$$

Hence, $(\mathbf{y}^\top A)\mathbf{x} \le \mathbf{y}^\top \mathbf{b} \implies \alpha^\top \mathbf{x} \le \beta$. $(\star)$: By strong duality we have dual of $\max\limits_{\mathbf{x}\in P} \alpha^\top \mathbf{x}$ as $\min \mathbf{b}^\top \mathbf{y},\ \mathbf{y} \ge \mathbf{0},\ \mathbf{y}^\top A = \alpha^\top$.

Remark 2.3.2: Strong duality theorem is equivalent to the statement: Every valid inequality for P is implied by an inequality derived via a suitable (i.e. non-negative) linear combination of constraints of P. ◁

2.3.1 Complementary Slackness (C-S) Conditions:

Complementary slackness (C-S) conditions are also referred as *structural characterization of optimal solutions*. Let (P), (D) be primal-dual pair where

$$\begin{aligned}
(\text{P}):\ &\max && \mathbf{c}^\top \mathbf{x}, &\qquad (\text{D}):\ &\min && \mathbf{b}^\top \mathbf{y},\\
&\text{subject to} && A\mathbf{x} \le \mathbf{b}, & &\text{subject to} && A^\top \mathbf{y} \ge \mathbf{c},\\
&\text{with} && \mathbf{x} \ge \mathbf{0}. & &\text{with} && \mathbf{y} \ge \mathbf{0}.
\end{aligned}$$

Let $\overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$ be feasible solutions to (P) and (D) respectively. We have

$$\begin{matrix} \overline{\mathbf{x}} \text{ is optimal solution for (P) and,} \\ \overline{\mathbf{y}} \text{ is optimal solution for (D)} \end{matrix} \underset{(\star\star)}{\iff} \mathbf{c}^\top \overline{\mathbf{x}} = \mathbf{b}^\top \overline{\mathbf{y}}.$$

$(\star\star)$: By weak and strong duality. By weak duality, we also have

$$\mathbf{c}^\top \overline{\mathbf{x}} \le (\overline{\mathbf{y}}^\top A)\overline{\mathbf{x}} = \overline{\mathbf{y}}^\top (A\overline{\mathbf{x}}) \le \overline{\mathbf{y}}^\top \mathbf{b} = \mathbf{b}^\top \overline{\mathbf{y}}.$$

For $\mathbf{c}^\top \overline{\mathbf{x}} = \mathbf{b}^\top \overline{\mathbf{y}}$, we must have

1 $\mathbf{c}^\top \overline{\mathbf{x}} = \overline{\mathbf{y}}^\top A \overline{\mathbf{x}}$.

2 $\overline{\mathbf{y}}^\top A \overline{\mathbf{x}} = \overline{\mathbf{y}}^\top \mathbf{b} \iff \overline{\mathbf{y}}^\top (A\overline{\mathbf{x}} - \mathbf{b}) = 0$.

We have $A \in M_{m\times n}, \overline{\mathbf{x}} \in \mathbb{R}^n, \overline{\mathbf{y}} \in \mathbb{R}^m$. Hence from 1 we get

$$\mathbf{c}^\top \overline{\mathbf{x}} = \overline{\mathbf{y}}^\top A\overline{\mathbf{x}} \iff (\mathbf{c}^\top - \overline{\mathbf{y}}^\top A)\overline{\mathbf{x}} = 0 \iff \sum_{j=1}^{n} \underbrace{\underbrace{\Big(c_j - (\overline{\mathbf{y}}^\top A)_j\Big)}_{\leq 0} \underbrace{\overline{x}_j}_{\geq 0}}_{\leq 0} = 0$$

$$\iff \sum_{j=1}^{n} \Big(c_j - (\overline{\mathbf{y}}^\top A)_j\Big)\overline{x}_j = 0.$$

i.e. $\forall\ j = 1,\ldots,n$ we have $\overline{x}_j = 0$ OR $(\overline{\mathbf{y}}^\top A)_j = c_j$. So $\forall\ j = 1,\ldots,n,\ \overline{x}_j = 0$ OR the dual corresponding to x_j must be tight (hold at equality for $\overline{\mathbf{y}}$) for $\overline{\mathbf{y}}$.

From 2 we get

$$\overline{\mathbf{y}}^\top A\overline{\mathbf{x}} = \overline{\mathbf{y}}^\top \mathbf{b} \iff \overline{\mathbf{y}}^\top (A\overline{\mathbf{x}} - \mathbf{b}) = 0 \iff \sum_{i=1}^{m} \underbrace{\overline{y}_i}_{\geq 0} \underbrace{((A\overline{\mathbf{x}})_i - b_i)}_{\leq 0} = 0$$

$$\iff \sum_{i=1}^{m} \overline{y}_i((A\overline{\mathbf{x}})_i - b_i) = 0.$$

i.e. $\forall\ i = 1,\ldots,m$ we have $\overline{y}_j = 0$ OR $(A\overline{\mathbf{x}})_i = b_i$. So $\forall\ i = 1,\ldots,m,\ \overline{y}_i = 0$ OR the dual corresponding to x_j must be tight (hold at equality for $\overline{\mathbf{x}}$) for $\overline{\mathbf{x}}$.

Theorem 2.3.3 (Complementary-Slackness (CS) Theorem)**:** Let (P), (D) be a primal-dual pair of LPs. Let $\overline{\mathbf{x}}, \overline{\mathbf{y}}$ be a feasible solutions to (P), (D) respectively. Then,

$\overline{\mathbf{x}}$ is an opt. soln. to (P) and, $\overline{\mathbf{y}}$ is an opt. soln. to (D) $\iff$
a: $\forall\ j = 1,\ldots,n, \overline{x}_j = 0$ or corresponding dual constraint is tight for $\overline{\mathbf{y}}$,
b: $\forall\ i = 1,\ldots,m, \overline{y}_i = 0$ or corresponding primal constraint is tight for $\overline{\mathbf{x}}$.

2.3.1.1 Example of Applying CS Conditions

THIS EXAMPLE IS INCOMPLETE

Let (P), (D) be primal-dual pair where

$$\begin{aligned} \text{(P)}:\ \max\quad & 5x_1 + 3x_2 + 5x_3 \\ \text{subject to}\quad & x_1 + 2x_2 - x_3 \leq 2 \\ & 3x_1 + x_2 + 2x_3 \leq 4 \\ & -x_1 + x_2 + x_3 \leq -1 \end{aligned}$$

$$\begin{aligned} \text{(D)}:\ \min\quad & 2y_1 + 4y_2 - y_3 \\ \text{subject to}\quad & y_1 + 3y_2 - y_3 = 5 \\ & 2y_1 + y_2 + y_3 \leq 3 \\ & -y_1 + 2y_2 + y_3 \geq 5 \\ \text{with}\quad & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Multiply (1), (2), (3) by $y_1, y_2, y_3 \geq 0$. We get the dual as

$$\min 2y_1 + 4y_2 - y_3 \quad \text{subject to} \quad y_1, y_2, y_3 \geq 0 \text{ and}$$

$$\begin{aligned} y_1 + 3y_2 - y_3 &= 5 \\ 2y_1 + y_2 + y_3 &\leq 3 \\ -y_1 + 2y_2 + y_3 &\geq 5. \end{aligned}$$

Question: Is $\overline{\mathbf{x}} = (1, -1, 1)^\top$ an optimal solution to (P)?

We first need to verify $\overline{\mathbf{x}}$ is feasible for (P). If $\overline{x}$ is an optimal solution to (P), then $\exists$ dual feasible solution $\overline{\mathbf{y}}$ such that $\overline{\mathbf{x}}, \overline{\mathbf{y}}$ satisfy the CS conditions.

$$\begin{array}{lll lll}
\overline{x}_1 = 0 & \text{or} & \overline{y}_1 + 3\overline{y}_2 - \overline{y}_3 = 5 & \overline{y}_1 = 0 & \text{or} & \overline{x}_1 + 2\overline{x}_2 + -\overline{x}_3 = 2 \\
\overline{x}_2 = 0 & \text{or} & 2\overline{y}_1 + \overline{y}_2 + \overline{y}_3 = 3 & \overline{y}_2 = 0 & \text{or} & 3\overline{x}_1 + \overline{x}_2 + 2\overline{x}_3 = 4 \\
\overline{x}_3 = 0 & \text{or} & -\overline{y}_1 + 2\overline{y}_2 + \overline{y}_3 = 5 & \overline{y}_3 = 0 & \text{or} & -\overline{x}_1 + \overline{x}_2 + \overline{x}_3 = -1.
\end{array}$$

Hence $\overline{\mathbf{y}} = (0, 2, 1)^\top$ *is the only* $\overline{\mathbf{y}}$ that satisfies the CS conditions with $\overline{\mathbf{x}}$. We need to verify this $\overline{\mathbf{y}}$ is feasible for D to show $\overline{x}$ is optimal for the primal and $\overline{\mathbf{y}}$ is optimal for the dual.

$\overline{\mathbf{y}}$ is the unique dual optimal solution. There can be examples that $\overline{\mathbf{y}}$ is not unique.

Chapter 3 – Geometry of Linear Programs: Lectures 6–9

Definition 3.0.1: A set $K \subseteq \mathbb{R}^n$ is called a ***cone*** if K satisfies the following properties:

- $\mathbf{0} \in K$.
- $\forall\, \mathbf{x} \in K, \forall\, 0 \le \lambda \in \mathbb{R}$ we have $\lambda\mathbf{x} \in K$.
- $\forall\, \mathbf{x}, \mathbf{y} \in K$ we have $\mathbf{x} + \mathbf{y} \in K$. $\lhd$

Claim 3.0.2: A cone is a convex set.

Proof: Exercise. $\lhd$

Lemma 3.0.3: An arbitrary intersection of cones is a cone.

Proof: Exercise. $\lhd$

Definition 3.0.4: For $S \subseteq \mathbb{R}^n$ we define cone(S) as ***the smallest cone that contains*** S. It is also called the *cone generated by* S. $\lhd$

Lemma 3.0.5: If $S = \{\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(n)}\} \subseteq \mathbb{R}^n$, then

$$\text{cone}(S) = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \exists\, \lambda_1, \dots, \lambda_k \ge 0 \text{ such that } \mathbf{x} = \sum_{i=1}^{k} \lambda_i \mathbf{a}^{(i)} \right\}$$

In this case, cone(S) is a polyhedron. The linear combination $\sum_{i=1}^{k} \lambda_i \mathbf{a}^{(i)}$ for $\lambda_i \ge 0$ is called ***conic combination*** of $\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(k)}$.

Proof: Exercise. $\lhd$

Example 3.0.6: Consider cone$(\{\mathbf{p}\}) = \{\lambda\mathbf{p} \mid \lambda \ge \mathbf{0}\}$ (this is also called a ray). $\lhd$

Example 3.0.7: The cone generated by vectors $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}$ where

$$\mathbf{a}^{(1)} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{a}^{(2)} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ and } \mathbf{a}^{(3)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

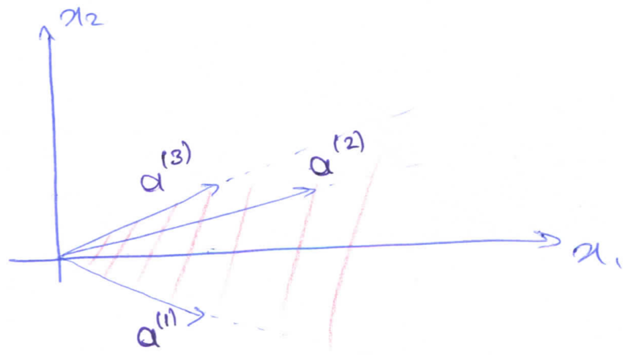


Figure 3.0.1: Cone generated by vectors $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}$ and $\mathbf{a}^{(3)}$.

◁

Example 3.0.8: The explanation for *theorem of the alternative* provided by Gilbert STRANG in Linear Algebra and Its Applications uses FTLA to illustrate a separating hyperplane separating $\mathbf{b}$ from the columns of A. Additional discussion from this Math SE question provides more insight on geometrical interpretation of duality. ◁

Theorem 3.0.9: Let $A \in M_{m\times n}$ with columns A_j for $j = 1, \ldots, n$. Then the following are equivalent.

1. $\mathbf{b} \notin \text{cone}(\{A_1, \ldots, A_n\})$.
2. The system $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$ has no solution.
3. $\exists\ \mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^\top A \geq 0, \mathbf{y}^\top \mathbf{b} < 0$. In other words, hyperplane $\{\mathbf{z} \in \mathbb{R}^m \mid \mathbf{y}^\top \mathbf{z} = 0\}$ separates $\mathbf{b}$ from $\text{cone}(\{A_1, \ldots, A_n\})$

Proof: Exercise. ◁

End of Lecture 6

Start of Lecture 7

3.1 Geometric Interpretation of Farkas' Lemma

Definition 3.1.1: We say that an inequality $\alpha^\top \mathbf{x} \le \beta$ is ***tight*** (or active) at $\overline{\mathbf{x}} \in \mathbb{R}^n$ if $\alpha^\top \overline{\mathbf{x}} = \beta$. ◁

3.1.1 Geometric Statement of Strong Duality/CS Conditions

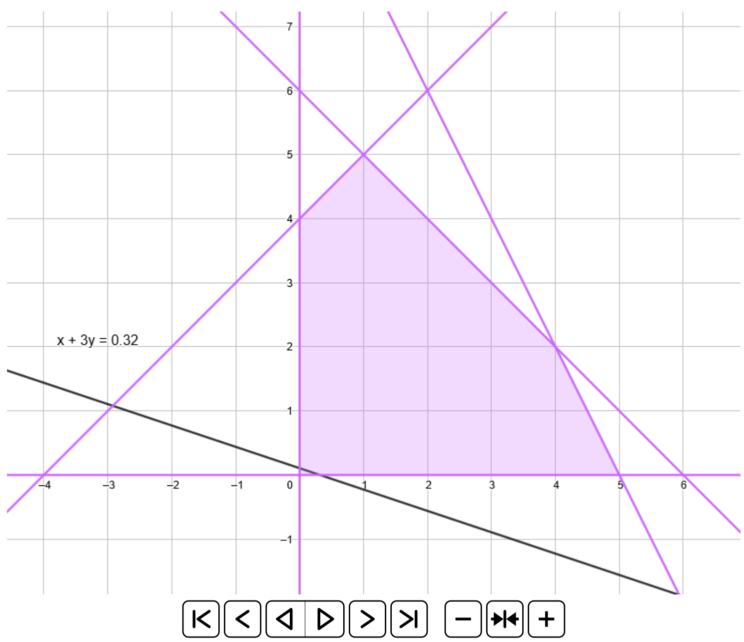
Theorem 3.1.2: Consider LP (P) : $\max \mathbf{c}^\top \mathbf{x}$ subject to $A\mathbf{x} \le \mathbf{b}$. Let $\overline{\mathbf{x}}$ be the feasible solution to (P). Let $A^{=}\mathbf{x} \le \mathbf{b}^{=}$ be the constraints of $A\mathbf{x} \le \mathbf{b}$ that are tight at $\overline{\mathbf{x}}$. Then, $\overline{\mathbf{x}}$ is an optimal solution if and only if $\mathbf{c} \in \text{cone}(\text{rows of } A^{=})$.

Proof: Let (D) be the dual of (P) where (D) : $\min \mathbf{b}^\top \mathbf{y}$ subject to $\mathbf{y}^\top A = \mathbf{c}^\top$, $\mathbf{y} \ge \mathbf{0}$. Then,

$$
\begin{aligned}
&\overline{\mathbf{x}} \text{ is an optimal solution} \\
&\qquad\qquad \iff \exists\, \mathbf{y} \text{ such that } \mathbf{y} \text{ is feasible to dual (D) and } \mathbf{y} \text{ satisfies the CS conditions with } \overline{\mathbf{x}}. \\
&\qquad\qquad \iff \exists\, \mathbf{y} \ge \mathbf{0} \text{ such that } A^\top \mathbf{y} = \mathbf{0} \text{ and } y_i > 0 \implies (A\overline{\mathbf{x}})_i = b_i \\
&\qquad\qquad\qquad \text{i.e. } (A\mathbf{x})_i = b_i \text{ is a constraint of } A^{=}\mathbf{x} \le \mathbf{b}^{=}. \\
&\qquad\qquad \iff \mathbf{c} = \sum_{i:(\star)} a_i^\top y_i \text{ where } (\star) \equiv (A\mathbf{x})_i \le b_i \text{ is a constraint of } A^{=}\mathbf{x} \le \mathbf{b}^{=}. \\
&\qquad\qquad\qquad \text{i.e. } \mathbf{c} \in \text{cone}(S)(\text{rows of } A^{=}).
\end{aligned}
$$
□

Example 3.1.3: Consider the LP, (P), and its geometrical representation.

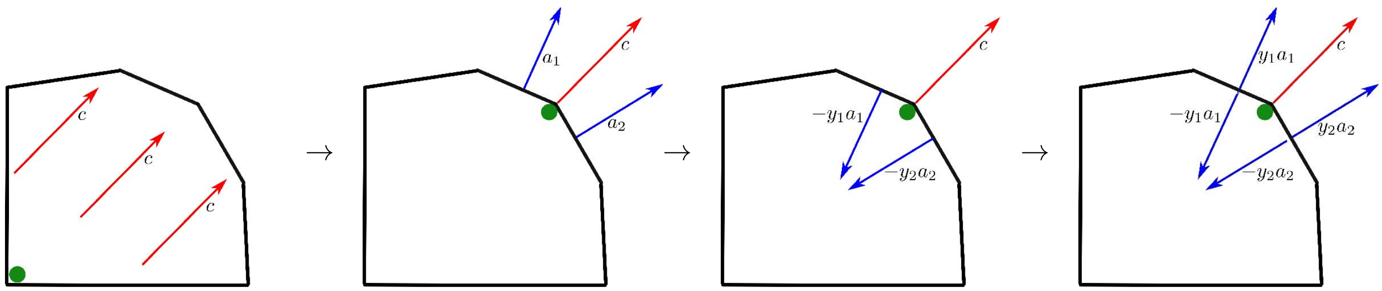
$$
\begin{aligned}
(\text{P}) : \max \quad & x_1 + 3x_2 \\
\text{subject to} \quad & 2x_1 + x_2 \le 10 \\
& x_1 + x_2 \le 6 \\
& -x_1 + x_2 \le 4 \\
\text{with} \quad & x_1, x_2, \ge 0.
\end{aligned}
$$



$\overline{\mathbf{x}}^\top = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ is $\text{OPT}_{(\text{P})}$. Moreover, $\mathbf{c} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \in \text{cone}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ since $\mathbf{c} = 2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. ◁

3.1.2 Physical Interpretation of Duality

Consider (P) : $\max \mathbf{c}^\top \mathbf{x}$ subject to $A\mathbf{x} \le \mathbf{b}$. Think of each hyperplane $a_i^\top = b_i$ as a wall. Let $\mathbf{p}$ be a free particle inside the feasible region subjected to the force $\mathbf{c}$.



(a) Force field $\mathbf{c}$ was applied to the particle inside the polytope $A\mathbf{x} \le \mathbf{b}$.

(b) Particle comes at rest when it reaches the wall.

(c) Wall exerts normal force to the particle.

(d) Since net force is zero, then $\mathbf{c} = \sum_{i:(\star\star)} y_i a_i^\top$.

Figure 3.1.1: Physical interpretation of duality. Source.[1]

The particle reaches equilibrium when the net force acting on it $\mathbf{c}$ is zero. So, if $\overline{\mathbf{x}}$ is resting position, then $\mathbf{c}$ is balanced by normal reaction from *walls that the particle touches* (exactly the constrains of $A\mathbf{x} \le \mathbf{b}$) that are tight at $\overline{\mathbf{x}}$. In other words,

$$-\mathbf{c} = \sum_{i:(\star\star)} y_i(-a_i^\top) \quad \text{where} \quad y_i \ge 0,$$

and where $(\star\star)$: particle rests on wall i. i.e. $\mathbf{c} \in \text{cone(rows of A}^=)$.

Remark 3.1.4: It is easy to see strong duality $\implies$ Farkas' lemma. Consider the system $A\mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge \mathbf{0}$ and suppose it is infeasible. Recall that Farkas' lemma states $A\mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge \mathbf{0}$ is infeasible if and only if $\exists\, \mathbf{y} \ge \mathbf{0}$ such that $\mathbf{y}^\top A \ge \mathbf{0}$ and $\mathbf{y}^\top \mathbf{b} < 0$. Consider the LP, (P) : $\max \mathbf{0}^\top \mathbf{x}$ subject to $A\mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge \mathbf{0}$ and its dual (D) : $\min \mathbf{b}^\top \mathbf{y}$ subject to $\mathbf{y}^\top A \ge \mathbf{0},\ \mathbf{y} \ge \mathbf{0}$. Hence, by strong duality we have

$$\begin{aligned} \text{(P) is feasible} &\iff \text{(D) is unbounded. Note (D) is always feasible } (\mathbf{y} = \mathbf{0}) \\ &\iff \exists\, \mathbf{y} \text{ such that } \mathbf{y} \text{ is feasible to (D) subject to } \mathbf{b}^\top \mathbf{y} < 0. \qquad \triangleleft \end{aligned}$$

3.1.3 Economic Interpretation of Duality and Sensitivity Analysis

Consider the LP

$$\begin{aligned} \text{(P)} : \max \quad & \mathbf{c}^\top \mathbf{x}, \\ \text{subject to} \quad & A\mathbf{x} \le \mathbf{b} \text{ where } A \in M_{m\times n}, \\ \text{with} \quad & \mathbf{x} \ge \mathbf{0} \end{aligned}$$

Assume (P) is modeling some production management problem with the variables

[1]Dughm's notes from USC. https://www-bcf.usc.edu/~shaddin/cs599fa13/slides/lec2.pdf

- x_j : how much of product j to produce.
- Every $i \in [m] \equiv$ resource.
- a_{ij} : number of resource i needed to produce 1 unit of product j.

$$(A\mathbf{x})_i \le b_i \equiv \sum a_{ij}x_j \le b_i \quad \text{where} \quad b_i \text{ is supply of resource } i.$$

Consider the dual of (P).

$$\begin{aligned} \text{(D)}: \min \quad & \mathbf{b}^\top \mathbf{y}, \\ \text{subject to} \quad & A\mathbf{y} \ge \mathbf{c}, \\ \text{with} \quad & \mathbf{y} \ge \mathbf{0}. \end{aligned}$$

Let $\overline{\mathbf{x}}, \overline{\mathbf{y}}$ be optimal solutions for the primal and the dual respectively. Consider perturbation by ε. By strong duality we have

$$\begin{aligned} &\mathbf{c}^\top \overline{\mathbf{x}} = \mathbf{b}^\top \overline{\mathbf{y}} \\ &b_i \to b_i + \varepsilon \\ &\mathbf{b}^\top \overline{\mathbf{y}} \to \mathbf{b}^\top \overline{\mathbf{y}} + \varepsilon \overline{y}_i \text{ where } \overline{y}_i \text{is the amount by which optimal value changes} \\ &b_{i'} \to b_{i'} \ \forall\, i' \ne i. \end{aligned}$$

i.e. $\overline{y}_i$ is the rate of change of optimal value with respect to change change in b_i. $\overline{y}_i$ is sometimes called the *shadow price* of resource i.

3.2 Geometry of Polyhedra

Definition 3.2.1: Let $S \subseteq \mathbb{R}^n$ be a convex set. We say that $\overline{\mathbf{x}}$ is an ***extreme point*** of S (or corner point) if $\overline{\mathbf{x}} \in S$ and there do not exist distinct $\mathbf{u}, \mathbf{v} \in S$ and $\lambda \in (0, 1)$ such that $\overline{\mathbf{x}} = \lambda\mathbf{u} + (1-\lambda)\mathbf{v}$. In other words, $\overline{\mathbf{x}}$ is not contained in the interior of any segment contain in S. This is equivalent to $\forall\, \mathbf{d} \in \mathbb{R}^n,\ \mathbf{d} \ne 0,\ \ \overline{\mathbf{x}} + \mathbf{d} \notin S$ or $\overline{\mathbf{x}} - \mathbf{d} \notin S$. ◁

Remark 3.2.2: In below figure points A, B, C are extreme points and D is not an extreme point.

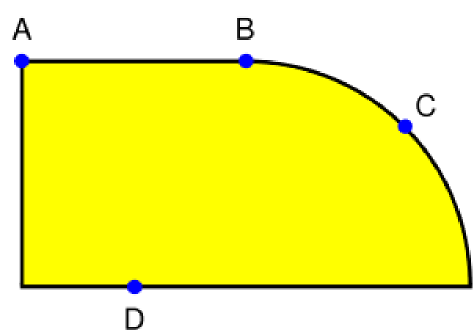


Figure 3.2.1: Extreme points of a closed set.[2]

◁

[2] https://math.stackexchange.com/q/959065

Other Natural ways of Defining Extreme Points

1. Unique intersection of some hyperplanes.
2. Unique optimal solution for some choice of objective function. This interpretation can be visualized with Example 3.1.3. Here $x_1 + 3x_2 = 16$ with $(x_1, x_2) = (1, 5)$ is a unique solution to the objective function $x_1 + 3x_2$ which is also an extreme point.

End of Lecture 7

Start of Lecture 8

3.2.1 Theorems about Extreme Points

Recall 3.2.3: Let $S \subseteq \mathbb{R}^n$ be convex. We say that $\overline{\mathbf{x}} \in \mathbb{R}^n$ is an extreme point of S if $\overline{\mathbf{x}} \in S$ and $\nexists \mathbf{u}, \mathbf{v} \in S, \mathbf{u} \neq \mathbf{v}, \lambda \in (0,1)$ such that $\overline{\mathbf{x}} = \lambda\mathbf{u} + (1-\lambda)\mathbf{v}$. ◁

Theorem 3.2.4: Let $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$ be a polyhedron. Let $\overline{\mathbf{x}} \in P$. Let $A^{=}\mathbf{x} \leq \mathbf{b}^{=}$ be constraints of $A\mathbf{x} \leq \mathbf{b}$ that are tight at $\overline{\mathbf{x}}$. Then, $\overline{\mathbf{x}}$ is an extreme point of P if and only if $\operatorname{rank} A^{=} = n$.

Proof: $\implies$. We prove the contrapositive. Suppose, $\operatorname{rank} A^{=} < n$. Then, $\exists\ \mathbf{d} \in \mathbb{R}^n, \mathbf{d} \neq \mathbf{0}$ such that $A^{=}\mathbf{d} = \mathbf{0}$. We will prove $\overline{\mathbf{x}}$ is not an extreme point. To to this we will show for suitable small $\varepsilon > 0$, both

$$\mathbf{x}^{(1)} = \overline{\mathbf{x}} + \varepsilon\mathbf{d} \quad \text{and} \quad \mathbf{x}^{(2)} = \overline{\mathbf{x}} - \varepsilon\mathbf{d}$$

lie in P. We have

$$\left.\begin{aligned} A^{=}\mathbf{x}^{(1)} &= A^{=}\overline{\mathbf{x}} + \varepsilon(A^{=}\mathbf{d}) = A^{=}\overline{\mathbf{x}} \\ A^{=}\mathbf{x}^{(2)} &= A^{=}\overline{\mathbf{x}} - \varepsilon(A^{=}\mathbf{d}) = A^{=}\overline{\mathbf{x}} \end{aligned}\right\} = \mathbf{b}^{=}.$$

So every constraint of $A^{=}\mathbf{x} \leq \mathbf{b}^{=}$ is tight at $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$. Consider a constraint $a_i^\top \mathbf{x} \leq b_i$ that is not tight at $\overline{\mathbf{x}}$.

$$\left.\begin{aligned} a_i^\top \mathbf{x}^{(1)} &= \underbrace{a_i^\top \overline{\mathbf{x}}}_{<\mathbf{b}} + \varepsilon(a_i^\top \mathbf{d}) < b_i + \varepsilon(a_i^\top \mathbf{d}) \\ a_i^\top \mathbf{x}^{(2)} &= \underbrace{a_i^\top \overline{\mathbf{x}}}_{<\mathbf{b}} - \varepsilon(a_i^\top \mathbf{d}) < b_i - \varepsilon(a_i^\top \mathbf{d}) \end{aligned}\right\} \quad \begin{array}{l} \text{We can take } \varepsilon > 0 \text{ small enough} \\ \text{so that RHS } \leq b_i \text{ for all non-tight constraints.} \end{array}$$

Note that in this step we're basically picking ε to offset the difference between $a_i^\top \overline{\mathbf{x}}$ and $\mathbf{b}$.

$\impliedby$. Let $\operatorname{rank} A^{=} = n$ and suppose, for contradiction, $\overline{\mathbf{x}}$ is not an extreme point. Then $\exists\ \mathbf{u}, \mathbf{v} \in P,\ \mathbf{u} \neq \mathbf{v},\ \lambda \in (0,1)$ such that $\overline{\mathbf{x}} = \lambda\mathbf{u} + (1-\lambda)\mathbf{v}$. Then,

$$\begin{aligned} \mathbf{b}^{=} = A^{=}\overline{\mathbf{x}} = A^{=}(\lambda\mathbf{u} + (1-\lambda)\mathbf{v}) &= \underbrace{\lambda}_{>0}\underbrace{(A^{=}\mathbf{u})}_{\leq \mathbf{b}^{=}} + \underbrace{(1-\lambda)}_{>0}\underbrace{A^{=}\mathbf{v}}_{\leq \mathbf{b}^{=}} \\ \implies \mathbf{0} = \underbrace{\lambda}_{>0}\underbrace{(A^{=}\mathbf{u} - \mathbf{b}^{=})}_{\leq \mathbf{0}} + \underbrace{(1-\lambda)}_{>0}\underbrace{(A^{=}\mathbf{v} - \mathbf{b}^{=})}_{\leq \mathbf{0}}. \end{aligned}$$

Here we used the fact that $\mathbf{b}^{=} = \lambda\mathbf{b}^{=} + (1-\lambda)\mathbf{b}^{=}$. Hence, $A^{=}\mathbf{u} = \mathbf{b}^{=} = A^{=}\mathbf{v} \implies A^{=}(\mathbf{v} - \mathbf{u}) = \mathbf{0}$. But if $\mathbf{v} - \mathbf{u} \neq \mathbf{0}$ then $\operatorname{rank} A^{=} < n$, which contradicts the assumption $\operatorname{rank} A^{=} = n$. Hence, $\overline{\mathbf{x}}$ is an extreme point. □

Remark 3.2.5: The system $A^{=}\mathbf{x} \leq \mathbf{b}^{=}$ has $\overline{\mathbf{x}}$ as the unique solution. ◁

Corollary 3.2.6: Every polyhedron has a finite number of extreme points.

Proof: Let $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$ be polyhedron and let $A \in M_{m\times n}$. If $\overline{\mathbf{x}}$ is an extreme point, and $A^{=}\mathbf{x} \leq \mathbf{b}^{=}$ are the tight constraints at $\mathbf{x}$ then $\operatorname{rank} A^{=} = n$ and $\overline{\mathbf{x}}$ is the unique solution to $A^{=}\mathbf{x} = \mathbf{b}^{=}$. Hence,

$$\#\text{ of extreme points} \leq \#\text{ of subsystems } A'\mathbf{x} \leq \mathbf{b}' \text{ of } A\mathbf{x} \leq \mathbf{b} \text{ with } \operatorname{rank} A' = n \leq \binom{m}{n}.$$

In other words, if a set has infinite number of extreme points, then it's not a polyhedron. $\square$

Theorem 3.2.7: Let $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$ be a polyhedron and $\overline{\mathbf{x}} \in P$. Then, $\overline{\mathbf{x}}$ is an extreme point of P if and only if $\exists\, \mathbf{c} \in \mathbb{R}^n$ such that $\overline{\mathbf{x}}$ is a unique optimal solution to $\max \mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{x} \in P$.

Proof: $\implies$. Let $A^= \mathbf{x} \leq \mathbf{b}^=$ are the tight constraints of the system $A\mathbf{x} \leq \mathbf{b}$ at $\overline{\mathbf{x}}$. Then, by Theorem 3.2.4, $\operatorname{rank} A^= = n$, so $\overline{\mathbf{x}}$ is the unique solution to $A^= \mathbf{x} = \mathbf{b}$ Let $a_1^\top \mathbf{x} \leq b_1, \ldots, a_k^\top \mathbf{x} \leq b_k$ be constraints of $A^= \mathbf{x} \leq \mathbf{b}^=$. Take

$$\mathbf{c}^\top = \sum_{i=1}^k a_i^\top.$$

Note that we have

$$\mathbf{c}^\top \mathbf{x} = \sum_{i=1}^k a_i^\top \mathbf{x} \leq \sum_{i=1}^k b_i \quad \forall\, \mathbf{x} \in P.$$

The equality case gives us

$$\mathbf{c}^\top \mathbf{x} = \sum_{i=1}^k b_i \iff a_i^\top \mathbf{x} = b_i \quad \forall\, i = 1, \ldots, k.$$

In other words, $\mathbf{x}$ is a solution to $A^= \mathbf{x} = \mathbf{b}^=$. Hence, $\overline{\mathbf{x}}$ is unique optimal solution to $\max \mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{x} \in P$.

$\impliedby$. Suppose $\overline{\mathbf{x}}$ is a unique optimal solution to $\max \mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{x} \in P$. Suppose, for contradiction, $\overline{\mathbf{x}}$ is not an extreme point. Then,

$$\overline{\mathbf{x}} = \lambda \mathbf{u} + (1-\lambda)\mathbf{v} \quad \text{where} \quad \mathbf{u}, \mathbf{v} \in P, \underbrace{\mathbf{u} \neq \mathbf{v}}_{(\star)}, \lambda \in (0,1).$$

Note that $(\star) \implies \mathbf{u} \neq \overline{\mathbf{x}}$ and $\mathbf{v} \neq \overline{\mathbf{x}}$. Then,

$$\mathbf{c}^\top \overline{\mathbf{x}} = \underbrace{\lambda}_{>0} \underbrace{(\mathbf{c}^\top \mathbf{u})}_{<\mathbf{c}^\top \overline{\mathbf{x}}} + \underbrace{(1-\lambda)}_{>0} \underbrace{\mathbf{c}^\top \mathbf{v}}_{<\mathbf{c}^\top \overline{\mathbf{x}}}$$

which is a contradiction. $\square$

Remark 3.2.8: The Theorem 3.2.7 does not hold for an arbitrary bounded convex set $\triangleleft$

Definition 3.2.9: Let $\overline{\mathbf{x}} \in \mathbb{R}^n, \mathbf{d} \in \mathbb{R}^n, \mathbf{d} \neq \mathbf{0}$. The set

1. $\{\overline{\mathbf{x}} + \lambda \mathbf{d} \mid \lambda \in \mathbb{R}\}$ is called a ***line***.
2. $\{\overline{\mathbf{x}} + \lambda \mathbf{d} \mid 0 \leq \lambda \in \mathbb{R}\}$ is called a ***ray***.

We say that $P \subseteq \mathbb{R}^n$ has a line if $\exists\, \overline{\mathbf{x}} \in P, \mathbf{d} \in \mathbb{R}^n, \mathbf{d} \neq \mathbf{0}$ such that $\{\overline{\mathbf{x}} + \lambda \mathbf{d} \mid \lambda \in \mathbb{R}\} \subseteq P$. $\triangleleft$

Definition 3.2.10: We call the polyhedra that do not contain a line ***pointed polyhedra***. $\triangleleft$

Theorem 3.2.11: Let P be non-empty and $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$. Then, the following are equivalent.

1. P has a line.

② $\exists\, \mathbf{d} \neq \mathbf{0}$ such that $A\mathbf{d} = \mathbf{0}$.

③ $\operatorname{rank} A < n$.

Proof: Exercise. $\triangleleft$

Theorem 3.2.12: Let $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$ be a polyhedron with no line. If the LP: $\max \mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{x} \in P$ has an optimal solution then it always has an optimal solution that is an extreme point of P.

Proof: Let $\overline{\mathbf{x}}$ be an optimal solution to the LP: $\max \mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{x} \in P$ that satisfies the maximum number of inequalities of $A\mathbf{x} \leq \mathbf{b}$ at equality. Let $A^= \mathbf{x} \leq \mathbf{b}^=$ be the tight constraints at $\overline{\mathbf{x}}$. If $\operatorname{rank} A^= = n$, then by Theorem 3.2.4, $\overline{\mathbf{x}}$ is an extreme point and we are done. If $\operatorname{rank} A^= < n$, then $\exists\, \mathbf{d} \neq \mathbf{0}$ such that $A^= \mathbf{d} = \mathbf{0}$. Then,

$$\exists\, \varepsilon > 0 \quad \text{such that} \quad \left.\begin{matrix} \overline{\mathbf{x}} + \varepsilon \mathbf{d} \\ \overline{\mathbf{x}} - \varepsilon \mathbf{d} \end{matrix}\right\} \in P$$

Then, $\mathbf{c}^\top \mathbf{d} = 0$ (otherwise one of $\overline{\mathbf{x}} \pm \varepsilon \mathbf{d}$ will have better objective value than $\overline{\mathbf{x}}$). Hence, all points $\mathbf{x}$ on the line $L = \{\overline{\mathbf{x}} + \lambda \mathbf{d} \mid \lambda \in \mathbb{R}\}$ have $\mathbf{c}^\top \mathbf{x} = \mathbf{c}^\top \overline{\mathbf{x}}$. Since $L \not\subseteq P$, there is a largest or smallest value of λ. Denote this value as λ^*. Note that λ^* satisfies $\overline{\mathbf{x}} + \lambda^* \in P$. Denote $\overline{\mathbf{x}} + \lambda^* = \mathbf{x}'$. Since $\lambda^* \neq 0$ then $\mathbf{x}' \neq \overline{\mathbf{x}}$. $\mathbf{x}'$ satisfies $\mathbf{c}^\top \mathbf{x}' = \mathbf{c}^\top \overline{\mathbf{x}}$ and $A^= \mathbf{x}' = A^= \overline{\mathbf{x}} = \mathbf{b}^=$ and $\mathbf{x}'$ satisfies at least one other constraint of P equality constraints choice of $\overline{\mathbf{x}}$. $\square$

Theorem 3.2.13: Let P be a non-empty polyhedron. Then, P has an extreme point if and only if P is pointed (has no line).

Proof: Let $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}$ be non-empty. Suppose P has an extreme point. Then we have $\operatorname{rank} A = n$. Then, by Theorem 3.2.11, P has no line. Conversely, suppose P has no line. Consider the LP max $\mathbf{0}^\top \mathbf{x}$ subject to $\mathbf{x} \in P$. This LP has an optimal solution. Since $P \neq \varnothing$, then by Theorem 3.2.12, P has an optimal solution that is an extreme point. $\square$

End of Lecture 8

Start of Lecture 9

3.2.2 Finite Generation of Polyhedra

Definition 3.2.14:

- We call a bounded polyhedron as a ***polytope***.
- If a set $K \subseteq \mathbb{R}^n$ is both a cone and a polyhedron, then we call K as a ***polyhedral cone***.

  Theorem 3.2.15: If $K \subseteq \mathbb{R}^n$ is a polyhedral cone, then there exists some matrix $A \in M_{m\times n}$ such that $K = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq 0\}$.

  Proof: Let C be a polyhedral cone. Then, C is a polyhedron and a cone. Then for all $x \in C \subseteq \mathbb{R}^n$, $\exists\ A \in \mathcal{M}_{m\times n}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^n$ such that $A\mathbf{x} \leq \mathbf{b}$. Since $\mathbf{0} \in C$ then $\mathbf{0} \leq \mathbf{b}$. Moreover, since $\forall\ \lambda \leq 0$ we have $\lambda\mathbf{x} \in C$, then $A(\lambda\mathbf{x}) \leq \lambda\mathbf{b} \leq \mathbf{b}$. Then $\mathbf{b} = \mathbf{0}$. Then $C = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{0}\}$. ◁

  Example 3.2.16: An ice-cream cone in $\mathbb{R}^3$ with its tip at origin is a cone but it's not a polyhedral cone. ◁

- **Minkowsky-Sum:** Given sets $S, T \in \mathbb{R}^n$, then we define the ***Minkowsky-Sum*** of S and T as

$$S + T \underset{\text{def}}{=} \{s + t \mid s \in S \text{ and } t \in T\}.$$

  Note that if one of the sets is empty then $S + T = \varnothing$. ◁

Theorem 3.2.17:

- If S, T are polyhedra then $S + T$ is also a polyhedron.
- If S, T are polytopes then $S + T$ is also a polytope.
- If S, T are polyhedral cones then $S + T$ is also a polyhedral cone.

Proof: Exercise. ◁

Theorem 3.2.18: Let $P \subseteq \mathbb{R}^n$. P is a polytope if and only if $P = \text{conv}(S)$ for a finite set $S \subseteq \mathbb{R}^n$. Moreover, if P is a polytope, then we can take S as the set of extreme points of P.

Proof: Suppose $P = \text{conv}(S)$. Then P is a polyhedron (shown in Assignment #1) and bounded (exercise). Then P is a polytope. Conversely, suppose P is a polytope. By Corollary 3.2.6, the set of extreme points of P is a finite set. Take S as the set of all extreme points of P. Clearly we have $P \supseteq \text{conv}(S)$ since $P \supseteq S$, P is convex. To show $P \subseteq \text{conv}(S)$. Suppose, for contradiction, $\exists\ \overline{\mathbf{x}} \in P \setminus \text{conv}(S)$. (check this sentence) Then, $\exists\ \alpha \in \mathbb{R}^n, \beta \in \mathbb{R}$ such that $\alpha^\top\overline{\mathbf{x}} > \beta$ and $\alpha^\top\overline{\mathbf{x}} \leq \beta$ $\forall\ \mathbf{x} \in \text{conv}(S)$. Then,

$$\underbrace{\begin{pmatrix} \max & \alpha^\top\mathbf{x} \\ \text{subject to} & \mathbf{x} \in P \end{pmatrix}}_{\text{LP}-1} > \begin{pmatrix} \max & \alpha^\top\mathbf{x} \\ \text{subject to} & \mathbf{x} \in \text{conv}(S) \end{pmatrix} \tag{$\star$}$$

LP-1 has an optimal solution (from Assignment #2 (since P is a polytope)). But then by Theorem 3.2.12, LP-1 has an optimal solution that is a point in S which contradicts ($\star$) □

Theorem 3.2.19: Let $P \subseteq \mathbb{R}^n$. P is a polyhedral cone if and only if $P = \text{cone}(S)$ for a finite set $S \subseteq \mathbb{R}^n$.

Proof: Suppose $P = \text{cone}(S)$. If $S = \{\mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(k)}\}$ then by Lemma 3.0.5,

$$\text{cone}(S) = \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{q}^{(i)} \;\middle|\; \lambda_1, \ldots, \lambda_k \geq 0 \right\}$$

which is a polyhedral cone. Conversely, suppose P is a polyhedral cone. Take $Q = \{\mathbf{x} \in P \mid -1 \leq x_j \leq 1 \quad \forall\, j = 1, \ldots, n\}$. Q is a polytope, so by Theorem 3.2.18, $Q = \text{conv}(S)$ for a finite set S. Let $S = \{\mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(k)}\} \subseteq \mathbb{R}^n$. We claim that $P = \text{cone}(S)$. Note that we have $P \supseteq Q \supseteq S$. P is a cone, so by definition $P \supseteq \text{cone}(S)$. For the other direction, let $\overline{\mathbf{x}} \in P$. Then, $\exists\, \gamma > 0$ such that $\frac{\overline{\mathbf{x}}}{\gamma} \in Q$. So

$$\frac{\overline{\mathbf{x}}}{\gamma} = \sum_{i=1}^{k} \lambda_i \mathbf{q}^{(i)} \quad \text{where} \quad \lambda_i \geq 0 \text{ and } \forall\, i = 1, \ldots, k \text{ and } \sum_{i=1}^{k} \lambda_i = 1.$$

Hence we have

$$\overline{\mathbf{x}} = \sum_{i=1}^{k} \underbrace{(\gamma \lambda_i)}_{\geq 0} \mathbf{q}^{(i)}.$$

Hence $\overline{\mathbf{x}} \in \text{cone}(S)$. Hence $P \subseteq \text{cone}(S)$. Hence $P = \text{cone}(S)$. □

Definition 3.2.20: For a set $S \subseteq \mathbb{R}^n$, we define the ***perp of*** S as

$$S^{\perp} = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{y} = \mathbf{y}^{\top}\mathbf{x} = 0\ \forall\, \mathbf{x} \in S\}. \qquad \triangleleft$$

Theorem 3.2.21: Let $P \subseteq \mathbb{R}^n$. P is a polyhedron if and only if $P = Q + C$ where $Q \subseteq \mathbb{R}^n$ is a polytope (so $Q = \text{cone(extreme points of } Q)$ by Theorem 3.2.18) and $C \subseteq \mathbb{R}^n$ is a polyhedral cone (so $C = \text{cone}(T)$ where T is finite). Moreover, if P is pointed, then we can take Q such that the set of extreme points of P is equivalent to the set of extreme points of Q.

Proof: The proof provided in lecture is included here. □

End of Lecture 9

Start of Lecture 10

Chapter 4 – Solving Linear Programs: Lectures 10–14

4.1 Simplex Method

Definition 4.1.1: If an LP is of the form

$$\begin{array}{ll} \max & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ \text{with} & \mathbf{x} \geq 0, \end{array}$$

we say the LP is in ***standard equality form*** (SEF). ◁

Remark 4.1.2: Every LP can be converted into an equivalent LP in SEF. We know every LP can be written in SIF. We can also write

$$A\mathbf{x} \leq \mathbf{b} \equiv A\mathbf{x} + I\mathbf{s} = \mathbf{b}, \quad \text{where} \quad \mathbf{s} \geq 0.$$

The $\mathbf{s}$ is called the ***slack variables***. ◁

Remark 4.1.3: We can assume A has full row rank, otherwise we can either drop a redundant constraint of $A\mathbf{x} = \mathbf{b}$ or determine infeasibility. ◁

Theorem 4.1.4: Show that if $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b} \text{ where } \mathbf{x} \geq 0\}$ is non-empty, then P has an extreme point.

Proof: Exercise. ◁

Definition 4.1.5: Consider the system $A\mathbf{x} = \mathbf{b}$ where $A \in M_{m\times n}$ and $\operatorname{rank} A = m$.

1. For $J \subseteq [n]$. We denote $A_J = [A_j]_{j\in J}$ for the columns of A corresponding indices in J. Similarly, if $\mathbf{v} \in \mathbb{R}^n$, we define $\mathbf{v}_J = (v_j)_{j\in J} \in \mathbb{R}^{|J|}$.
2. We say that $\mathcal{B} \subseteq [n]$ is a ***basis of*** A if $A_\mathcal{B}$ is square and non-singular (i.e. $A_\mathcal{B}^{-1}$ exists). Equivalently, $|\mathcal{B}| = m$ and $\operatorname{rank} A_\mathcal{B} = m$.
3. Let $\mathcal{B} \subseteq [n]$ be a basis of A. Let $N = [n] \setminus \mathcal{B}$. The system $A\mathbf{x} = \mathbf{b}$, $\mathbf{x}_N = 0$ has a unique solution, $\mathbf{x}_\mathcal{B}$ where

$$A\mathbf{x} = A_\mathcal{B}\mathbf{x}_\mathcal{B} + \underbrace{A_N\mathbf{x}_N}_{=\mathbf{0}} = \mathbf{b} \implies \mathbf{x}_\mathcal{B} = A_\mathcal{B}^{-1}\mathbf{b},\ \mathbf{x}_N = 0.$$

   $\mathbf{x}_\mathcal{B}$ is called the basic solution corresponding to $\mathcal{B}$.
4. We say that $\overline{\mathbf{x}}$ is a ***basic solution*** to $A\mathbf{x} = \mathbf{b}$ if there exists a basis $\mathcal{B}$ such that

$$A\overline{\mathbf{x}} = \sum_{j\in\mathcal{B}} A_j x_j + \sum_{k\in N=[n]\setminus\mathcal{B}} A_k x_k = A_\mathcal{B}\mathbf{x}_\mathcal{B} + A_N\mathbf{x}_N = A_\mathcal{B}\mathbf{x}_\mathcal{B} = \mathbf{b}.$$

   In other words, a solution to $A\mathbf{x} = \mathbf{b}$ is called basic if at most m of its entries are non-zero.
5. We say that $\overline{\mathbf{x}}$ is a ***basic feasible solution*** (BFS) if $\overline{\mathbf{x}}$ is a basic solution and if $\overline{\mathbf{x}} \geq \mathbf{0}$.

6. If $\mathcal{B}$ is a basis, we call the variables x_j for $j \in \mathcal{B}$ as ***basic variables*** and x_i for $i \in N$ as ***non-basic variable***. ◁

Theorem 4.1.6: Let $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ where A has full row rank. Then,

$$\begin{aligned} \overline{\mathbf{x}} \text{ is an extreme point of } P &\iff \overline{\mathbf{x}} \text{ is a BFS of } A\mathbf{x} = \mathbf{b} \\ &\iff \left[A_j\right]_{j \in \operatorname{supp}(\overline{\mathbf{x}})} \text{ has full column rank and } \overline{\mathbf{x}} \geq \mathbf{0}. \end{aligned}$$

Proof: Exercise. ◁

4.1.1 Idea Behind Simplex Method

Let LP be the linear program where $A \in M_{m \times n}$ has full row rank

$$\begin{aligned} (\text{LP}):\ \max \quad & \mathbf{c}^\top \mathbf{x}, \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b}, \\ \text{with} \quad & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

1. Start with a BFS $\overline{\mathbf{x}}$.
2. Repeat until $\overline{\mathbf{x}}$ is optimal (or detect (LP) is unbounded) by moving to a "nearby extreme point" of better objective value.

Example 4.1.7: Consider the LP (P) shown below.

$$\begin{aligned} (\text{LP}):\ \max \quad & 2x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 3 \\ & x_1 \leq 2 \\ & x_2 \leq 2 \\ \text{with} \quad & x_1, x_2 \geq 0 \end{aligned} \quad \equiv \quad \begin{aligned} (\text{LP}):\ \max \quad & 2x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 + s_1 = 3 \\ & x_1 + s_2 = 2 \\ & x_2 + s_3 = 2 \\ \text{with} \quad & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

We obtained an equivalent version of (P) by adding the slack variables $s_1, s_2, s_3 \geq 0$. ◁

End of Lecture 10

Start of Lecture 11

University was closed and the classes were canceled on 12 February 2019.

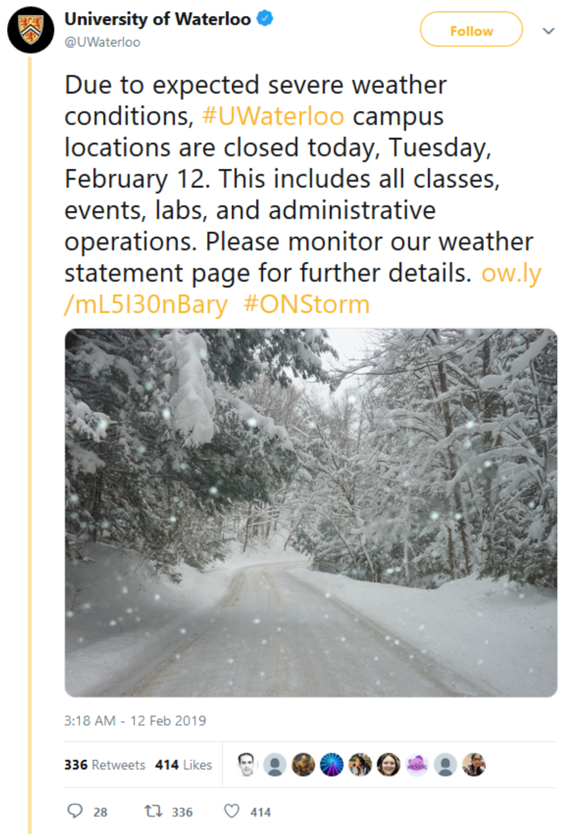


Figure 4.1.1: https://twitter.com/UWaterloo/status/1095281098911707136.

End of Lecture 11

Start of Lecture 12

Recall the discussion of simplex method from the previous lecture.

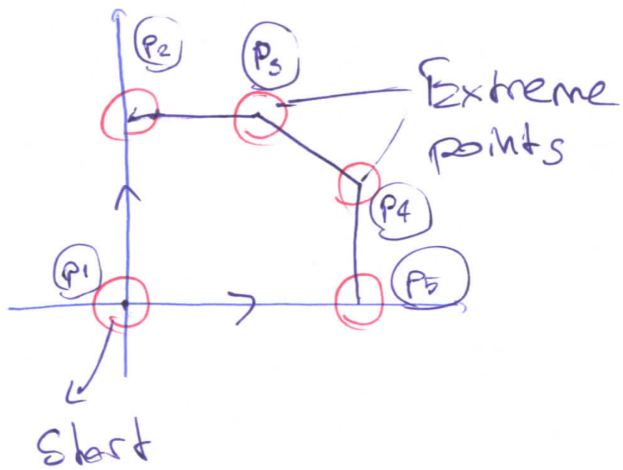


Figure 4.1.2: Starting from an extreme points and jumping the neighboring extreme points.

Recall 4.1.8: $\overline{\mathbf{x}}$ is optimal if and only if there exists dual feasible solution $\overline{\mathbf{y}}$ such that $\overline{\mathbf{x}}, \overline{\mathbf{y}}$ satisfy the CS conditions. ◁

We want to find a generalized way to detect extreme points and define what it means to be "nearby". We will maintain a $\overline{\mathbf{y}}$ that satisfies the CS conditions with $\overline{\mathbf{x}}$.

- If $\overline{\mathbf{y}}$ is dual feasible, then we are done.
- Otherwise, we will use $\overline{\mathbf{y}}$ to move to an extreme point of no smaller objective value.

Definition 4.1.9: Let $P \subseteq \mathbb{R}^n$ be a polyhedron. We say that the extreme points $\overline{\mathbf{x}}^{(1)}, \overline{\mathbf{x}}^{(2)}$ of P are ***neighbors*** if $\exists\ \mathbf{c} \in \mathbb{R}^n$ such that $\{\lambda\overline{\mathbf{x}}^{(1)} + (1-\lambda)\overline{\mathbf{x}}^{(2)} \mid \lambda \in [0,1]\}$ is the set of optimal solutions to the LP $\max \mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{x} \in P$. Note that this set can be considered as a line connecting $\overline{\mathbf{x}}^1$ and $\overline{\mathbf{x}}^2$. ◁

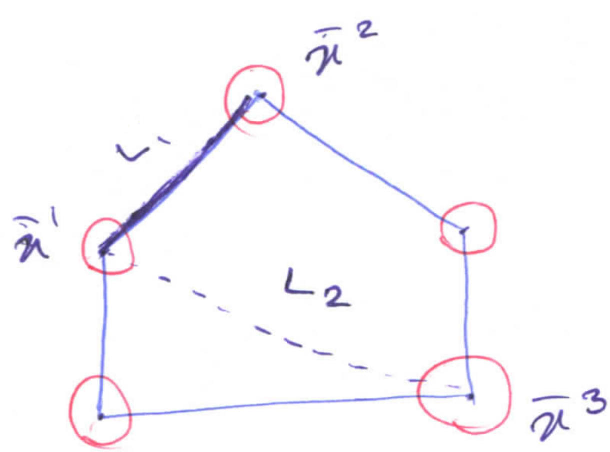


Figure 4.1.3: Extreme points $\overline{\mathbf{x}}^{(1)}$ and $\overline{\mathbf{x}}^{(2)}$ are neighbors with each other but not with $\overline{\mathbf{x}}^{(3)}$.

Definition 4.1.10: Consider $A\mathbf{x} = \mathbf{b}$ with $\mathbf{x} \geq \mathbf{0}$ and where A has full row rank. We say that bases $\mathcal{B}$ and $\mathcal{B}'$ of A are neighbors if $[\mathcal{B} \setminus \mathcal{B}'] = 1 = [\mathcal{B}' \setminus \mathcal{B}]$. We say that two basic solutions $\overline{\mathbf{x}}^{(1)}, \overline{\mathbf{x}}^{(2)}$ are neighbors if $\exists$ bases $\mathcal{B}_1, \mathcal{B}_2$ such that

$$\overline{\mathbf{x}}^{(1)}: \text{ is a basic soltuion corresponding to } \mathcal{B}_1, \overline{\mathbf{x}}^{(2)}: \text{ is a basic soltuion corresponding to } \mathcal{B}_2.$$

In this case, we denote the bases $\mathcal{B}_1, \mathcal{B}_2$ as ***neighboring bases***. ◁

Remark 4.1.11: If $\overline{\mathbf{x}}^{(1)}, \overline{\mathbf{x}}^{(2)}$ are neighboring extreme points, then $\overline{\mathbf{x}}^{(1)}, \overline{\mathbf{x}}^{(2)}$ are neighboring basic feasible solutions. Conversely, if $\overline{\mathbf{x}}^{(1)}, \overline{\mathbf{x}}^{(2)}$ are neighboring basic feasible solutions, $\overline{\mathbf{x}}^{(1)} = \overline{\mathbf{x}}^{(2)}$ or $\overline{\mathbf{x}}^{(1)}, \overline{\mathbf{x}}^{(2)}$ are neighboring extreme points. ◁

4.1.2 Details of Simplex Method

Let $A \in M_{m \times n}$ with $\operatorname{rank} A = m$ and consider the LP, (P) and its dual (D)

$$\begin{aligned} \text{(P)}: \max \quad & \mathbf{c}^\top \mathbf{x}, \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b}, \\ \text{with} \quad & \mathbf{x} \geq \mathbf{0}. \end{aligned} \qquad \begin{aligned} \text{(D)}: \min \quad & \mathbf{b}^\top \mathbf{y}, \\ \text{subject to} \quad & \mathbf{y}^\top A \geq \mathbf{c}. \end{aligned}$$

1. Assume we have a starting BFS $\overline{\mathbf{x}}$, corresponding to basis $\mathcal{B}$.
2. We want to find $\mathbf{y}$ such that $\forall\, j \in \operatorname{supp}(\overline{\mathbf{x}}) \subseteq \mathcal{B}$ we have $(\mathbf{y}^\top A)_j = c_j$. In fact, we find $\mathbf{y}$ such that $(\mathbf{y}^\top A)_j = c_j$ for all $j \in \mathcal{B}$. In other words,
$$\mathbf{y}^\top A_{\mathcal{B}} = \mathbf{c}_{\mathcal{B}}^\top,$$
has a unique solution since $A_{\mathcal{B}}$ is non-singular. Let this unique solution be $\overline{\mathbf{y}}$. Define
$$\overline{c}_j = c_j - (\overline{\mathbf{y}}^\top A)_j,$$
this is called the *reduced cost* of j. We observe that if $\overline{c}_j \leq 0$, for all $j \in [n]$, then $\overline{\mathbf{x}}$ is optimal.
3. If $\overline{\mathbf{x}}$ is not optimal, then $\exists\, j \in [n]$ such that $\overline{c}_j > 0$. Note that since $\overline{c}_{\mathcal{B}} = 0$ then we have $[n] = N$. Pick some $k \in [n] = N$ such that $\overline{c}_k > 0$.

   Claim: If we increase $\mathbf{x}_k$ while keeping all other non-basic variables are zero (and changing basic variables so that $A\mathbf{x} = \mathbf{b}$ holds), then $\mathbf{c}^\top \mathbf{x}$ increases.

   Proof: Since $A_{\mathcal{B}}$ is non-singular, then we have
$$A_{\mathcal{B}}\mathbf{x}_{\mathcal{B}} + A_N \mathbf{x}_N = \mathbf{b} \implies \mathbf{x}_{\mathcal{B}} = A_{\mathcal{B}}^{-1}(\mathbf{b} - A_N \mathbf{x}_N).$$
So we have $\mathbf{c}^\top = \mathbf{c}_{\mathcal{B}}^\top \mathbf{x}_{\mathcal{B}} + \mathbf{c}_N^\top \mathbf{x}_N$. Which gives us
$$\begin{aligned} \mathbf{c}^\top \mathbf{x} &= \mathbf{c}_{\mathcal{B}}^\top (A_{\mathcal{B}}^{-1}\mathbf{b} - A_{\mathcal{B}}^{-1} A_N \mathbf{x}_N) + \mathbf{c}_N^\top \mathbf{x}_N \\ &= \underbrace{\mathbf{c}_{\mathcal{B}}^\top A_{\mathcal{B}}^{-1}}_{\overline{\mathbf{y}}^\top} \mathbf{b} + \left(\mathbf{c}_N^\top - \underbrace{\mathbf{c}_{\mathcal{B}}^\top A_{\mathcal{B}}^{-1}}_{\overline{\mathbf{y}}^\top} A_N \right) \mathbf{x}_N \\ &= \overline{\mathbf{y}}^\top \mathbf{b} = \sum_{j \in N} \underbrace{\left(c_j - \overline{\mathbf{y}}^\top A_j \right)}_{\overline{c}_j} x_j \\ &\implies \overline{c}_k > 0 \implies \uparrow x_k. \end{aligned}$$

□

Notation 4.1.12: We denote the standard basis vector $\mathbf{e}_j$ as the vector with 1 in j^{th} position and 0's everywhere else. ◁

Our goal is to $\uparrow x_k$ as much as possible while maintaining feasibility (satisfying the necessary conditions). In other words, we want to increase x_k as much as possible while not breaking the conditions

$$A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \implies \mathbf{x}_{\mathcal{B}} = A_{\mathcal{B}}^{-1}(\mathbf{b} - A_N\mathbf{x}_N).$$

So, if we set $x_k = t \in \mathbb{R}_+$ and $x_j = 0\ \forall\ j \in N \setminus \{k\}$, then $\mathbf{x}_N^{\text{new}}(t)$ is a function of t and

$$\mathbf{x}_N^{\text{new}}(t) = \mathbf{x}_N^{\text{old}} + t\mathbf{e}_k.$$

Which gives us

$$\begin{aligned}\mathbf{x}_{\mathcal{B}}^{\text{new}} &= A_{\mathcal{B}}^{-1}(\mathbf{b} - A_N\mathbf{x}_N^{\text{new}}(t)) \\ &= A_{\mathcal{B}}^{-1}(\mathbf{b} - A_N\mathbf{x}_N^{\text{old}} - tA_k) \\ &= \mathbf{x}_{\mathcal{B}}^{\text{old}} - t(A_{\mathcal{B}}^{-1}A_k).\end{aligned}$$

Denote $(A_{\mathcal{B}}^{-1}A_k)$ as $\mathbf{d}_{\mathcal{B}}$. Then, for $\mathbf{x}_{\mathcal{B}}^{\text{new}}(t) \geq \mathbf{0}$, we want $t\mathbf{d}_{\mathcal{B}} \leq \mathbf{x}_{\mathcal{B}}^{\text{old}}$. Note that if $\mathbf{d}_{\mathcal{B}} \leq \mathbf{0}$, then (P) is unbounded since

$$(\mathbf{x}_{\mathcal{B}}^{\text{new}}(t), \mathbf{x}_N^{\text{new}}(t))$$

is a family of feasible solutions whose objective value goes to infinity.

4. Otherwise, the largest value of t, denoted t^*, ensuring $\mathbf{x}^{\text{new}}(t \geq \mathbf{0})$ is

$$\min\left\{ \frac{x_j^{\text{old}}}{d_j} \;\middle|\; j \in \mathcal{B},\ d_j > 0 \right\}. \tag{$\star$}$$

Let $r \in \mathcal{B}$ be the index that attains minimum in ($\star$).

Claim: $\mathcal{B}' = \mathcal{B} \cup \{k\} \setminus \{r\}$ is also a basis and $\mathbf{x}^{\text{new}}(t^*)$ is a BFS corresponding to $\mathcal{B}'$ neighbor of $\mathbf{x}^{\text{old}}$.

4.1.3 Summary of Simplex Method

Given a linear program (P),

$$\begin{aligned}(\text{P}):\ \max \quad & \mathbf{c}^\top\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b}, \\ \text{with} \quad & \mathbf{x} \geq \mathbf{0},\end{aligned}$$

where $A \in \mathcal{M}_{m\times n}$ and A has full row rank with initial BFS $\overline{\mathbf{x}}$ with corresponding to basis $\mathcal{B}$, the simplex method can be summarized in 7 steps below.

1. Compute $\overline{\mathbf{y}}$ such that $\overline{\mathbf{y}}^\top A_{\mathcal{B}} = \mathbf{c}_{\mathcal{B}}^\top$. Let $\overline{c}_j = c_j - A_j^\top\overline{\mathbf{y}}$ for all $j \in N$.
2. If $\overline{\mathbf{c}} \leq \mathbf{0}$, STOP. We have $\overline{\mathbf{x}}$ is the optimal solution.
3. Otherwise, let $k \in N$ with $\overline{c}_k > 0$.

4. Solve $A_{\mathcal{B}}\mathbf{d}_{\mathcal{B}} = A_k$.
5. If $\mathbf{d}_{\mathcal{B}} \leq \mathbf{0}$, STOP. We have (P) as unbounded.
6. Otherwise let $r \in \mathcal{B}$ be such that $d_r > 0$ and

$$t^* = \frac{\overline{x}_r}{d_r} = \min\left\{ \frac{\overline{x}_j}{d_j} \;\middle|\; j \in B \text{ and } d_j > 0 \right\}.$$

7. Update the following

$$\begin{aligned} \mathcal{B} &\leftarrow \mathcal{B} \cup \{k\} \setminus \{r\}, \\ \overline{\mathbf{x}}_{\mathcal{B}} &\leftarrow \overline{\mathbf{x}}_{\mathcal{B}} - t^* \mathbf{d}_{\mathcal{B}}, \\ \overline{\mathbf{x}}_N &\leftarrow \overline{\mathbf{x}}_N + t^* \mathbf{e}_k. \end{aligned}$$

   Go back to step 1.

Remark 4.1.13: We often say that k is the ***entering variable*** and r is the ***leaving variable***. Moreover, the updating step described in 7 is called ***pivoting*** on (r, k). $\triangleleft$

End of Lecture 12

Start of Lecture 13

4.1.4 Implementation of Simplex Method

We want to maintain $\bar{\mathbf{c}}$ so that the we have $A_{\mathcal{B}}^{-1}A\mathbf{x} = A_{\mathcal{B}}^{-1}\mathbf{b}$ where $A_{\mathcal{B}}^{-1}A\mathbf{x} = I\mathbf{x}_{\mathcal{B}} + A_{\mathcal{B}}^{-1}A_N\mathbf{x}_N$.

Example 4.1.14: Consider the LP

$$\begin{array}{ll} \max & (2,3,0,0,0)\mathbf{x}, \\ \text{subject to} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 6 \\ 10 \\ 4 \end{bmatrix} \\ \text{with} & \mathbf{x} \geq \mathbf{0}. \end{array}$$

Starting $\overline{\mathbf{x}} = (0,0,6,10,4)^\top$ (corresponds to basis $\mathcal{B} = \{3,4,5\}$) we introduce a variable z to denote objective function.

$$\overline{A}\mathbf{x} = \overline{\mathbf{b}} \equiv \begin{aligned} z - 2x_1 - 3x_2 \qquad &= 0 \\ x_1 + x_2 + x_3 &= 6 \\ 2x_1 + x_2 + x_4 &= 10 \\ -x_1 + x_2 + x_5 &= 4 \end{aligned}$$

Here we have x_1 as the entering variable and x_4 as the leaving variable. We have the new basis as $\mathcal{B}' = \{1,3,5\}$. We use elementary operations to get $\overline{c}_{\mathcal{B}'} = 0, \overline{A}_{\mathcal{B}'} = I$. We obtain

$$\begin{array}{cccccc} z & -2x_2 & & +x_4 & & = 10 \\ & \frac{x_2}{2} & +x_3 & -\frac{x_4}{2} & & = 1 \\ x_1+ & \frac{x_2}{2} & & +\frac{x_4}{2} & & = 5 \\ & 3\frac{x_2}{2} & & +\frac{x_4}{2} & +x_5 & = 9 \end{array}$$

continue... ◁

4.1.4.1 Finding an Initial Basis

We will construct the following auxiliary LP (denoted by AuxLP) with $A \in M_{m\times n}(\mathbb{R})$ and rank $A = m$ and $\mathbf{b} \geq \mathbf{0}$. We create auxiliary variables $w_1, \ldots, w_m$

$$\begin{array}{lll} (\text{AuxLP}):\ \max & -\sum_{i=1}^{n} w_i, & \\ \text{subject to} & A\mathbf{x} + \mathbf{w} = \mathbf{b}, & \equiv \quad \min \sum_{i=1}^{n} w_i \\ \text{with} & \mathbf{x}, \mathbf{w} \geq \mathbf{0}. & \end{array}$$

We have the following observations.

1. (AuxLP) is feasible. We can see this by choosing auxiliary variables as basic variables. This gives $\mathbf{w} = \mathbf{b} \geq \mathbf{0}$ and $\mathbf{x} = 0$ as BFS.

2 (AuxLP) is not unbounded (by FTLA).

($\star$) From 1 and 2, we can run simplex method on (AuxLP) (starting with BFS in 1) and obtain an optimal solution $(\overline{\mathbf{x}}, \overline{\mathbf{w}})$, corresponding to basis $\overline{\mathcal{B}} = \mathcal{B}_1 \cup \mathcal{B}_2$ where $\mathcal{B}_1 = \overline{\mathcal{B}} \cap \{1, \ldots, n\}$ and $\mathcal{B}_2 =$ values of $\overline{\mathcal{B}}$ corresponding to auxiliary variables.

3 $\text{OPT}_{(\text{AuxLP})} = 0$ if and only if the system $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ has a feasible solution. Moreover, if $\text{OPT}_{(\text{AuxLP})} = 0$, then (by exercise the student should do) $\overline{\mathbf{x}}$ obtained from ($\star$) is a BFS to $A\mathbf{x} = \mathbf{b}$, correspond to some basis $\mathcal{B} \supseteq \mathcal{B}_1$.

4.1.4.2 2-phase Simplex Method

Phase 1: Run simplex method on (AuxLP) to detect if (LP) is infeasible of find a BFS $\overline{\mathbf{x}}$.

Phase 2: Run simplex method on (LP) starting with $\overline{\mathbf{x}}$ to detect if (LP) is unbounded, or find an optimal solution.

4.1.4.3 Termination of Simplex Method

Does the simplex method terminate in a finite (ideally "small" number of iterations? We consider the cases for $t = 0$ and $t > 0$.

Remark 4.1.15: If $t > 0$ in every iteration of simplex method, then the objective value is strictly increasing. Then we never repeat a basis. Then the simplex method terminates in a finite number of iterates. $\triangleleft$

Remark 4.1.16: $t = 0$ can happen only if $x_j = 0$ for some $j \in \mathcal{B}$. $\triangleleft$

Definition 4.1.17: For a basis $\mathcal{B}$ with corresponding to basic solution $\overline{\mathbf{x}}$, we say $\overline{\mathbf{x}}$ is ***degenerate*** if $\exists\, j \in \mathcal{B}$ such that $\overline{x}_j = 0$. Otherwise, we say $\overline{\mathbf{x}}$ is ***non-degenerate***. More links in degeneracy in linear programming 1, 2, 3. $\triangleleft$

Example 4.1.18: Consider the system where the constraints form a pyramid in $\mathbb{R}^3$.

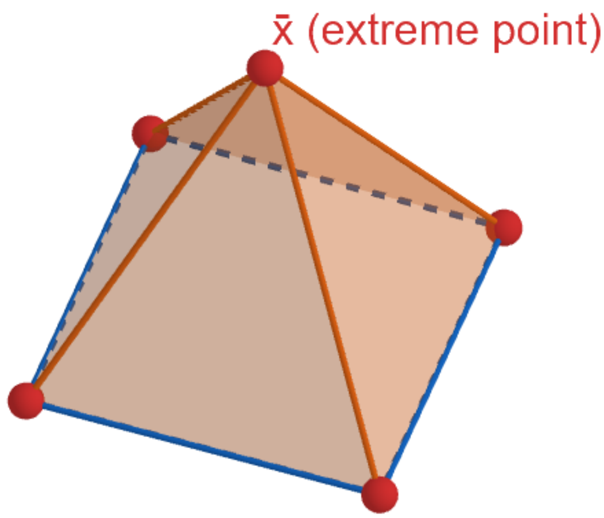


Figure 4.1.4: Pyramid with $\overline{\mathbf{x}}$ at the top vertex.[3]

This problem has 5 inequalities (one for the base of the pyramid and four for its sides), so $A\mathbf{x} \leq \mathbf{b}$ has 5 constraints. At $\overline{\mathbf{x}}$ all of the four vertical cases are tight ◁

Remark 4.1.19: There is no known method of avoiding degeneracy but under suitable tie-breaking rules, the simplex method always terminates. ◁

4.1.4.4 Tie-breaking Rules

Smallest-Subscript Rule: This rule is also known as Bland's rule. Break ties in favor of smaller index.

Perturb RHS by a Small Amount: For $\varepsilon \in \mathbb{R}$, define $\mathbf{e} = (\varepsilon, \varepsilon^2, \dots, \varepsilon^m)^\top$. Suppose we have a BFS $\overline{\mathbf{x}}$ corresponding to basis $\mathcal{B}$. We perturb the system and obtain the following.

$$\begin{aligned} \overline{\mathbf{x}}_{\mathcal{B}} &\to \overline{\mathbf{x}}_{\mathcal{B}} + \mathbf{e} \underset{\text{def}}{=} \mathbf{x}_{\mathcal{B}}(\varepsilon), \\ \mathbf{b} &\to \mathbf{b} + A_{\mathcal{B}}\mathbf{e} \underset{\text{def}}{=} \mathbf{b}(\varepsilon). \end{aligned}$$

Hence, we obtain the perturbed LP, (LP(ε)) as

$$\begin{aligned} (LP(\varepsilon)):\ \max \quad & \mathbf{c}^\top \mathbf{x}, \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b}(\varepsilon) = \mathbf{b} + A_{\mathcal{B}}\mathbf{e}, \\ \text{with} \quad & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Hence for sufficiently small $\varepsilon > 0$ we have

1. All bases of $A\mathbf{x} = \mathbf{b}(\varepsilon)$ are non-degenerate. To see why this is true consider the basis $\mathcal{B}'$. We have $\mathbf{x}_{\mathcal{B}'}(\varepsilon) = A_{\mathcal{B}'}^{-1}\mathbf{b}(\varepsilon)$. Which gives us

$$\mathbf{x}_{\mathcal{B}'}(\varepsilon) = \underbrace{A_{\mathcal{B}'}^{-1}\mathbf{b}}_{\mathbf{x}_{\mathcal{B}'}(0)} + A_{\mathcal{B}'}^{-1}A_{\mathcal{B}}\mathbf{e}.$$

[3] Geogebra pyramid: https://ggbm.at/a2kaq7dc.

Consider some $j \in \mathcal{B}'$. So $x_j(\varepsilon) = x_j(0) + p(\varepsilon) = q(\varepsilon)$. Note that $q(\varepsilon)$ is a polynomial in ε. If $x_j(\varepsilon) = 0$ for all suitable small $\varepsilon > 0$, then $q(\varepsilon) = 0$ for all suitably small $\varepsilon > 0$. But then q is identically zero. Then p is identically zero. But then we have $(A_{\mathcal{B}'}^{-1} A_{\mathcal{B}} \mathbf{e})_j = 0$. Since $\varepsilon \neq 0$, then the matrix $A_{\mathcal{B}'}^{-1} A_{\mathcal{B}}$ must have a zero row. This is a contradiction since $A_{\mathcal{B}'}^{-1} A_{\mathcal{B}}$ is non-singular.
Instructor came unprepared and wasn't able explain the contradiction without the help of the students.

2. If the simplex method terminates on (LP(ε)) with an optimal solution corresponding to basis $\mathcal{B}$, then $\mathcal{B}$ also yields optimal solution to LP(0).

3. If the simplex method determines that (LP(ε)) is unbounded, then (LP(0)) is also unbounded.

4.1.5 Efficiency of Simplex Method

We want to check if the simplex method is efficient. For that, we need to define efficiency.

4.1.5.1 Algorithms and Order of Functions

Notation 4.1.20: Given functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ we say that $f(n) = O(g(n))$ (f is of order g) if $\exists$ constants $n_0 \geq 0$ and $c > 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$. $\triangleleft$

Example 4.1.21: Some examples with big-O.

- $2n + 10 = O(n)$, $n = O(2n + 10)$.
- $n \log_2 n = O(n^2)$ but not $O(n)$. In fact, $n \log_2 n = O(n^2) = O(n^{1+\varepsilon})$ for any $\varepsilon > 0$.
- If $p(n)$ and $q(n)$ are polynomials of degrees c and d then $p(n) = O(g(n)) \iff c \leq d$.
- 2^n is not $O(p(n))$ for any polynomial $p(n)$.
- $\log_c n = O(\log_d n)$ if $c, d > 1$.
- $p(n) = O(n^{O(1)}) \equiv p(n)$ is bounded by a polynomial function of n for n large enough. $\triangleleft$

End of Lecture 13

Start of Lecture 14

Remark 4.1.22: If $p(n) = O(n^{O(1)})$ then we say $p(n)$ is upper bounded by some fixed polynomial in n. In other words, $\exists\, g(n) = O(1)$ such that $p(n) = O(n^{g(n)})$. Since $g(n) = O(1)$, then $g(n) \leq c_1$ for all $n \geq n_0$. Moreover, if $p(n) = O(n^{g(n)})$ then $p(n) \leq c_2 \cdot n^{g(n)}$ for all $n \geq n_0'$. Then,

$$p(n) \leq c_2 \cdot n^{c_1} \quad \forall\, n \geq \max(n_0, n_0'). \qquad \triangleleft$$

Definition 4.1.23: We define the number of bits needed to specify the input data as the ***input size***. ◁

Definition 4.1.24: For an integer $x \geq 0$, we call the the number of bits needed to specify x as the ***size of*** x, and denote it by $\text{size}(x)$. ◁

Remark 4.1.25: For an integer $x \geq 0$, $\text{size}(x) = O(\log x)$. ◁

Remark 4.1.26: For a rational number $x = \frac{p}{q}$ where p, q are co-prime, we have $\text{size}(x) = \text{size}(p) + \text{size}(q)$. ◁

Example 4.1.27: For an LP $\max \mathbf{c}^\top \mathbf{x}$ such that $A\mathbf{x} \leq \mathbf{b}$ where $A \in M_{m\times n}(\mathbb{Q})$ and $\mathbf{c} \in \mathbb{Q}^m, \mathbf{b} \in \mathbb{Q}^n$ we have the input size of the LP as

$$O\left(\sum_{j=1}^{n} \text{size}(c_j) + \sum_{i=1}^{m}\sum_{j=1}^{n} \text{size}(A_{ij}) + \sum_{i=1}^{m} \text{size}(b_i) + m_n\right). \qquad \triangleleft$$

Definition 4.1.28: The mathematical or logical operations involving a combination basic arithmetic $(+, -, \times, \div)$, comparisons, if-then-else statements and assignments are denotes as ***elementary operations***. ◁

Definition 4.1.29: We define the ***running time*** of an algorithm as the number of elementary operations executed by the algorithm as a function of input size. ◁

Example 4.1.30: To find the running time for the algorithm of finding minimum of n numbers $a_1, \ldots, a_n \geq 0$.

$$\left.\begin{array}{l} v \leftarrow -1 \\ \text{For } i = 1 \text{ to } n: \\ \quad \text{if } v < a_i, \\ \quad v \leftarrow a_i \end{array}\right\} \; O(n) \text{ running time.}$$

◁

Remark 4.1.31: We can sort n numbers in $O(n \log n)$ running time. ◁

Definition 4.1.32: We say that an algorithm is (theoretically) ***efficient*** if its running time is bounded by some *fixed polynomial* of input size. In other words, the running time is $O(n^{O(1)}) = O(\text{poly}(n))$ where n is the input size. If this is the case, we say the algorithm is in ***polynomial time*** (polytime). ◁

We want to check if the simplex method is in polytime.

Remark 4.1.33: There is no known a tie-breaking rule under which the simplex method always runs in $\text{poly}(m + n)$ number of iterations. Solving a system of n equations in n variables (Gaussian elimination) can be done in polytime. ◁

Aside: Some historical remarks about efficiency of simplex method:

- Friedman and others in 2011,2013 found problems that give bad examples (number of iterations is exponential in m+n so $c^{O(m+n)}$) for some tie-breaking rules.
- Work by Klee-Minty 1970 (or 1971).
- Spielman-Tang (1990s): Perturbing an instance of LP by a "small" amount can make the simplex method run in polynomial time.

Moreover, below are the work done to determine if there are polytime algorithms for solving LPs:

- Ellipsoid method (1970s) [highly impractical]
- Interior-Point Methods (1988) $\triangleleft$

4.1.5.2 Polynomial Hirsch Conjecture

Given a polytope $P \subseteq \mathbb{R}^n$ described by m constraints, with any two extreme points $\overline{\mathbf{x}}^{(1)}$ and $\overline{\mathbf{x}}^{(2)}$ of P, we can move from $\overline{\mathbf{x}}^{(1)}$ to $\overline{\mathbf{x}}^{(2)}$ in poly$(m+n)$ steps where each step is moving from one extreme point to a neighboring extreme point.

Remark 4.1.34: This is an open conjecture. $\triangleleft$

Chapter 5 – Combinatorial Optimization: Lectures 14–20

5.1 Integer Programming and Discrete Optimization

Definition 5.1.1: An integer program (IP) is a problem os the form

$$\left.\begin{array}{ll} \max & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} & A\mathbf{x} \le \mathbf{b}, \\ \text{with} & \mathbf{x} \in \mathbb{Z}^n, \end{array}\right\} \quad \begin{array}{l} \text{LP} \ \equiv \ \text{(canonical) LP-relaxation} \\ \text{of integer program.} \end{array}$$
$$\underbrace{x_i \in \mathbb{Z},\ \forall\, i \in I \subseteq [n]}_{\text{integrality constraints}}$$

It is generally considered that the set of integer programs contain the set of linear programs. $\triangleleft$

Definition 5.1.2: If $I = [n]$, then the IP is called the ***pure IP***. In general, for an arbitrary I, we say that IP is a ***mixed IP***. $\triangleleft$

Remark 5.1.3: The feasible region of a pure IP is $P \cap \mathbb{Z}^n \underset{\text{def}}{=} \mathbb{Z}(\text{P})$ where $P \subseteq \mathbb{R}^n$ is a polyhedron. $\triangleleft$

5.2 Reducing IPs to LPs

Let $P \subseteq \mathbb{R}^n$ be a polyhedron. So $\mathbb{Z}(\text{P}) = P \cap \mathbb{Z}^n$. Define $P_I = \text{conv}(\mathbb{Z}(\text{P}))$.

Definition 5.2.1: We say the vector $\mathbf{b} \in \mathbb{R}^m$ is a ***rational vector*** if $b_i \in \mathbb{Q}$ for all $i = 1, \ldots, m$. Similarly we say the matrix $A \in \mathcal{M}_{m \times n}(\mathbb{R})$ is a rational matrix if $a_{ij} \in \mathbb{Q}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. We show the rational vectors and polytopes as $A \in \mathcal{M}_{m \times n}(\mathbb{Q})$ and $\mathbf{b} \in \mathbb{Q}^m$.

We say that the polyhedron $P = \{\mathbf{x} \in \mathbb{R}^m\}$ is a ***rational polyhedron*** if A and $\mathbf{b}$ are rational. Similarly for polytopes and polyhedral cones. ◁

Theorem 5.2.2: Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron. Then P_I is also a rational polyhedron.

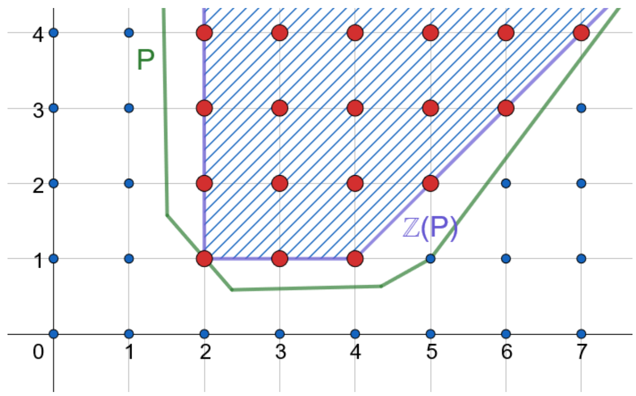


Figure 5.2.1: The polyhedra P, in green, and $\mathbb{Z}$(P), in purple.

Remark 5.2.3: The rational condition is essential. Consider $P = \{(x_1, x_2) \mid x_2 \leq \sqrt{2}\,x_1\}$.

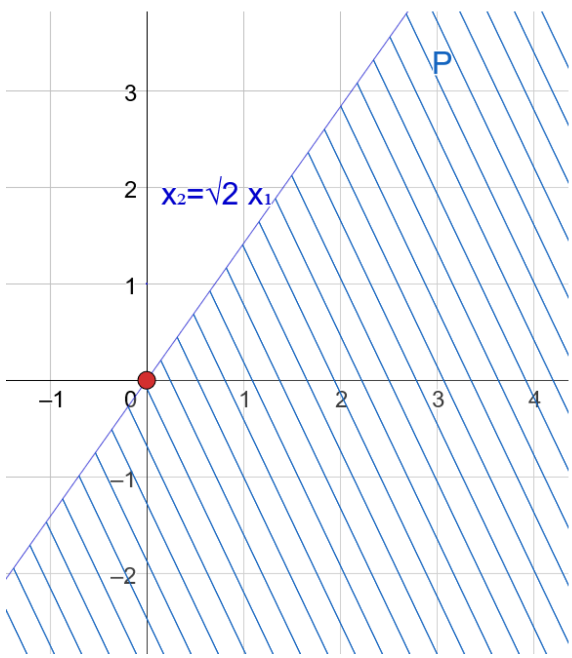


Figure 5.2.2: Shaded regions shows $\mathbf{x} \in P = \{(x_1, x_2) \mid x_2 \leq \sqrt{2}\,x_1\}$.

We can show that $P_I = \{(0,0)\} \cup \{(x_1, x_2) \mid x_2 < \sqrt{2}\,x_1\}$ which is not rational. ◁

End of Lecture 14

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Definition 5.2.4: A vector $\mathbf{x} \in \mathbb{Z}^n$ is called an ***integral vector*** is $\mathbf{x} \in \mathbb{Z}^n$. Similarly we say a matrix $A \in \mathbb{R}^{m\times n}$ is an ***integral matrix*** if $A \in \mathcal{M}_{m\times n}(\mathbb{Z})$. ◁

Recall 5.2.5: Theorem 5.2.2 states for a rational polyhedron $P \subseteq \mathbb{R}^n$, P_I is also a rational polyhedron. ◁

Proof: We have $P = Q + C$. Since P is rational then Q is a rational polytope and C is a rational polyhedral cone (as shown in Assignment #4). So $Q = \operatorname{conv}(\{\mathbf{s}^{(1)}, \ldots, \mathbf{s}^{(k)}\})$ where $\mathbf{s}^{(i)}$ are extreme points for $i = 1, \ldots, k$.

Claim 5.2.6*:* If Q is a rational polytope, then every extreme point of Q is also rational.

Proof: This is shown in Question #2 of Assignment #4. ◀

So, $\mathbf{s}^{(1)}, \ldots, \mathbf{s}^{(k)}$ are rational. We have $C = \operatorname{cone}(\{\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(\ell)}\})$ where $\mathbf{t}^{(j)}$ are extreme points of an associated rational polytope for $j = 1, \ldots, \ell$. Hence, by the Claim 5.2.6, $\mathbf{t}^{(j)}$ are rational. Since scaling $\mathbf{t}^{(j)}$ does not change the cone, we can assume, by scaling, $\mathbf{t}^{(j)}$ are integral vectors (i.e. they lie in $\mathbb{Z}^n$). Let

$$B = \left\{ \sum_{j=1}^{\ell} \lambda_j \mathbf{t}^{(j)} \;\middle|\; 1 \geq \lambda_j \geq 0 \quad \forall\, j = 1, \ldots, \ell \right\}.$$

Claim 5.2.7*:* $P_I = (Q + B)_I + C$. Note that $Q + B$ is a polytope. So $\mathbb{Z}(Q + B)$ is finite which implies $(Q + B)_I$ is a polytope.

Proof: Note that we have $Q = \operatorname{conv}(\mathbf{s}^{(1)}, \ldots, \mathbf{s}^{(k)})$ and $C = \operatorname{cone}(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(\ell)})$. Suppose $P = \varnothing$. Then $P_I = \operatorname{conv}(\mathbb{Z}(\varnothing)) = \varnothing$. Now suppose $P \neq \varnothing$. To show $P_I \subseteq (Q + B)_I + C$, it is sufficient to show $\mathbb{Z}(\mathrm{P}) \subseteq (Q + B)_I + C$. Let $\overline{\mathbf{x}} \in \mathbb{Z}(\mathrm{P})$. So $\overline{\mathbf{x}} = \mathbf{u} + \mathbf{v}$ where

$$\begin{aligned} Q \ni \mathbf{u} &= \sum_{i=1}^{k} u_i \mathbf{s}^{(i)} \text{ such that } \sum_{i=1}^{k} u_i = 1, \quad &&\text{where} \quad u_i \geq 0, \quad \forall\, i \in [k], \\ C \ni \mathbf{v} &= \sum_{i=1}^{\ell} \lambda_i \mathbf{t}^{(i)}, &&\text{where} \quad \lambda_i \geq 0, \quad \forall\, i \in [\ell]. \end{aligned}$$

Notation: For $a \in \mathbb{R}$, we define the

- ***floor*** of a as the largest integer that is less than or equal to a and show it as $\lfloor a \rfloor$,
- ***ceiling*** of a as the smallest integer that is greater than or equal to a and show it as $\lceil a \rceil$. ◁

This gives us

$$\overline{\mathbf{x}} = \sum_{i=1}^{k} u_i \mathbf{s}^{(i)} + \sum_{i=1}^{\ell} \lambda_i \mathbf{t}^{(i)} = \underbrace{\sum_{i=1}^{k} u_i \mathbf{s}^{(i)} + \sum_{i=1}^{\ell} \lambda_i \mathbf{t}^{(i)} - \lfloor \lambda_i \rfloor \mathbf{t}^{(i)}}_{\mathbf{x}^{(1)}} + \underbrace{\sum_{i=1}^{\ell} \lfloor \lambda_i \rfloor \mathbf{t}^{(i)}}_{\mathbf{x}^{(2)}}.$$

Since $\lfloor \lambda_i \rfloor \in \mathbb{Z}$ and $\mathbf{t}^{(i)} \in \mathbb{Z}^n$, then $\mathbf{x}^{(2)} \in \mathbb{Z}^n$. Since $\overline{\mathbf{x}} \in \mathbb{Z}^n$ then $\mathbf{x}^{(1)} = \overline{\mathbf{x}} - \mathbf{x}^{(2)} \in \mathbb{Z}^n$. Also $\mathbf{x}^{(1)} = \mathbf{q} + \mathbf{b}$ where $\mathbf{q} = \mathbf{u} \in Q$ and $\mathbf{b} \in B$. So $\mathbf{x}^{(1)} \in Q + B$ and since $\mathbf{x}^{(1)} \in \mathbb{Z}^n$, then

$\mathbf{x}^{(1)} \in \mathbb{Z}(Q+B)$. So $\overline{\mathbf{x}} \in \mathbb{Z}(Q+B)+C \subseteq (Q+B)_I + C$. To show $(Q+B)_I + C \subseteq P_I$, note that since any point $\lambda_i \mathbf{t}^{(i)} \in C$ can be written as a convex combination of $\lfloor \lambda_i \rfloor \mathbf{t}^{(i)}$ and $\lceil \lambda_i \rceil \mathbf{t}^{(i)}$ where $\lfloor \lambda_i \rfloor \mathbf{t}^{(i)}, \lceil \lambda_i \rceil \mathbf{t}^{(i)} \in \mathbb{Z}(C)$ then $C = C_I$. We now make the following claim.

Claim 5.2.8*:* For any $F, G \subseteq \mathbb{R}^n$, $\operatorname{conv}(F) + \operatorname{conv}(G) \subseteq \operatorname{conv}(F+G)$.

Proof: Let $\mathbf{z}_1, \mathbf{z}_2$ defined as follows.

$$\mathbf{z}_1 = \sum_{i=1}^{k} \alpha_i \mathbf{f}^{(i)}, \quad \text{where} \quad \mathbf{f}^{(i)} \in F \text{ and } \quad \alpha_i \geq 1 \text{ with } \sum_{i}^{k} \alpha_i = 0, \quad \forall\, i \in [k],$$
$$\mathbf{z}_2 = \sum_{j=1}^{\ell} \beta_j \mathbf{g}^{(j)}, \quad \text{where} \quad \mathbf{g}^{(j)} \in G \text{ and } \beta_j \geq 1 \text{ with } \sum_{j}^{\ell} \beta_i = 0, \quad \forall\, j \in [\ell].$$

Hence we can write

$$\mathbf{z}_1 + \mathbf{z}_2 = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \alpha_i \beta_j \left(\underbrace{\mathbf{f}^{(i)} + \mathbf{g}^{(j)}}_{\in F+G} \right).$$

Finish the wording for proof of Claim 5.2.8 (exercise). ◂

This proves Claim 5.2.7 (exercise). ◂

Hence we have,

$$(Q+B)_I + C \underset{\text{def}}{=} \operatorname{conv}(\mathbb{Z}(Q+B)) + C \underset{(\star)}{=} \operatorname{conv}(\mathbb{Z}(Q+B)) + \underbrace{\operatorname{conv}(\mathbb{Z}(C))}_{C_I} \underset{(\star\star)}{\subseteq} \operatorname{conv}(\underbrace{\mathbb{Z}(Q+B) + \mathbb{Z}(C)}_{\in \mathbb{Z}(\mathrm{P}) \quad \text{exercise}}) \subseteq P_I,$$

where in $(\star)$ we used the fact that $C = C_I$, and in $(\star\star)$ we used Claim 5.2.8. Since $P_I \subseteq (Q+B)_I$ and $P_I \supseteq (Q+B)_I$ then $P_I = (Q+B)_I$. Exercise (reword the end of the proof). ☐

Corollary 5.2.9: Consider the pure (IP) $\max \mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{x} \in \mathbb{Z}(\mathrm{P})$ where P is a rational polyhedron and the (LP) $\max \mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{x} \in P_I$ where $P_I \underset{\text{def}}{=} \operatorname{conv}(\mathbb{Z}(\mathrm{P}))$ which is a rational polyhedron by Theorem 5.2.2. Then,

1. (IP) is infeasible $\iff$ (LP) is infeasible.
2. (IP) is unbounded $\iff$ (LP) is unbounded.
3. (IP) has optimal solution $\iff$ LP has an optimal solution.
   a. $\mathrm{OPT}_{(\mathrm{IP})} = \mathrm{OPT}_{(\mathrm{LP})}$.
   b. Every extreme point of P_I is in $\mathbb{Z}(\mathrm{P})$ and so if $\overline{\mathbf{x}}$ is an extreme point which is a solution to (LP), then it is also an optimal solution to (IP).

Proof: 1: $\mathbb{Z}(\mathrm{P}) = \varnothing \iff P_I = \varnothing$ (exercise).

2: Exercise.

3: Suppose (LP) has an optimal solution $\overline{\mathbf{x}}$. Then, by FTLP, (IP) is not infeasible and not unbounded and $\mathbf{c}^\top \overline{\mathbf{x}}$ is an upper bound on objective value of feasible solutions to (IP). We have

$\overline{\mathbf{x}} \in \text{conv}(\mathbb{Z}(\text{P}))$. So,

$$\overline{\mathbf{x}} = \sum_{i=1}^{k} \lambda_i \mathbf{x}^{(i)}, \quad \text{where} \quad \lambda_i \geq 0 \text{ and } \sum_{i=1}^{k} \lambda_i = 1 \text{ and } \mathbf{x}^{(i)} \in \mathbb{Z}(\text{P}), \quad \forall\, i \in [k].$$

Hence we have

$$\mathbf{c}^\top \overline{\mathbf{x}} = \sum_{i=1}^{k} \lambda_i (\mathbf{c}^\top \mathbf{x}^{(i)}) \quad \text{and} \quad \mathbf{c}^\top \mathbf{x}^{(i)} \leq \mathbf{c}^\top \overline{\mathbf{x}}, \quad \forall\, i \in [k].$$

So there exists $i \in [k]$ such that $\mathbf{c}^\top \mathbf{x}^{(i)} = \mathbf{c}^\top \overline{\mathbf{x}}$. Then $\mathbf{x}^{(i)}$ is an optimal solution to (IP). also, $\text{OPT}_{(\text{IP})} = \text{OPT}_{(\text{LP})}$. If (IP) has an optimal solution, then by 1 and 2, (LP) has also an optimal solution and we showed above $\text{OPT}_{(\text{IP})} = \text{OPT}_{(\text{LP})}$. □

5.2.1 Modeling Using IPs

Example 5.2.10: Consider an investment problem with 5 projects.

	P_1	P_2	P_3	P_4	P_5	Cash available (\$)
Year 1	40	20	25	80	30	100
Year 2	10	30	30	40	20	70
Returns	100	90	120	160	100	

Here each entry represents the amount needed if we choose to invest. Our goal is to find investment strategy to maximize the return at the end of Year 2.

For $i = 1, \ldots, 5$, define variables $x_i \in \{0, 1\}$ where

$$x_i = \begin{cases} 1 & \equiv \text{ invest in } P_i, \\ 0 & \equiv \text{ do not invest in } P_i. \end{cases}$$

We have the objective function as $\max 100x_1 + 90x_2 + 120x_3 + 160x_4 + 100x_5$ and the constraints as

$$\begin{aligned} Y_1 &\leftarrow 40x_1 + 20x_2 + 25x_3 + 80x_4 + 30x_5 \leq 100, \\ Y_2 &\leftarrow 10x_1 + 30x_2 + 30x_3 + 40x_4 + 20x_5 \leq 70, \end{aligned}$$

where $x_i \in \{0, 1\}$ for all $i \in [5]$. i.e. $0 \leq x_i \leq 1$ where $x_i \in \mathbb{Z}$ for $i \in [5]$. ◁

End of Lecture 15

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Things that can be modeled using IPs: We have binary variables $x_1, \dots, x_n$.

1. **Cardinality Constraints:** "At least k of x_i's are 1". i.e. $\sum_{i=1}^{n} x_i \geq k$.
2. **Boolean Logic:**
   1. NOT $x_i \equiv 1 - x_i$.
   2. $(x_1 = 1)$ OR $x_2 = 1$ OR $\dots$ OR $(x_k = 1) \equiv \underbrace{\max(x_1, \dots, x_k)}_{\in\{0,1\}} \geq 1 \equiv \sum_{i=1}^{k} x_i \geq 1$.
   3. $(x_1 = 1)$ AND $(x_2 = 1)$ AND $\dots$ AND $(x_k = 1) \equiv \min(x_1, \dots, x_k) \geq 1$ $\equiv x_i = 1 \quad \forall\, i \in [k] \equiv x_1 \geq 1, \dots, x_k \geq 1$.
   4. IF $(x_i = 1)$ THEN $(x_j = 1) \equiv x_i \leq x_j$.

Example 5.2.11: An example of rewriting a boolean expression.

if (P_2 and P_3 and P_5 are selected), then (P_1 or P_4 are selected).

$$\begin{aligned}
&\equiv \text{ if } (x_2 = 1 \text{ and } x_3 = 1 \text{ and } x_5) \text{ then } (x_1 \text{ or } x_4) \\
&\equiv \text{ if } (\min(x_2, x_3, x_5) = 1) \text{ then } (\max(x_1, x_4) \geq 1) \\
&\equiv \min(x_2, x_3, x_5) \leq \underbrace{\max(x_1, x_4)}_{(\star)} \\
&\equiv \min(x_2, x_3, x_5) \leq x_1 + x_4
\end{aligned}$$

We can replace $(\star)$ by sum. We negate the above expression and add 1 both sides. We get

$$\begin{aligned}
&\equiv 1 - \min(x_2, x_3, x_5) \geq 1 - x_1 - x_4 \\
&\equiv \max(1 - x_2, 1 - x_3, 1 - x_5) \geq 1 - x_1 - x_4 \\
&\equiv 1 - x_2 + 1 - x_3 + 1 - x_5 \geq 1 - x_1 - x_4 \\
&\equiv x_2 + x_3 + x_5 \leq 2 + x_1 + x_4.
\end{aligned}$$

$\triangleleft$

Remark 5.2.12: We can use binary variables to encode statements about arbitrary linear expressions. Suppose we have some linear expression $f(\mathbf{x})$ in terms of (arbitrary) variables $x_1, \dots, x_n$ and for all feasible choices of $\mathbf{x}$, we have $f(\mathbf{x}) \in [L, M]$. We can define a binary variable $z \in \{0, 1\}$ such that

- if $(f(\mathbf{x}) > k)$ then $z = 1$ where $k \in [L, M]$. We can encode this as follows

$$f(\mathbf{x}) - k \leq (M - k)z.$$

- if $(z = 1)$ then $f(\mathbf{x}) \geq K$ where $K \in [L, M]$. We can encode this as

$$f(\mathbf{x}) \geq Kz + (1 - z)L.$$

$\triangleleft$

Example 5.2.13: Suppose we have constrains $a_i^\top \mathbf{x} \geq b_i$ and $a_i^\top \mathbf{x} \in [L_i, M_i]$ for all $i = 1, \ldots, m$. We want to encode that at least k constraints hold. Our strategy is to

1. introduce $\{0, 1\}$ variables z_i for all $i = 1, \ldots, m$.
2. encode if $z_i = 1$ then $(a_i^\top \mathbf{x} \geq \mathbf{b}_i) \equiv a_i^\top \mathbf{x} \geq b_i z_i + L_i(1 - z_i)$
3. have at least k of z_i's are 1 (equivalently $\sum_{i=1}^m z_i \geq k$).

Assuming $a_i^\top \mathbf{x}$ and $\mathbf{b}_i$ are integers for all choices of $\mathbf{x}$. Question: What if we have to encode at most k constraints hold? Note that $\sum z_i \leq k$ does not encode this. ◁

5.2.1.1 Integral Polyhedra

Recall for a set $S \subseteq \mathbb{R}^n$ we defined $\mathbb{Z}(S) = S \cap \mathbb{Z}^n$ and $S_I = \operatorname{conv}(\mathbb{Z}(S))$. When do we have $P = P_I$? If $P = P_I$ then we can solve (IP): $\max \mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{x} \in \mathbb{Z}(\mathrm{P})$ by solving the (LP): $\max \mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{x} \in P$.

Definition 5.2.14: A polyhedron P is called ***integral*** if $P = P_I$. ◁

Theorem 5.2.15: Let $P \subseteq \mathbb{R}^n$ be a pointed polyhedron and let Ext(P) be the set of extreme points of P.

1. If $P = P_I$ then $\mathrm{Ext(P)} \subseteq \mathbb{Z}^n$.
2. If P is a rational polyhedron or if P is a polytope, and $\mathrm{Ext(P)} \subseteq \mathbb{Z}^n$, then $P = P_I$.

Proof: 1: Let $\overline{\mathbf{x}} \in \mathrm{Ext(P)}$. If $\overline{\mathbf{x}} \notin \mathbb{Z}(\mathrm{P})$, then since $\overline{\mathbf{x}} \in P = P_I = \operatorname{conv}(\mathbb{Z}(\mathrm{P}))$, then $\overline{\mathbf{x}}$ can be written as convex combination of points of P distinct from $\overline{\mathbf{x}}$ which is a contradiction because $\overline{\mathbf{x}}$ is an extreme point.

2: Since P is pointed then $P = \operatorname{conv}(\mathrm{Ext(P)}) + C$ where C is a polyhedral cone. If P is a polytope, then $C = \{\mathbf{0}\}$ (since P is bounded). So $P = \operatorname{conv}(\mathrm{Ext(P)}) \subseteq \operatorname{conv}(\mathbb{Z}(\mathrm{P})) = P_I$. Hence $P = P_I$. If P is a rational polyhedron, then we know $C = \operatorname{cone}(\{\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(\ell)}\})$ where $\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(\ell)} \in \mathbb{Z}^n$ and $C = C_I$. So, we have

$$P = \operatorname{conv}(\mathrm{Ext(P)}) + C \subseteq \operatorname{conv}(\mathbb{Z}(\mathrm{P})) + \operatorname{conv}(\mathbb{Z}(C)) \subseteq \operatorname{conv}(\mathbb{Z}(\mathrm{P}) + \mathbb{Z}(C)) = \operatorname{conv}(\mathbb{Z}(\mathrm{P})) = P_I.$$

□

5.3 Total Unimodularity

Definition 5.3.1: A matrix $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ is called ***totally unimodular*** if for every square submatrix M of A, $\det M \in \{-1, 0, 1\}$. In particular, all entries of A are 0 or ± 1. ◁

Lemma 5.3.2: If $B \in \mathcal{M}_{k\times k}(\mathbb{R})$ is an integral matrix with $\det B \neq 0$. Then, $\det B \in \{-1, 1\}$ if and only if B^{-1} is an integral matrix.

Proof: Exercise. ◁

Theorem 5.3.3: Let $A \in \{0, \pm 1\}^{m\times n}$ be a totally unimodular matrix. Then, for every $\mathbf{b} \in \mathbb{Z}^m$, all extreme points of $P = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$ (if any) are integral.

Proof: Let $\overline{\mathbf{x}}$ be an extreme point of P. So $A^{=}\mathbf{x} \leq \mathbf{b}^{=}$ is the set of tight constraints at $\overline{\mathbf{x}}$. Then, $\operatorname{rank}(A^{=}) = n$. So we can extract a square submatrix M of $A^{=}$ and A, and $\mathbf{b}'$ such that $\operatorname{rank} M = n$ and $\overline{\mathbf{x}}$ is the unique solution to $M\mathbf{x} = \mathbf{b}'$. Since $\det M \in \{-1, 1\}$ then M^{-1} is integral. So $\overline{\mathbf{x}} = M^{-1}\mathbf{b}'$ where $\mathbf{b}'$ is also integral. Hence $\overline{\mathbf{x}}$ is integral. □

Lemma 5.3.4: Let $D \in \mathcal{M}_{k\times k}(\mathbb{R})$ with entries (d_{ij}) where $i, j \in [k]$. Then the determinant of D is equivalent to

$$\det D = \sum_{i,j=1}^{k} (-1)^{i+j} d_{ij} \det(D_{-i,-j}),$$

where $D_{-i,-j}$ is the matrix D with row i and column j deleted.

Proof: Exercise. This is beyond the scope of this course (M136/145 material). This is also known as the minor-cofactor method. ◁

Lemma 5.3.5: Let $A \in \{0, \pm 1\}^{m\times n}$ be totally unimodular. Then, the matrices (or the matrices obtained from the operations) below

1. $A^\top$,
2. permuting rows and columns,
3. multiplying a row or column by -1,
4. duplicating a row or column,
5. adding a row or columns to A with exactly one non-zero entry that is ± 1,
6. $[A \quad I]$ (this can be obtained by repeating 5),
7. $[A \quad -A]$ (this can be obtained by repeating 3 and 4),

are totally unimodular matrices.

Proof: The steps 1-3 may affect only sign of determinants. For step 4, any new square submatrices, can have a duplicate row or column, which has determinant zero. For step 5, let

$$A' = \left[\begin{array}{c|c} \mathbf{0} & \\ \pm 1 & A \\ \mathbf{0} & \end{array}\right] \in \mathcal{M}_{m\times n+1}.$$

Here the entry with ± 1 is in i^{th} row. Similarly let D be the square submatrix of A'. If D is a square submatrix of A, then $\det D \in \{0, \pm 1\}$. Otherwise,

$$D = \left[\begin{array}{c|c} \mathbf{0} & \\ \pm 1 & M \\ \mathbf{0} & \end{array}\right] \in \mathcal{M}_{k\times k}.$$

Then, by Lemma 5.3.4, we have $\det D = \pm \det(M_{-i})$ where M_{-i} is the square submatrix obtained from M by deleting row i. Since M is a submatrix of A then $\det M \in \{0, \pm 1\}$. Then $\det D \in \{0, \pm 1\}$. Then A' is a totally unimodular matrix. □

Theorem 5.3.6: Let $A \in \{0, \pm 1\}^{m \times n}$ be totally unimodular and let $\mathbf{b} \in \mathbb{Z}^m$. The following polyhedra are integral.

1. $\{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b} \text{ where } \mathbf{x} \geq \mathbf{0}\}$. Note that the constrains in this system can be written as
$$\begin{bmatrix} A \\ -I \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix},$$
2. $\{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b} \text{ where } \mathbf{x} \geq \mathbf{0}\}$.
3. $\{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b} \text{ where } \boldsymbol{\ell} \leq \mathbf{x} \leq \mathbf{u} \text{ and } \boldsymbol{\ell}, \mathbf{u} \in \mathbb{Z}^n\}$. Here we have the constraints as
$$\begin{bmatrix} A \\ -I \\ I \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} \mathbf{b} \\ \boldsymbol{\ell} \\ \mathbf{u} \end{bmatrix}.$$

This theorem is a corollary of Theorem 5.3.3 and Lemma 5.3.5.

Corollary 5.3.7: Let $A \in \{0, \pm 1\}^{m \times n}$ be a totally unimodular and $\mathbf{b} \in \mathbb{Z}^m$, $\mathbf{c}, \boldsymbol{\ell}, \mathbf{u} \in \mathbb{Z}^n$. Then, the LP (P) and its dual (D) always have optimal integral solutions where

$$\begin{aligned} (\text{P}): \max \quad & \mathbf{c}^\top \mathbf{b}, \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b}, \\ & -\mathbf{w}^\top \mathbf{x} \leq -\mathbf{w}^\top \boldsymbol{\ell}, \\ & -\mathbf{z}^\top \mathbf{x} \leq \mathbf{w}^\top \mathbf{u}, \\ \text{with} \quad & \mathbf{w}, \mathbf{z} \geq \mathbf{0}. \end{aligned} \qquad \text{(D) : exercise}$$

Proof: We provide a proof sketch. Feasible region of (P) is a polytope. So (P) and (D) always have optimal solutions. Feasible regions of (P) and (D) are pointed polyhedra defined by totally unimodular matrices. This implies that there always exists an extreme point that is an optimal solution and every extreme point is integral. ◀

End of Lecture 16

Start of Lecture 17

Lemma 5.3.8: Let $A \in \{0, \pm 1\}^{m \times n}$ be such that every column of A has at most one 1 and at most one -1. A is a totally unimodular matrix. An example of such a matrix is given below.

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Proof: We will use induction on number of rows and columns of the square submatrix M of A. Let $M \in \mathcal{M}_{k \times k}(\mathbb{R})$ and let $k \leq \min(m, n)$ be a submatrix of A.

Base Case: $k = 1$. Then M is either ± 1 or 0. Then $\det M \in \{0, \pm 1\}$. Suppose the statement holds for all $M_{\ell \times \ell}$ where $\ell \leq k$. Consider $M_{k+1 \times k+1}$. If M has a column with all zeros, then $\det M = 0$. If M has a column with one non-zero entry, then M is of the form

$$M = \left[\begin{array}{c|ccc} & * & * & * \\ [\pm 1] & & M' & \\ & & & \end{array}\right] \in \mathcal{M}_{k+1 \times k+1},$$

here $[+1] \in \mathbb{R}^{k+1}$ is a column vector with at most one $+1$ or -1. Then, by Lemma 5.3.4, we have $\det M = |\det M'|$.[4] Hence $\det M \in \{0, \pm 1\}$. If every column of M has exactly one 1 and one -1, then adding all rows gives us a row whose sum is zero. We then add columns obtain a zero row, which gives $\det M = 0$. □

5.4 Graph Theory

Definition 5.4.1: A ***graph*** is a tuple (V, E) where V is the set of notes and E is the set of edges. In a graph every edge joins two notes. If e joins u, v we write $e = uv$. Equivalently we say uv : ends of e or e is incident on u, v. ◁

Example 5.4.2: An example of a graph $G = (V, E)$.

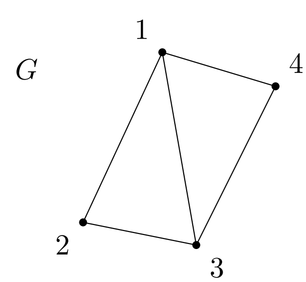


Figure 5.4.1: The graph $G = (V, E)$.

Here we have $V = \{1, 2, 3, 4\}$ and $E = \{12, 23, 31, 14, 34\}$. ◁

[4] CK: This is subtle but easy to understand. Fix first row and recall that adding rows doesn't change the determinant.

Definition 5.4.3: A ***directed graph*** (digraph) G is a pair (V, E) where edges have directions. Every edge goes from a node u to a node v, indicated by (u, v). The order of nodes matter.

If a graph is not a directed graph (i.e. if the order of nodes doesn't matter) then we say it is an ***undirected graph***. In an undirected graph $G = (V, E)$ we have $E \ni e = uv = vu$ where $u, v \in V$. $\triangleleft$

Example 5.4.4: An example of a digraph G.

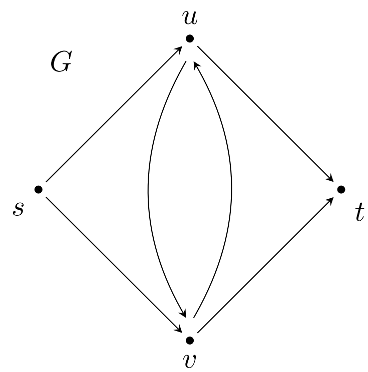


Figure 5.4.2: Directed graph.

Here we have $G = (V = \{s, u, v, t\}, E = \{(s, u), (s, v), (u, v), (u, t), (v, u), (v, t)\})$. $\triangleleft$

Remark 5.4.5: We have the following remarks about graphs.

1. Graphs in this course contain no loops. That is, the end points of an edge are distinct nodes.
2. Graphs contain no parallel edges. That is, if $e = uv$ and $\widetilde{e} = \widetilde{u}\widetilde{v}$ are distinct edges, then $\{u, v\} \neq \{\widetilde{u}, \widetilde{v}\}$. $\triangleleft$

Definition 5.4.6: A $u - v$ ***path*** in a graph G where $u, v \in V$ is a sequence of nodes starting at u and ending at v, where each vertex is unique and consecutive vertices in the sequence are adjacent vertices (connected by an edge) in the graph. $\triangleleft$

Definition 5.4.7: For an undirected graph $G = (V, E)$ and $u \in V$, we will use $\delta(u)$ to denote the ***set of edges incident to*** u. The size of this set is called the ***degree*** of u. We use $\delta^{\text{in}}(u)$ $[\delta^{\text{out}}(u)]$ to denote the set of edges entering [leaving] the vertex u. $\triangleleft$

Example 5.4.8: Consider the vertex u below.

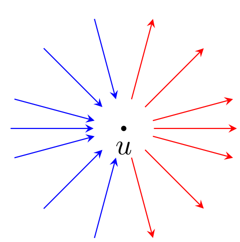


Figure 5.4.3: Edges entering and leaving u.

Here we have $\delta^{\text{in}}(u) = \{\text{edges in blue}\}$ and $\delta^{\text{out}}(u) = \{\text{edges in red}\}$. $\triangleleft$

Definition 5.4.9: Let $G = (V, E)$ be a graph. The node-edge incidence matrix (just incidence matrix for short) of G is a matrix A whose rows are indexed by V and whose columns are indexed by E. $\triangleleft$

Remark 5.4.10: Let A be the incidence matrix of G. If G is undirected, then for all $e = uv \in E$, the column A_e of A is

$$\begin{array}{cl} \begin{bmatrix} \mathbf{0} \\ 1 \\ \mathbf{0} \\ 1 \\ \mathbf{0} \end{bmatrix} & \begin{array}{l} \\ \rightarrow u \\ \\ \rightarrow v \\ \\ \end{array} \end{array}$$

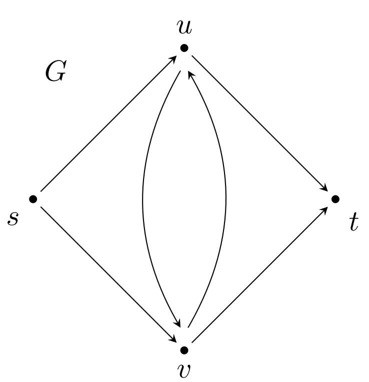
Every column consists of 0's and 1's and each contain only two 1's.

If G is directed and if $e = u \to v \in E$, the column A_e of A is

$$\begin{array}{cl} \begin{bmatrix} \mathbf{0} \\ -1 \\ \mathbf{0} \\ +1 \\ \mathbf{0} \end{bmatrix} & \begin{array}{l} \\ \rightarrow u \\ \\ \rightarrow v \\ \\ \end{array} \end{array}$$

So, $A_{u,e} = -1$, $A_{v,e} = 1$ and $A_{w,e} = 0$ for all $w \in V \setminus \{u, v\}$. $\triangleleft$

Example 5.4.11:



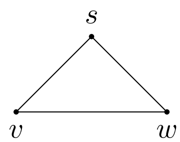
$$\implies A = \begin{array}{c} \\ s \\ u \\ v \\ t \end{array} \begin{array}{c} \begin{array}{cccccc} (s,u) & (s,v) & (u,v) & (v,u) & (u,t) & (v,t) \end{array} \\ \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Figure 5.4.4: Directed graph.

An example of a digraph G and its incidence matrix A. $\triangleleft$

Proposition 5.4.12: If A is the incidence matrix of a digraph, then A is totally unimodular.

Remark 5.4.13: Proposition 5.4.12 is false for an undirected graph. Consider the graph G (on the left) and its incidence matrix (on the right). We have $\det A = 2$.



$$A = \begin{array}{c c} & \begin{array}{ccc} uv & vw & wu \end{array} \\ \begin{array}{c} u \\ v \\ w \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \implies \det A = 2.$$

Figure 5.4.5: Undirected graph with $\det A = 2$.

This is a counter example to Proposition 5.4.12. ◁

Definition 5.4.14: An (undirected) graph is called ***bipartite*** with bipartition X, Y is $X \cup Y = V$ is a *partition* of V. We say a pair $X, Y \subsetneq V$ is a partition of V if

① $X, Y \neq \varnothing$,

② $X \cap Y = \varnothing$,

③ $X \cup Y = V$.

Informally, a graph G is bipartite if every edge has one end in X and one end in Y. ◁

Example 5.4.15: Example of a bipartite graph G.

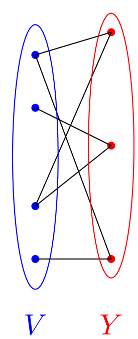


Figure 5.4.6: A bipartite graph.

Here X, Y is a partition of V. ◁

Lemma 5.4.16: Incidence matrix A of a bipartite graph is totally unimodular.

Proof: We can partition the rows of A into X, Y such that for each e, every column A_e has a 1 in a row of X and a 1 in a row of Y. So multiplying rows in X by -1 gives an incidence matrix of a digraph, which gives a totally unimodular matrix. Hence A is totally unimodular. □

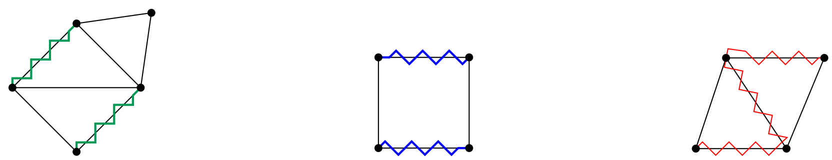
5.4.1 Matchings and Assignment Problems

Let $G = (V, E)$ be an undirected graph.

Definition 5.4.17: A ***matching*** in G is a subset $M \subseteq E$ such that no two edges of M share an end point. In other words, $|\delta(v) \cap M| \leq 1$ for all $v \in V$.

We say a matching M is a ***perfect*** if $|\delta(v) \cap M| = 1$ for all $v \in V$. ◁

Example 5.4.18: Examples of matchings.



(a) : matching (not perfect).

(b) : perfect matching.

(c) :not a matching.

Figure 5.4.7: Three undirected with graphs with a sets of edges.

Note that there is another perfect matching for the graph in Figure 5.4.7b. ◁

Example 5.4.19: Recall that job assignment problem discussed in subsection 1.4.4. We have

- set J of jobs,
- set W of workers,
- $|J| = |J|$, that is, the number of jobs are same of number of available workers.

We are given a list of (i, j) pairs where $i \in W$ and $j \in J$. The list indicates that job j can be assigned to worker i, for each such (i, j) entry we have a cost $c_{ij} \in \mathbb{R}$. This is the cost of assigning j to i. We want to know

1. if there is an assignment of jobs to workers such that every job compatible with list is assigned to exactly one worker. That is, every worker is assigned exactly one job,
2. and if there is, the minimum cost of such an assignment.

These translate to matching related questions on bipartite graphs.

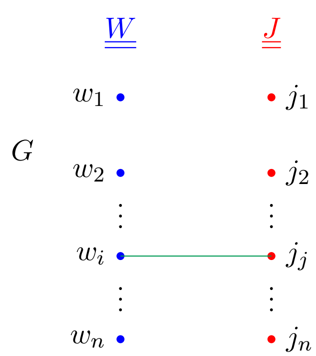


Figure 5.4.8: Bipartite graph $G = (V, E)$ where $V = W \cup J$.

Let $|J| = |W| = n$. If (i, j) is in the list, then c_{ij} is the cost of $w_i j_j$. So we have the following equivalent problem.

1. Does G have a matching M that is a perfect matching?
2. If so, we want to minimize the cost of M, $c(M)$, such that

$$c(M) \underset{\text{def}}{=} \sum_{e \in M} c_e.$$

This problem is called "min-cost perfect matching problem" (MCPM).

3. Another related problem is finding a matching M of G that maximizes $|M|$. This problem is called the "maximum cardinality (size) matching problem".

◁

Theorem 5.4.20: If G is a bipartite graph then $M(G)$ and $PM(G)$ are integral polytopes.

Proof: Exercise. ◁

We will use LP theory to prove various statements.

Notation 5.4.21: For a graph $G = (V, E)$, we define the sets $M(G)$ and $PM(G)$ as follows.

$$M(G) = \left\{ \mathbf{x} \in \mathbb{R}^{|E|} \;\middle|\; \sum_{e \in \delta(v)} x_e \leq 1, \quad \forall\, v \in V \text{ and } \mathbf{x} \geq \mathbf{0} \right\},$$

$$PM(G) = \left\{ \mathbf{x} \in \mathbb{R}^{|E|} \;\middle|\; \sum_{e \in \delta(v)} x_e = 1, \quad \forall\, v \in V \text{ and } \mathbf{x} \geq \mathbf{0} \right\}.$$

The integer points in $M(G)$ and $PM(G)$ correspond to matching and perfect matching in G respectively. Moreover, we can write the constraints of these sets as follows

$$\sum_{e \in \delta(v)} x_e \leq 1 \equiv A\mathbf{x} \leq \mathbb{1},$$

$$\sum_{e \in \delta(v)} x_e = 1 \equiv A\mathbf{x} = \mathbb{1},$$

where A is the incidence matrix and where $\mathbb{1}$ is the vector with all entries being 1's. ◁

Theorem 5.4.22: Let $G = (V, E)$ be a bipartite graph with bipartition X, Y where $|X| = |Y|$. Suppose $|\delta(v)| = d$ for all $v \in V$. Recall this is the degree of v. Then G has a perfect matching.

Proof: Setting $x_e = \frac{1}{d}$ for all $e \in E$ gives a point in $PM(G)$. So $PM(G) \neq \varnothing$. Hence it has an integral extreme point which gives a perfect matching. □

Example 5.4.23: Show that if G satisfies the conditions in Theorem 5.4.22 then there are d perfect matchings $M_1, \ldots, M_d$ such that

$$E = \bigcup_{i \in [d]} M_i \quad \text{and} \quad M_i \cap M_j = \begin{cases} M_i & \text{if } i = j, \\ \varnothing & \text{if } i \neq j, \end{cases} \quad \text{for} \quad i = 1, \ldots, d.$$ ◁

End of Lecture 17

Start of Lecture 18

Definition 5.4.24: Let $G = (V, E)$ be a graph. We say that $C \subseteq V$ is a ***vertex cover*** if for all $e = uv \in E$, $C \cap e \neq \varnothing$. ◁

Lemma 5.4.25: For any graph G, the maximum size of a matching is at most the minimum size of a vertex cover of G. That is,

$$(\max |M| \text{ s.t } M \text{ is a matching in } G) \leq (\min |C| \text{ s.t } C \text{ is a vertex cover of } G).$$

Proof: For any matching M and vertex cover C, C must include at least one node from every edge of M. Since the edges of M do not share any end-points, then we have $|C| \geq |M|$. We will use weak duality. Consider the LP and its dual

$$\begin{aligned} \text{(LP): } \max \quad & \sum_{e \in E} x_e, \\ \text{subject to} \quad & \sum_{e \in \delta(v)} x_e \leq 1, \\ \text{with} \quad & \mathbf{x} \geq \mathbf{0}. \end{aligned} \qquad \begin{aligned} \text{(DLP): } \min \quad & \sum_{v \in V} y_v, \\ \text{subject to} \quad & y_u + y_v \geq 1 \\ \text{with} \quad & \mathbf{y} \geq \mathbf{0}. \end{aligned}$$

Since by strong duality $\text{OPT}_{(\text{LP})} = \text{OPT}_{(\text{LP})}$, then we get

$$\begin{aligned} (\max |M| \text{ s.t } M \text{ is a matching in } G) &\underset{(\star)}{\leq} \text{OPT}_{(\text{LP})} = \text{OPT}_{(\text{LP})} \\ &\underset{(\star\star)}{\leq} \text{Opt. value among integer solutions to (DLP)} \\ &\underset{(\star\star\star)}{=} (\min |C| \text{ s.t } C \text{ is a vertex cover of } G). \end{aligned}$$

We used

$(\star)$: LHS is optimal value among integer solutions to (LP).

$(\star\star)$: Something. Exercise.

$(\star\star\star)$: An optimal integer solution to (DLP) can be assured to be a $\{0,1\}$-solution and if $\overline{\mathbf{y}} \in \{0,1\}^E$ is a solution to (DLP), then $C = \{u \mid \overline{\mathbf{y}}_u = 1\}$ is a vertex cover (and vice versa).

□

Remark 5.4.26: We check to see if we can find conditions that makes Lemma 5.4.25 hold in equality. Consider the graph G below.

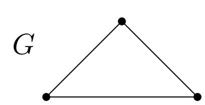


Figure 5.4.9: Graph G.

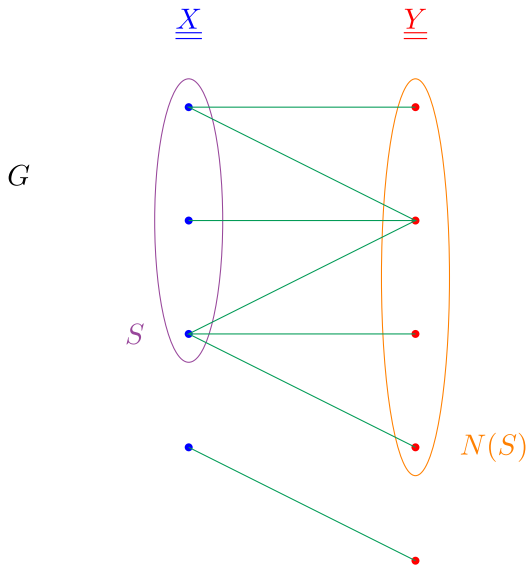
Here every matching has maximum size 1 and every vertex cover has size at least 2. ◁

Theorem 5.4.27 (Kőnig's theorem)**:** If G is bipartite, then

$$(\max |M| \text{ s.t } M \text{ is a matching in } G) = (\min |C| \text{ s.t } C \text{ is a vertex cover of } G).$$

Proof: Since the constraint matrices of (LP) and (DLP) are totally unimodular, then the in equalities ($\star$) and ($\star\star$) hold at equality $\square$

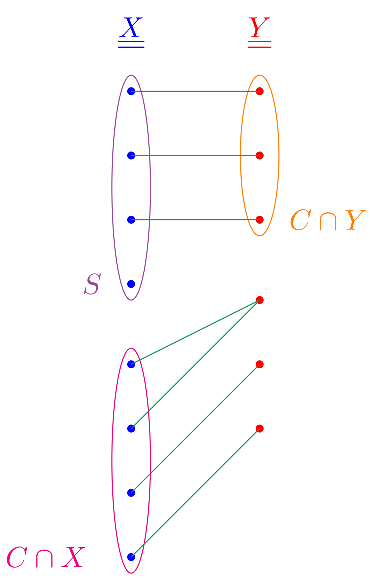
Corollary 5.4.28 (Hall's theorem)**:** Let $G = (V = X \cup Y, E)$ be bipartite graph with bipartition $X \cup Y$. Then, G has a perfect matching if and only if $|X| = |Y|$ and $|S| \leq |N(S)|$ for all $S \subseteq X$. Here S is a subset of X and $N(S)$ is the set of neighbors of S, i.e. the set of vertices that are adjacent to vertices in S.



For $S \subseteq X,\ N(S) \subseteq Y$,
$S \subseteq Y,\ N(S) \subseteq X$.

Figure 5.4.10: A bipartite graph G with $S \subseteq X$ in purple and $N(S)$ in orange.

Proof: Clearly, if $|X| \neq |Y|$, then G does not have a perfect matching. So assume $|X| = |Y|$. Suppose G has a perfect matching M. Then, for all $v \in S$, there is an edge of M that matches to v to a distinct node of $N(S)$. So $|S| \leq |N(S)|$.



For the converse, we prove the contrapositive. Suppose G does not have a perfect matching. Then, by Theorem 5.4.27, there is a vertex cover C such that $|C| < |X|$. Take $S = X \setminus C$. Then $N(S) \subseteq C \cap Y$ (since C is a vertex cover). Hence we have $|S| = |X| - |C \cap X|$. This gives us

$$\begin{aligned} |N(S)| &\leq |C \cap Y| \\ &= |C| - |C \cap X| \\ &< |X| - |C \cap X| = |S|. \end{aligned}$$ $\square$

Figure 5.4.11: G with the vertex vertex cover C.

We look for ways to increase the size of a matching in a bipartite graph?

Definition 5.4.29: We say that a node v is ***M-exposed*** if no edge of M is incident to v. If v is not M-exposed, then we say v is ***M-matched***. $\triangleleft$

Example 5.4.30: Consider the digraph D with perfect matching M.

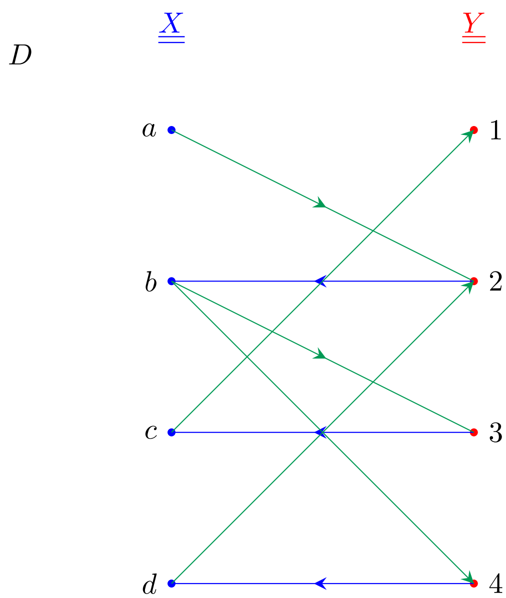
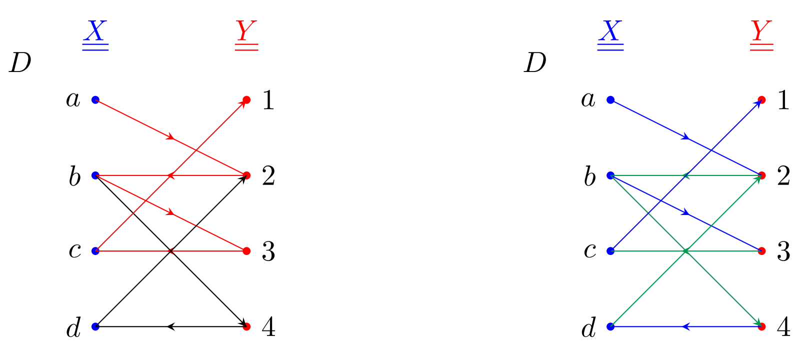


Figure 5.4.12: Directed graph D with perfect matching $\leftarrow : M$.

D is obtained by directing edges of M from Y to X and edges not in M from X to Y. Here a and 1 are M-exposed nodes. Here we have a directed path P from a to 1, where

$$P : a \longrightarrow 2 \longrightarrow b \longrightarrow 3 \longrightarrow c \longrightarrow 1.$$



(a) Path P (in red).

(b) New perfect matching M' (in blue).

Figure 5.4.13: A larger perfect marching M' from the path P in D.

Note that by switching edges along P (by swapping out edges of M in P and swapping in edges of P not in M), we get a larger matching $M' = \{a2, b3, c1, d4\}$. $\triangleleft$

Definition 5.4.31: A path in a directed graph D that goes from $M-$exposed node in X to $M-$exposed node in Y is called an M***-augmenting path***. ◁

Notation 5.4.32: For a bipartite graph $G = (V = X \cup Y, E)$, we denote

$$\widetilde{X} = \{x \in X \mid x \text{ is } M\text{-exposed}\},$$
$$\widetilde{Y} = \{y \in Y \mid y \text{ is } M\text{-exposed}\}.$$ ◁

Lemma 5.4.33: M is max-size matching if and only if there is no M-augmenting path.

Proof: $\implies$: Exercise.

$\impliedby$: Let T be the set of nodes reachable from $\widetilde{X}$ in a digraph D constructed from M. That is,

$$T = \{v \in V \mid \exists\, u \in \widetilde{X} \text{ such that there is a } u \rightsquigarrow v \text{ path in } D\}.$$

Note that we have $T \supseteq \widetilde{X}$ if $\widetilde{X} \neq \varnothing$ then $T = \varnothing$.

Claim 5.4.34*:* $C = (X \setminus T) \cap (T \cap Y)$ is a vertex cover and $|C| = |M|$.

Proof: Exercise. ◀

Then, by Theorem 5.4.27 and above claim, we are done. □

Remark 5.4.35: By running this algorithm on a bipartite graph $G = (X \cap U, E)$ where $|X| = |Y|$, we can find a perfect matching or find a vertex cover C with $|C| < |X|$, and hence we can find $S \subseteq X$ such that $|S| > |N(S)|$. If this is the case, we call the set S a ***deficient set***. ◁

5.4.2 Min-cost Perfect Matching Problem

We will consider a min-cost perfect matching problem in a bipartite graph $G = (V = X \cup Y, E)$ such that $|X| = |Y|$ and with costs $\{c_e\}_{e \in E}$. Consider the following LP and its dual.

$$\text{(P)}: \quad \min\ c_e x_e, \qquad \text{PM(G)}: \begin{cases} \displaystyle\sum_{e \in \delta(v)} x_e = 1, & \forall\, v \in V, \\ \text{with } \mathbf{x} \geq \mathbf{0}. \end{cases}$$

$$\text{(D)}: \max \sum_{v \in V} y_v, \qquad \text{subject to} \quad y_u + y_v \leq c_{uv}, \quad \forall\, e = uv \in E.$$

Note that (D) is always feasible.

Remark 5.4.36:

1. G has a perfect matching if and only if $\text{PM}(G) \neq \varnothing$.
2. M is a min-cost perfect matching if and only if $\chi^M \in \{0,1\}^E$ is an optimal solution to (P). Here χ^M is defined as

$$\chi^M_e = \begin{cases} 1 & \text{if } e \in M, \\ 0 & \text{if } e \notin M, \end{cases}$$

and it is called as the ***characteristic vector of*** M. Equivalently, this is true if and only if there exists a dual feasible $\overline{\mathbf{y}}$ such that χ^m and $\overline{\mathbf{y}}$ satisfy the CS conditions. That is, for all $e \in M$, we have $\overline{y}_u + \overline{y}_v = c_{uv}$. So given a dual feasible $\mathbf{y}$, define the equality subgraph of $\mathbf{y}$, $G^=(\mathbf{y})$, as

$$G^=(\mathbf{y}) \underset{\text{def}}{=} (V, E^=(\mathbf{y}) = \{uv \in E \mid y_u + y_v = c_{uv}\}).$$

If M is a perfect matching in $G^=(\mathbf{y})$, then M is a min-cost perfect matching. $\triangleleft$

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Out strategy is to maintain a dual feasible $\mathbf{y}$. We will look at $G^{=}(\mathbf{y})$. This gives us two cases.

1. We find a perfect matching M in $G^{=}(\mathbf{y})$ (in this case we are done).
2. There exists a deficient set $S \subseteq X$. In other words, $|S| > \left|N_{G^{=}(\mathbf{y})}\right|$ (where $N_{G^{=}(\mathbf{y})}$ is the neighbors of S in $G^{=}(\mathbf{y})$).
   a. If $N_G(S) = N_{G^{=}(\mathbf{y})}(S)$ then S is deficient in G. So G does not have a perfect matching.
   b. Otherwise, we have $N_G(S) \supsetneqq N_{G^{=}(\mathbf{y})}(S)$. So there exists $e = uv \in E \setminus E^{=}(\mathbf{y})$.

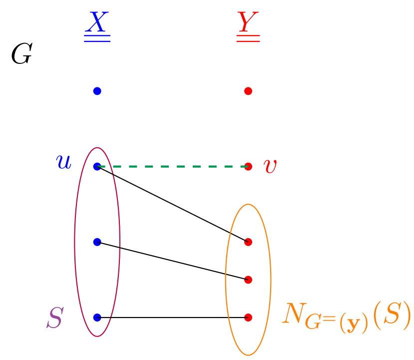

Figure 5.4.14: The edge in $E \setminus E^{=}(\mathbf{y})$ is shown in green.

So $y_u + y_v < c_{uv}$. Let

$$\varepsilon = \min\{c_{uv} - y_u - y_v \mid uv \in E, u \in S, v \notin N_{G^{=}(\overline{\mathbf{y}})}(S)\}.$$

Update y to

$$y \leftarrow \begin{cases} y_v + \varepsilon & \text{if } v \in S, \\ y_v - \varepsilon & \text{if } v \in N_{G^{=}(\overline{\mathbf{y}})}(S), \\ y_v & \text{if otherwise.} \end{cases}$$

New y is a feasible dual solution and dual objective increases by $\varepsilon(|S| - \left|N_{G^{=}(\mathbf{y})}(S)\right|) > 0$. Repeat the algorithm until termination.

Termination: Take the upper bound as $\sum_{e \in E} c_e$ where $c_e > 0$. If (rational input) $\sum_{v \in V} \overline{y}_v$ is strictly greater than the upper bound on min-cost PM, then LP-relaxation for min-cost PM is infeasible. So we can terminate.

Remark 5.4.37: There is a way of choosing augmenting paths and deficient set S to terminate in $O(|V|^2)$ steps. ◁

5.4.3 Min-cost Vertex Cover in General Graphs

Let $w_v \geq 0$ be the cost of vertex v. We want to find a vertex cover of minimum total cost. i.e. we have the following LP.

$$\begin{aligned} \text{(LP)}: \min \quad & \sum_{v \in V} w_v x_v, \\ \text{subject to} \quad & x_u + x_v \geq 1, \quad \forall\, uv \in E, \\ \text{with} \quad & \mathbf{x} \geq \mathbf{0}, \\ & \mathbf{x} \in \mathbb{Z}^{|V|} \text{ (integrality constraint)}. \end{aligned}$$

Note that we cannot drop integrality constraints from the above LP. Consider the graph with 3 vertices and 3 edges. We have

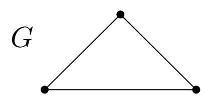

Figure 5.4.15: Triangle graph G.

Here we have

$$\text{(LP)}_{\text{OPT}} = 1.5, \quad \text{and} \quad \text{(IP)}_{\text{OPT}} = 2.$$

Remark 5.4.38: There is no known efficient algorithm. We take this as a fact. ◁

We have two approaches to the above problem.

1. Drop the polynomial requirement: Get algorithms that work fine in practice.
2. Drop optimality requirement: We want a polytime algorithm that always returns a solution of cost $\leq \alpha \cdot$ OPT, where $\alpha \geq 1$. This is referred as $\alpha-$approximation algorithm.

5.4.4 2-approximation for Vertex Cover

This is the best currently known approximation algorithm. We have the LP below and its dual.

$$\begin{aligned} \text{(VC-LP)}: \min \quad & \sum_{v \in V} w_v v_v, \\ \text{subject to} \quad & x_u + x_v \geq 1, \quad \forall\, e = uv \in E, \\ \text{with} \quad & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

$$\begin{aligned} \text{(D)}: \max \quad & \sum_{e \in E} z_e, \\ \text{subject to} \quad & \sum_{e \in S(V)} z_e \leq w_u, \\ \text{with} \quad & \mathbf{z} \geq \mathbf{0}. \end{aligned}$$

Algorithm 1:

1. Solve (VC-LP) to get optimal solution $\mathbf{x}^*$.
2. Take
$$C^{(1)} = \{u \in V \mid x_u^* \geq \frac{1}{2}\},$$
and let OPT_I be the optimal value of the IP.

Theorem 5.4.39: $C^{(1)}$ is a vertex cover and

$$w(C^{(1)}) = \sum_{v \in C^{(1)}} w_v \leq 2 \cdot \text{OPT}_{\text{(VC-LP)}} \leq 2 \cdot \text{OPT}_I.$$

Proof: For every $e = uv \in E$, at least one of $x_u^*, x_v^* \geq \frac{1}{2}$. So at least one of $u, v \in C^{(1)}$. For every $v \in C^{(1)}$ we have $w_v \leq 2w_v x_v^*$. □

Remark 5.4.40: These LPs can be solved in polytime. We take this as a fact. ◁

Algorithm 2: Suppose $\overline{\mathbf{z}}$ is a maximal dual feasible solution. That is, $\overline{\mathbf{z}}$ is a feasible solution to (D) and for all $\mathbf{z}' \geq \mathbf{z}$, $\mathbf{z}' \neq \mathbf{z}$ (so $\mathbf{z}'$ is not feasible to (D)). Take

$$C^{(2)} = \{u \in V \mid \sum_{e \in \delta(u)} \overline{z}_e = w_u\}.$$

Theorem 5.4.41: $C^{(2)}$ is a vertex cover and

$$w(C^{(2)}) \leq 2 \sum_{e \in E} \overline{z}_e \leq 2 \cdot \mathrm{OPT}_{(\text{VC-LP})} \leq 2 \cdot \mathrm{OPT}_I.$$

Proof: We have

$$w(C^{(2)}) = \sum_{v \in C^{(2)}} w_v = \sum_{v \in C^{(2)}} \left(\sum_{e \in \delta(v)} \overline{z}_e \right).$$

We can construct $\overline{z}$ by starting with $\overline{z} = 0$ and choosing edges in any order and then increasing $\overline{z}_e$ as much as possible while preserving dual feasibility. □

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5.4.5 Max-weight Matching in General Graphs

We have the below LP and its dual.

$$\begin{aligned} \text{(Mat-LP)}: \max \quad & \sum_{e\in E} w_e x_e, \\ \text{subject to} \quad & \sum_{e\in\delta(v)} x_e \le 1, \quad \forall\, v \in V, \\ \text{with} \quad & \mathbf{x} \ge \mathbf{0}. \end{aligned} \qquad \begin{aligned} \text{(Mat-D)}: \min \quad & \sum_{v\in V} y_v, \\ \text{subject to} \quad & y_u + y_v \ge w_{uv}, \quad \forall\, e = uv \in E, \\ \text{with} \quad & \mathbf{y} \ge \mathbf{0}. \end{aligned}$$

We have $\frac{1}{2}$-approximation algorithms.

5.4.5.1 Greedy Algorithm for Max-weight Matching

① Sort edges so that

$$w_{e_1} \ge \cdots \ge w_{e_k} > 0 \ge w_{e_{k+1}} \ge \cdots \ge w_{e_n}.$$

② Initialize $M \leftarrow \varnothing$. For $i = 1$ to k, if $M \cup \{e_i\}$ is a matching, set $M \leftarrow M \cup \{e_i\}$.

Theorem 5.4.42:

$$w(M) \ge \frac{1}{2} \cdot \mathrm{OPT}_{(\text{Mat-D})} \ge \frac{1}{2} \cdot \mathrm{OPT}_I.$$

Proof: Define $\overline{y}_u = \overline{y}_v = w_{uv} > 0$ if $uv \in M$ and if u is M-exposed then let $\overline{y}_u = 0$. By constriction, $w(M) = \frac{1}{2}\sum_{v\in V} \overline{y}_v$ and $\overline{\mathbf{y}} \ge \mathbf{0}$. We need to show $\overline{\mathbf{y}}$ satisfies the dual constraint, that is $y_u + y_v \ge w_{uv}$ for all $e = uv \in E$. Consider $e = uv = e_i$ with $w_e > 0$, so $i \le k$.

- If $e \in M$ then $\overline{y}_u = \overline{y}_v = w_{uv}$.
- If $e \notin M$, then at start of iteration, at least one of u, v is matched by M using some edge e' such that $w_{e'} \ge w_e$. So at least one of $\overline{y}_u$ OR $\overline{y}_v$ is set to $w_{e'} \ge w_e$.

□

Notation 5.4.43: For $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ we define the floor and ceiling of $\mathbf{v}$ as

$$\lfloor \mathbf{v} \rfloor = (\lfloor v_i \rfloor)_{i\in[n]}, \quad \text{and} \quad \lceil \mathbf{v} \rceil = (\lceil v_i \rceil)_{i\in[n]}.$$

That is, when we say floor of a vector we mean the vector obtained by taking the floor each component of the original vector. ◁

5.4.6 Methods for General IPs

Recall the definition of *valid inequality* that we introduced in section 2.3. Consider (IP) with objective function $\max \mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{x} \in \mathbb{Z}(\mathrm{P}) \underset{\text{def}}{=} P \cap \mathbb{Z}^n$ where $P \subseteq \mathbb{R}^n$ is a rational polyhedron. This is equivalent to the (LP) $\max \mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{x} \in P_I \underset{\text{def}}{=} \mathrm{conv}(\mathbb{Z}(\mathrm{P}))$.

The idea is to start with $Q = P$, and repeatedly add valid inequalities for P_I (that are also not valid for Q), until we find an optimal solution to our current (LP) $\max \mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{x} \in Q$ that is integral.

We want to find a valid inequality for P_I. Suppose

$$Q = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}.$$

This is our current polyhedron. Note that we have $\mathbb{Z}(Q) = \mathbb{Z}(\mathrm{P})$. By duality, any valid inequality $\alpha^\top \mathbf{x} \leq \beta$ for Q is implied by an inequality of the form

$$\mathbf{y}^\top A\mathbf{x} = \mathbf{y}^\top \mathbf{b}. \tag{$\star$}$$

We know ($\star$) is valid for Q. Then since $\mathbf{x} \geq \mathbf{0}$, the inequality

$$\lfloor \mathbf{y}^\top A \rfloor \mathbf{x} \leq \mathbf{y}^\top \mathbf{b},$$

is also valid for Q. Then,

$$\lfloor \mathbf{y}^\top A \rfloor \mathbf{x} \leq \lfloor \mathbf{y}^\top \mathbf{b} \rfloor, \tag{$\star\star$}$$

is valid for $\mathbb{Z}(\mathrm{P})$ and hence for P_I. This is because the LHS of ($\star\star$) is an integer for all $\mathbf{x} \in \mathbb{Z}(\mathrm{P})$. Note that this need not be valid for P. ($\star\star$) is referred as *Chvátal-Gomory cut/inequality*. We refer to the procedure of obtaining this inequality as the (CG) procedure.

5.4.7 Cutting Plane Algorithm

1. Start with $Q \leftarrow P$. We maintain $\mathbb{Z}(Q) = \mathbb{Z}(\mathrm{P})$. Note that we can consider $Q = \{\mathbf{x} \in \mathbb{R}^n \mid A'\mathbf{x} = \mathbf{b}', \mathbf{x} \geq \mathbf{0}\}$.
2. Consider the current (LP): $\max \mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{x} \in Q$.
3. Assume (LP) is not unbounded. If (LP) is infeasible, then STOP. (IP) is infeasible.
4. Otherwise, let $\mathrm{OPT}_{(\mathrm{LP})} = \overline{\mathbf{x}}$ found using the simplex method, corresponding to basis $\mathcal{B}$.
5. If $\overline{\mathbf{x}} \in \mathbb{Z}^n$, then $\overline{\mathbf{x}} \in \mathbb{Z}(Q) = \mathbb{Z}(\mathrm{P})$ and $\overline{\mathbf{x}} = \mathrm{OPT}_{(\mathrm{IP})}$.
6. Otherwise, consider the system
$$\underbrace{A'^{-1}_{\mathcal{B}} A'}_{\overline{A}} \mathbf{x} = \underbrace{A'^{-1}_{\mathcal{B}} \mathbf{b}}_{\overline{\mathbf{b}}}.$$
Since $\overline{\mathbf{x}} \notin \mathbb{Z}^n$, then $\exists\, i \in \mathcal{B}$ such that $\overline{x}_i = \overline{b}_i \notin \mathbb{Z}$ (so $\overline{x}_i$ is a fraction). Consider the constraint of $\overline{A}\mathbf{x} = \overline{\mathbf{b}}$ containing x_i.
$$x_i + \sum_{j \in N} \overline{a}_{ij} x_j = \overline{b}_i. \tag{$\dagger$}$$
This is the equation obtained by taking a suitable linear combination of $A\mathbf{x} = \mathbf{b}$. So applying (CG) procedure gives
$$x_i + \sum_{j \in N} \lfloor \overline{a}_{ij} \rfloor x_j \leq \lfloor \overline{b}_i \rfloor. \tag{$\dagger\dagger$}$$
This is the CG cut derived from x_i. Note that ($\dagger\dagger$) is valid for $Q_I = P_I$ but is violated by $\overline{\mathbf{x}}$ since
$$\overline{x}_i = \overline{b}_i > \lfloor \overline{b}_i \rfloor.$$
This is true because $\overline{b}_i \notin \mathbb{Z}$.

(7) Assign $Q \leftarrow Q \cap \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \text{ satisfies } (\dagger\dagger)\}$.

Chapter 6 – Computational Complexity: Lectures 20–21

Recall the definitions we made for *input size*, *size* and *polytime* in Definition 4.1.32.

Notation 6.0.1: We denote the ***class of problems that can be solved in polytime*** as P. ◁

Example 6.0.2: Some examples of polytime problems.

- Sorting n numbers.
- Solving a system of equations $A\mathbf{x} = \mathbf{b}$ where $A \in \mathcal{M}_{m\times n}(\mathbb{Z})$ and $\mathbf{b} \in \mathbb{Z}^m$. ◁

Example 6.0.3: Gaussian elimination (GE) is a polytime algorithm. GE reduces $A\mathbf{x} = \mathbf{b}$ to a simpler system $A'\mathbf{x} = \mathbf{b}'$ using row operations. This involves poly(m, n) arithmetic operations. But also we have to ensure that the intermediate entries we have also polynomial size. That is, they most have size poly(size$(A, \mathbf{b})$) where

$$\text{size}(A, \mathbf{b}) \underset{\text{def}}{=} \sum_{i,j} \text{size}(A_{ij}) + \sum_i \text{size}(b_i). \qquad \triangleleft$$

Remark 6.0.4: Every intermediate entry is a ratio of determinants of two square matrices of $A|\mathbf{b}$ (the vector $\mathbf{b}$ augmented to A). Moreover, if $M \in \mathcal{M}_{k\times k}(\mathbb{Q})$, then $\log|\det M| = \text{poly}(\text{size}(M))$. ◁

Definition 6.0.5: Problems that can be answered by "YES" or "NO" are called ***decision problems***. ◁

Example 6.0.6: The problem below is a decision problem.

> Does graph G have a vertex cover of size $\leq k$.

Note that this is the decision version of vertex cover. This is also problem is also in NP. ◁

Definition 6.0.7: We say a decision problem is in ***non-deterministic polynomial time*** (NP) if there exists a polytime verifier algorithm $B(\cdot,\cdot)$ and a polynomial p such that

(1) For every YES-instance x, there exists a certificate y with

$$\text{size}(y) \leq p(\text{size}(x)) \text{ such that } B(x, y) = \text{YES}.$$

(2) For every NO-instance x, $B(x, y) = \text{NO}$ for all y. ◁

Example 6.0.8: The problems below are in NP.

LP feasibility: Given $A, \mathbf{b}$, is $A\mathbf{x} \leq \mathbf{b}$ feasible?
LP infeasibility: Given $A, \mathbf{b}$, is $A\mathbf{x} \leq \mathbf{b}$ infeasible?

Here both the YES and NO instances for both problems have certificates. ◁

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Example 6.0.9: Consider the problem of LP-feasibility.

> Given rational $A, \mathbf{b}$, does $A\mathbf{x} \le \mathbf{b}$ have a feasible solution?

We can write the S.I.F in S.E.F as follows.

$$A\mathbf{x} \le \mathbf{b} \equiv \underbrace{[A \ -A \ I]}_{A'} \underbrace{\begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{s} \end{bmatrix}}_{\mathbf{x}'} = \mathbf{b}, \quad \text{where} \quad \mathbf{x}^+, \mathbf{x}^-, \mathbf{s} \ge \mathbf{0}.$$

Hence we obtain

$$A'\mathbf{x}' = \mathbf{b}, \text{ where } \mathbf{x}' \ge \mathbf{0} \text{ is feasible} \iff A'\mathbf{x}' = \mathbf{b}, \text{ where } \mathbf{x}' \ge \mathbf{0} \text{ has a BFS.}$$

Hence, if $\overline{\mathbf{x}}$ is a BFS, then $\text{size}(\overline{\mathbf{x}}) = \text{poly}(\text{size}(A', \mathbf{b}))$ (this follows from Remark 6.0.4). ◁

Remark 6.0.10: From the above example, we infer that the problems below are in NP.

1. **LP Infeasibility:** Is $A\mathbf{x} \le \mathbf{b}$ is infeasible? (This is by Farkas' Lemma in Remark 2.1.8)
2. **LP Optimality:** Does the LP $\max \mathbf{c}^\top \mathbf{x}$ subject to $A\mathbf{x} \le \mathbf{b}$ have an optimal solution? (This is by duality)
3. **LP Unboundedness:** Is $\max \mathbf{c}^\top \mathbf{x}$ subject to $A\mathbf{x} \le \mathbf{b}$ unbounded? (This is by using the fact that if LP is unbounded then there exists a certificate of unboundedness)
4. $\mathsf{P} \subseteq \mathsf{NP}$: Given an instance x and verifier $B(\cdot, \cdot)$, the problem ignores the certificate and runs polytime algorithm for the problem on x. i.e. A decision problem in P is also in NP. ◁

Remark 6.0.11: This gives rise to the question. Is $\mathsf{NP} \subseteq \mathsf{P}$? i.e. Is $\mathsf{P} = \mathsf{NP}$?

The consensus to this question is no. ◁

Definition 6.0.12: Given problems A, B, we say that A ***polytime reduces to*** B, denoted $A \le_p B$ if given an algorithm $\mathcal{B}$ for B, we can solve problem A by making a polynomial number of calls to $\mathcal{B}$ and a polynomial number of elementary operations. Note that if $A \le_p B$, then A is no harder than B. So if we can solve B in polytime, then we can also solve A in polytime as well. ◁

Example 6.0.13: The problem of LP infeasibility polytime reduces to problem of LP feasibility. ◁

6.0.1 NP-completeness and NP-hardness

Definition 6.0.14: Given a problem A, we say

- A is ***NP-hard*** if $X \le_p A$ for all $X \in \mathsf{NP}$,
- A is ***NP-complete*** if $A \in \mathsf{NP}$ and A is NP-hard. ◁

Remark 6.0.15: Let A be a problem.

- If A is NP-hard and $A \in \mathsf{P}$ then $\mathsf{NP} = \mathsf{P}$.

- If A is NP-complete, and $A \notin \mathsf{P}$, then $\mathsf{NP} \not\subseteq \mathsf{P}$.

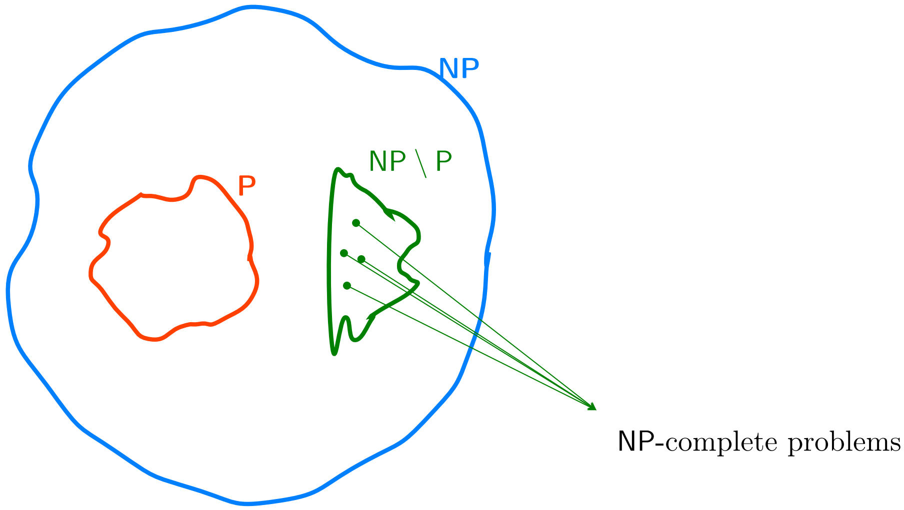

Figure 6.0.1: P and NP.

◁

6.0.1.1 Examples of NP-complete Problems

Example 6.0.16: Some examples of NP-complete problems.

1. Decision version of the vertex cover problem in Example 6.0.6.
2. $\{0,1\}-$IP feasibility: Given $A \in \mathcal{M}_{m\times n}(\mathbb{Z})$ and $\mathbf{b} \in \mathbb{Z}^m$, does $A\mathbf{x} \le \mathbf{b}$ have a $\{0,1\}$-solution?

We want to show 2 is NP-complete. To show A is NP-hard, we can take any other NP-hard problem B and show $B \le_p A$. This is because for all $X \in \mathsf{NP}$, we have $X \le_p B$. So B is NP-hard. Since $\le_p$ is transitive, so $X \le_p B \le_p A$, which gives us $X \le_p A$ for all $X \in \mathsf{NP}$. ◁

Theorem 6.0.17 (Cook-Levin 1971)**:** We show a brief introduction of the theorem. If a problem has

$$n\ \{0,1\}\ \text{(binary) variables } x_1,\dots,x_n \text{ and } m \text{ constraints } (z_{i_1}=1)\ \text{OR}\ \dots\ \text{OR}\ (z_{i_k}=1)\ \forall\, i \in [m],$$

then we classify this problem as a SAT problem. We ask if this problem have a feasible solution. The idea $X \le_p \mathsf{SAT}$ where $X \in \mathsf{NP}$ (verifier B). The theorem states that SAT is NP-complete. So, if we can show SAT is in P, then we show $\mathsf{P} = \mathsf{NP}$.

Example 6.0.18: Show the problem below is NP-complete.

> Given $A \in \mathcal{M}_{m\times n}(\mathbb{Z})$ and $\mathbf{b} \in \mathbb{Z}^m$, does $A\mathbf{x} \le \mathbf{b}$ have an integer solution.

This problem is known as the IP- feasibility problem. ◁

Chapter 7 – Non-Linear Programming: Lectures 21–x

Definition 7.0.1: We call programs of the form

$$\begin{array}{ll} \min & f(x), \\ \text{subject to} & g_i(x) \le 0, \\ \text{where} & f, g_i : \mathbb{R}^n \to \mathbb{R}, \quad \forall\, i \in [m], \end{array}$$

as ***non-linear programs*** (NLP). ◁

Remark 7.0.2: If the functions f and g_i are of the form $\alpha^\top \mathbf{x} + \beta$ (linear expressions) then we get an LP. ◁

Remark 7.0.3: In general, we can consider any optimization problem as

$$\begin{array}{ll} (\star) : \min & f(\mathbf{x}), \\ \quad\ \text{subject to} & \mathbf{x} \in S \subseteq \mathbb{R}^n. \end{array}$$

We can define

$$g(\mathbf{x}) = \begin{cases} 0 & \text{if } x \in S, \\ 1 & \text{if } x \notin S. \end{cases}$$

So $(\star)$ is equivalent to

$$\begin{array}{ll} (\star) : \min & f(\mathbf{x}), \\ \quad\ \text{subject to} & g(\mathbf{x}) \le 0. \end{array}$$

◁

7.1 Convex Analysis

Remark 7.1.1: We might experience some issues while dealing with NLPs.

1. Feasible region might be "complicated". e.g. we can have a non-convex set (as in IPs).
2. The function f could be "complicated". For example, consider the function $f(x)$ below.

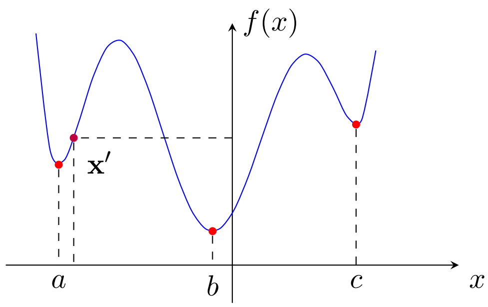

Figure 7.1.1: Complicated function. Here we have $f(b) < f(a) < f(c)$.

Suppose we start the algorithm at $\mathbf{x}' = (x', f(x'))$. We will tend to move closer to a since $f(a) < f(x')$ which will make us stuck there. So, an example of a completed function f is a function that has local minima that are not global minima.

$\triangleleft$

Notation 7.1.2: For $\mathbf{x} \in \mathbb{R}^n$, we have the Euclidean norm (ℓ-2 norm) as

$$\|\mathbf{x}\| \underset{\text{def}}{=} \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{\mathbf{x}^\top \mathbf{x}}.$$

$\triangleleft$

Definition 7.1.3: Consider the non-linear program below

$$\begin{aligned}\text{(NLP)}: \min \quad & f(\mathbf{x}),\\ \text{subject to} \quad & \mathbf{x} \in S \subseteq \mathbb{R}^n,\\ \text{where} \quad & f : \mathbb{R}^n \to \mathbb{R}.\end{aligned}$$

We say that $\overline{\mathbf{x}} \in S$ is a ***local minimum*** of (NLP) if there exists $\varepsilon > 0$ such that $f(\overline{\mathbf{x}}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} \in S$ and $\|\mathbf{x} - \overline{\mathbf{x}}\| \leq \varepsilon$.

We say $\overline{\mathbf{x}} \in \mathbb{R}^n$ is a ***global minimum*** of (NLP) if $f(\overline{\mathbf{x}}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$. $\triangleleft$

Definition 7.1.4: Let $S \subseteq \mathbb{R}^n$ be a convex set. A function $f : S \to \mathbb{R}$ is called a ***convex function*** (over S) if $\forall\, \mathbf{x}, \mathbf{y} \in S$ and $\forall\, \lambda \in [0, 1]$, we have $f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y})$. We say a function g is ***concave*** if $-g$ is convex. $\triangleleft$

Exercise 7.1.5: Let $\mathbf{x} \in \mathbb{R}^n$. Show $f(\mathbf{x}) = \|\mathbf{x}\|$ is convex. $\triangleleft$

Definition 7.1.6: For a convex function $f : \mathbb{R}^n \to \mathbb{R}$. The ***level set*** of f for $\alpha \in R$, $L_\alpha(f)$, is defined as

$$L_\alpha(f) \underset{\text{def}}{=} \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}.$$ $\triangleleft$

Proposition 7.1.7: Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Then $L_\alpha(f)$ is a convex set for all $\alpha \in \mathbb{R}$.

Proof: For any $\mathbf{x}, \mathbf{y} \in L_\alpha(f)$ and any $\lambda \in [0, 1]$, we have

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \leq \alpha \implies \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in L_\alpha(f).$$ $\square$

Definition 7.1.8: We say an (NLP) of the form

$$\begin{aligned}\min \quad & f(x),\\ \text{subject to} \quad & g_i(x) \leq 0,\\ \text{where} \quad & f, g_i : \mathbb{R}^n \to \mathbb{R}, \quad \forall\, i \in [m],\end{aligned}$$

is ***convex*** if the functions f, g_i are convex for $i = 1, \ldots, m$. $\triangleleft$

Lemma 7.1.9: Let (CP) be a convex program where

$$\begin{aligned}\text{(CP)}: \min \quad & f(\mathbf{x}),\\ \text{subject to} \quad & g_i(\mathbf{x}) \leq 0,\\ \text{where} \quad & f, g_i : S \subseteq \mathbb{R}^n \to \mathbb{R} \text{ are convex}, \quad \forall\, i \in [m],\end{aligned}$$

Then

1. The feasible region of (CP) is convex.
2. Any local minimum of (CP) is also a global minimum.

Proof: Exercise. ◁

End of Lecture 21

Start of Lecture 22

Remark 7.1.10:

- Every LP is a convex program.
- An IP is not a convex program (because we cannot write its feasible region as a convex set).

In general the we will work on convex programs of the form

$$\begin{array}{ll} \min & f(\mathbf{x}), \\ \text{subject to} & \mathbf{x} \in S \subseteq \mathbb{R}^n, \\ \text{where} & f : S \subseteq \mathbb{R}^n \to \mathbb{R} \text{ is convex.} \end{array}$$

Let S be the domain of f. If S is convex, then the program with objective function as f and feasible region S is a convex program. ◁

Definition 7.1.11: We say $f : S \to \mathbb{R}^n$ is ***strictly convex*** if for all $\mathbf{x}, \mathbf{y} \in S$ and for all $\lambda \in [0, 1]$ we have

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) < \lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y}).$$ ◁

Proposition 7.1.12: If the objective function of a convex program is strictly convex, and if the function has a global maximum then the function has a unique global minimum.

Proof: Exercise. ◁

7.1.1 Definitions from Calculus and Analysis

Definition 7.1.13: Let $x^{(1)}, x^{(2)} \cdots \equiv \left\{x^{(i)}\right\}_{i\in\mathbb{N}}$ be a sequence of real numbers. We say that $\overline{x} \in \mathbb{R}$ is the ***limit*** of this sequence, denoted by

$$\lim_{i\to\infty} x^{(i)} = \overline{x}, \quad \text{or} \quad \{x^{(i)}\}_{i\in\mathbb{N}} \to \overline{x},$$

if $\forall\, \varepsilon > 0$, $\exists\, n$ such that $\left|x^{(i)} - \overline{x}\right| < \varepsilon$ for all $i \geq n$.

We say that the limit is ∞ $[-\infty]$ if for all $t \in \mathbb{R}$, there exists n such that $x^{(i)} > t$ $[x^{(i)} < t]$ for all $i \geq n$.

Let $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \cdots \in \mathbb{R}^n$ be a sequence of vectors. We say the vector $\overline{\mathbf{x}} \in \mathbb{R}^n$ is the limit of this sequence if

$$\left\{\mathbf{x}^{(i)} \in \mathbb{R}^n\right\}_i \to \overline{\mathbf{x}} \in \mathbb{R}^n \text{ if } \forall\, j \in [n], \left\{x_j^{(i)}\right\}_{i\in\mathbb{N}} \to \overline{x}_j \in \mathbb{R}.$$

That is, the limit of a vector is taken component wise. ◁

Definition 7.1.14: Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$. We say f is ***continuous*** at $\overline{\mathbf{x}} \in S$ if whenever a sequence $\{\mathbf{x}^{(i)} \in S\}_{i\in\mathbb{N}} \to \overline{\mathbf{x}}$, we have $\{f(\mathbf{x}^{(i)})\}_{i\in\mathbb{N}} \to f(\overline{\mathbf{x}})$.

If f is continuous on all $\mathbf{x} \in S$, we say f is continuous on S. ◁

Definition 7.1.15: We define an ***open ball*** $B(\overline{\mathbf{x}}, \delta)$ as a set around $\overline{\mathbf{x}} \in \mathbb{R}^n$ of radius $\delta > 0$. i.e.

$$B(\overline{\mathbf{x}}, \delta) \underset{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \overline{\mathbf{x}}\| < \delta\}.$$ ◁

Definition 7.1.16: Let $S \subseteq \mathbb{R}^n$. We define the ***interior*** of S, denoted $\operatorname{int} S$, as

$$\operatorname{int} S \underset{\text{def}}{=} \{\mathbf{x} \in S \mid \exists\, \delta > 0 \text{ such that } B(\mathbf{x}, \delta) \subseteq S\}. \qquad \triangleleft$$

Definition 7.1.17: Let $S \subseteq \mathbb{R}^n$. We define the ***closure*** of S, denoted $\operatorname{cl} S$, as

$$\operatorname{cl} S \underset{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^n \mid \exists \text{ sequence } \{\mathbf{x}^{(i)} \in S\}_{i \in \mathbb{N}} \text{ that has } \mathbf{x} \text{ as its limit}\}. \qquad \triangleleft$$

Example 7.1.18: If $S = (0, 1]$, then $\operatorname{int} S = (0, 1)$ and $\operatorname{cl} S = [0, 1]$. $\triangleleft$

Definition 7.1.19: Let $S \subseteq \mathbb{R}^n$. We say

- S is ***open*** if $S = \operatorname{int} S$.
- S is ***closed*** if $S = \operatorname{cl} S$. $\triangleleft$

Remark 7.1.20: $\varnothing$ and $\mathbb{R}^n$ are both open and closed. $\triangleleft$

Definition 7.1.21: We say that a set $S \subseteq \mathbb{R}^n$ is ***bounded*** if there exists some $\gamma \in \mathbb{R}$ such that $\max_{j \in [n]} |x_j| \leq \gamma$ for all $\mathbf{x} \in S$. i.e. For all $\mathbf{x} = (x_1, \ldots, x_n) \in S \subseteq \mathbb{R}^n$ and for all $j \in [n]$, $x_j \in [-\gamma, \gamma]$. i.e. $\exists\, r > 0$ such that $\|\mathbf{x}\| \leq r$ for all $\mathbf{x} \in S$ $\triangleleft$

Definition 7.1.22: We say a set $S \subseteq \mathbb{R}^n$ is ***compact*** if it is closed and bounded. This characterization is a result of Heine-Borel theorem. $\triangleleft$

Theorem 7.1.23 (Bolzano-Weierstrass)**:** Let $S \subseteq \mathbb{R}^n$ be compact. Then every sequence $\{\overline{\mathbf{x}}^{(i)} \in S\}_{i \in \mathbb{N}}$ has an infinite subsequence $\{\mathbf{x}^{(i_k)}\}_{k \in \mathbb{N}}$ that converges to a point in S.

Definition 7.1.24: Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$. We define the ***infimum*** of f over S, denoted

$$\inf f \underset{\text{def}}{=} \inf\{f(\mathbf{x}) \mid \mathbf{x} \in S\},$$

as the largest $z \in \mathbb{R}$ such that $z \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$.

We define the ***supremum*** of f over S, denoted

$$\sup f \underset{\text{def}}{=} \sup\{f(\mathbf{x}) \mid \mathbf{x} \in S\},$$

as the smallest $z \in \mathbb{R}$ such that $z \geq f(\mathbf{x})$ for all $\mathbf{x} \in S$. $\triangleleft$

Remark 7.1.25: Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$. By convention, if

- $S = \varnothing$, then $\inf f(\mathbf{x}) = \infty$ and $\sup f(\mathbf{x}) = -\infty$.
- $\{f(\mathbf{x}) \mid \mathbf{x} \in S\}$ is unbounded from below, i.e. $\forall\, t \in \mathbb{R}$, $\exists\, \mathbf{x} \in S$ such that $f(\mathbf{x}) < t$, then $\inf f(x) = -\infty$.
- $\{f(\mathbf{x}) \mid \mathbf{x} \in S\}$ is unbounded from above, i.e. $\forall\, t \in \mathbb{R}$, $\exists\, \mathbf{x} \in S$ such that $f(\mathbf{x}) > t$, then $\sup f(x) = \infty$. $\triangleleft$

Remark 7.1.26: If there exists $\overline{\mathbf{x}} \in S$ such that $f(\overline{\mathbf{x}}) = \inf f(\mathbf{x})$ then we say that infimum is attained and we can replace inf by min. Similarly for sup (we replace sup by max). $\triangleleft$

Theorem 7.1.27: Let $S \subseteq \mathbb{R}^n$. Suppose $S \neq \varnothing$ and S is compact. Let $f : S \to \mathbb{R}$ be continuous. Then for all $\mathbf{x} \in S$, $\inf f(\mathbf{x})$ and $\sup f(\mathbf{x})$ are both attained.

Proof: Consider the infimum. Since S is compact, then S is closed and bounded. We want to show $\{f(\mathbf{x})\}_{\mathbf{x}\in S}$ is bounded. Suppose, for contradiction, $\{f(\mathbf{x})\}_{\mathbf{x}\in S}$ is unbounded from below. We can take the sequence $\{\mathbf{x}^{(i)} \in S\}_{i\in\mathbb{N}}$ such that $\{f(\mathbf{x}^{(i)})\}_{i\in\mathbb{N}} \to -\infty$. But by Bolzano-Weierstrass theorem, we also have a convergent infinite subsequence that converges to $\mathbf{x}' \in S$ which gives us a contradiction. Similarly if we assume f is unbounded from above. Hence, $\{f(\mathbf{x})\}_{\mathbf{x}\in S}$ is bounded.

Since $S \neq \varnothing$, then $\inf f(\mathbf{x}) = z$, for some $z \in \mathbb{R}$. By definition, for all $\varepsilon > 0$, there exists $\mathbf{x} \in S$ such that $z \leq f(x) \leq z + \varepsilon$. Define a sequence $\{\mathbf{x}^{(i)} \in S\}_{i\in\mathbb{N}}$ where $\mathbf{x}^{(i)}$ satisfies

$$z \leq f(\mathbf{x}^{(i)}) \leq z + \frac{1}{2}i, \quad \forall\, i \in \mathbb{N}.$$

We observe that

1. $\{f(x^{(i)})\}_{i\in\mathbb{N}} \to z$.
2. By Bolzano-Weierstrass theorem, there exists an infinite subsequence $\{\mathbf{x}^{(i_k)}\}_{k\in\mathbb{N}}$ that converges to (has a limit of) some $\overline{\mathbf{x}} \in S$.
3. By 1, $\{f(\mathbf{x}^{(i_k)})\}_{k\in\mathbb{N}} \to z$.

Since f is continuous, then by 2 and 3 we have $f(\overline{\mathbf{x}}) = z$. So $\overline{\mathbf{x}}$ is a *minimizer* of f over S. $\square$

7.1.2 Convex Programs

We will use the definition of nearest point and infimum/supremum to determine if a program has a unique optimal solution.

Definition 7.1.28: let $S \subseteq \mathbb{R}^n$. A point $\mathbf{x}' \in S$ is called a ***nearest point*** to $\mathbf{z} \in \mathbb{R}^n$ if $\|\mathbf{x}' - \mathbf{z}\| \leq \|\mathbf{x} - \mathbf{z}\|$ for all $\mathbf{x} \in S$. $\triangleleft$

Lemma 7.1.29: Let $S \subseteq \mathbb{R}^n$ be non-empty and closed. Then, any $\mathbf{z}$ has a nearest point in S.

Proof: Define $f(\mathbf{x}) = \|\mathbf{x} - \mathbf{z}\|$.

Exercise 7.1.30: Show that f is continuous. $\triangleleft$

Define $S' = \{\mathbf{x} \in S \mid f(\mathbf{x}) \leq f(\mathbf{x}_0)\}$ where $\mathbf{x}_0$ is some point in $S \neq \varnothing$. Note that S' is bounded. Since S is closed, then S' is also closed. Then S' is compact. Clearly we also have

$$\inf f(\mathbf{x}) \text{ s.t } \mathbf{x} \in S' = \inf f(\mathbf{x}) \text{ s.t } \mathbf{x} \in S.$$

By Theorem 7.1.27, $\inf f(\mathbf{x})$ such that $\mathbf{x} \in S'$ is attained by some $\overline{\mathbf{x}} \in S$ which is a nearest point to $\mathbf{z} \notin S$. $\square$

Proposition 7.1.31: Let $S \subseteq \mathbb{R}^n$ be a closed, convex set. Then every $\mathbf{z} \in \mathbb{R}^n$ has a unique nearest point in S.

Proof: Exercise. *Hint:* $f(\mathbf{x}) = \|\mathbf{x} - \mathbf{z}\|$ is strictly convex. $\triangleleft$

Definition 7.1.32: Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$. Let $\overline{\mathbf{x}} \in \operatorname{int} S$. We say that f is ***differentiable*** at $\overline{\mathbf{x}}$ if there exists a vector, $\nabla f(\overline{\mathbf{x}}) \in \mathbb{R}^n$, called the ***gradient*** of f (at $\overline{\mathbf{x}}$), such that

$$\lim_{\mathbb{R}^n \ni \mathbf{d} \to \mathbf{0}} \frac{f(\overline{\mathbf{x}} + \mathbf{d}) - f(\overline{\mathbf{x}}) - \mathbf{d}^\top \nabla f(\overline{\mathbf{x}})}{\|\mathbf{d}\|} = 0. \tag{7.1.1}$$

One line... ◁

Remark 7.1.33:

1. When $n = 1$ we have $\nabla f(\overline{x}) = f'(\overline{x})$ and we refer gradient as the derivative.
2. When we take $\mathbf{d}$'s of the form $\varepsilon \mathbf{e}_i$ where $\mathbf{e}_i$ is the i^{th} standard basis we get

$$(\nabla f(\overline{\mathbf{x}}))_i = \lim_{\varepsilon \to 0} \frac{f(\overline{\mathbf{x}} + \varepsilon \mathbf{e}_i) - f(\overline{\mathbf{x}})}{\varepsilon} = \left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}}.$$

3. If $S \underset{\text{open}}{\subseteq} \mathbb{R}^n$ and if f is differentiable at all $\mathbf{x} \in S$ then we say f is differentiable on S.

◁

Definition 7.1.34: Suppose f is differentiable at $\overline{\mathbf{x}} \in \operatorname{int} S$. Let $\mathbf{d} \in \mathbb{R}^n$. Since $\overline{\mathbf{x}} \in \operatorname{int} S$, then there exists ε such that for all $\delta \in (-\varepsilon, \varepsilon)$ we have

$$\overline{\mathbf{x}} \pm \delta \mathbf{d} \in S.$$

Define $g : (-\varepsilon, \varepsilon) \to \mathbb{R}$ as

$$g(t) \underset{\text{def}}{=} f(\overline{\mathbf{x}} + t\mathbf{d}).$$

This is called the ***projection*** of f along $\mathbf{d}^\top$. ◁

Proposition 7.1.35: Using (7.1.1), we can infer that

$$\beta \underset{\text{def}}{=} g'(0) = \nabla f(\overline{\mathbf{x}})^\top \mathbf{d}.$$

If $\beta < 0$ then there exists $\delta \in \mathbb{R}$ such that $\varepsilon \geq \delta > 0$ such that for all $t \in [0, \delta)$, we have $g(t) < g(0)$.

Proof: Suppose, for contradiction, the hypothesis is false. Then for all $\delta > 0$, there exists $t \in [0, \delta)$ such that $g(t) \geq g(0)$. So we can define a sequence $\{t_i \in (0, \varepsilon]\}_{i \in \mathbb{N}}$ such that $\{t_i\}_{i \in \mathbb{N}} \to 0$ and $g(t_i) \geq g(0)$ for all $i \in \mathbb{N}$. Since $\{t_i\}$ is a sequence that converges to zero for which $\frac{g(t_i) - g(0)}{t_i} \geq 0$ for all $i \in \mathbb{N}$ then

$$g'(0) = \lim_{\varepsilon \to 0} \frac{g(\varepsilon) - g(0)}{\varepsilon} \neq 0.$$

So if $\nabla f(\overline{\mathbf{x}})^\top \mathbf{d} < 0$, then there exists $\delta > 0$ such that for all $t \in (0, \delta]$ we have $f(\overline{\mathbf{x}} + t\mathbf{d}) < f(\overline{\mathbf{x}})$. □

End of Lecture 22

Start of Lecture 23

Proposition 7.1.36: Suppose $f : \mathbb{R}^n \to \mathbb{R}$. For any $\overline{\mathbf{x}} \in \mathbb{R}^n$ and any $\mathbf{d} \in \mathbb{R}^n$, define $g_{\overline{\mathbf{x}},\mathbf{d}} : \mathbb{R} \to \mathbb{R}$ as $g_{\overline{\mathbf{x}},\mathbf{d}}(t) = f(\overline{\mathbf{x}} + t\mathbf{d})$. Note that this is same as projecting f. f is convex if and only if $g_{\overline{\mathbf{x}},\mathbf{d}}$ is convex for all $\overline{\mathbf{x}}, \mathbf{d} \in \mathbb{R}^n$.

Proof: Exercise. ◁

Proposition 7.1.37: Suppose $f : S \to \mathbb{R}$ is differentiable at $\overline{\mathbf{x}} \in \operatorname{int}(S)$. Consider the function $g(t) = f(\overline{\mathbf{x}} + t\mathbf{d})$ for $t \in (-\varepsilon, \varepsilon)$ for $\varepsilon > 0$ small enough. Since g is differentiable, then it is continuous at $t = 0$. For any $\gamma > 0$, there exists $\delta > 0$ such that $|g(t) - g(0)| < \gamma$ for all $t \in [-\delta, \delta]$.

i.e. $\forall\, \mathbf{d} \in \mathbb{R}^n$ and for all $\gamma > 0$, there exists $\delta > 0$ such that

$$|f(\overline{\mathbf{x}} + t\mathbf{d}) - f(\overline{\mathbf{x}})| \leq \gamma, \qquad \forall\, t \in [0, \delta].$$

Proof: Exercise. ◁

Theorem 7.1.38: Let $I \subseteq \mathbb{R}$ be an interval. Let $g : I \to \mathbb{R}$ be differentiable everywhere on I. Then the following are equivalent.

1. g is convex.
2. $\forall\, \mathbf{x}, \mathbf{y} \in I$, $g(\mathbf{y}) - g(\mathbf{x}) \geq g'(\mathbf{x})(y - x)$. The tangent curve (in red) always lies below $g(x)$. Consider the example in 2-D.

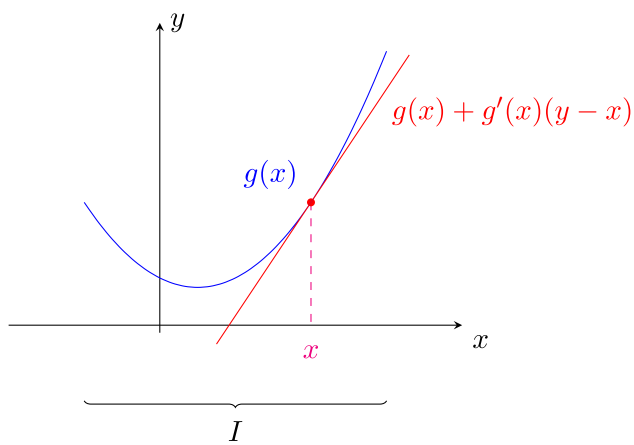

Figure 7.1.2: Tangent curve in red.

3. $g'(\mathbf{x})$ is non-decreasing in $\mathbf{x}$. We will not prove 3.

Proof: 2 $\Longrightarrow$ 1: Take $\mathbf{x}, \mathbf{y} \in I$ and let $\mathbf{z} = \lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in I$ for $\lambda \in [0, 1]$. We have

$$\begin{aligned} g(\mathbf{x}) - g(\mathbf{z}) &\geq g'(\mathbf{z})(\mathbf{x} - \mathbf{z})\big] \cdot \lambda \\ g(\mathbf{y}) - g(\mathbf{z}) &\geq g'(\mathbf{z})(\mathbf{y} - \mathbf{z})\big] \cdot (1 - \lambda) \\ \Longrightarrow \lambda g(\mathbf{x}) + (1 - \lambda)g(\mathbf{y}) - g(\mathbf{z}) &\geq 0. \end{aligned}$$

Hence g is convex.

1 $\implies$ 2: Fix $\mathbf{x}, \mathbf{y} \in I$. Consider the sequence

$$\left\{ \frac{g\big(\mathbf{x} + \frac{\mathbf{d}}{2^i}\big) - g(\mathbf{x})}{2^{-i}\mathbf{d}} \right\}_{\substack{i \geq 0, \\ i \in \mathbb{Z}_+}}.$$

Take $\mathbf{d} = \mathbf{y} - \mathbf{x}$. By definition, this sequence converges to $g'(\mathbf{x})$. Due to convexity, the sequence is non-increasing with i. We have

$$\frac{g\big(\mathbf{x} + \frac{\mathbf{d}}{2^i}\big) - g(\mathbf{x})}{2^{-i}\mathbf{d}} \leq \frac{\big[\frac{1}{2}g(\mathbf{x}) + \frac{1}{2}g\big(\mathbf{x} + \frac{\mathbf{d}}{2^{i-1}}\big)\big] - g(\mathbf{x})}{2^{-i}\mathbf{d}}.$$

This is because

$$\mathbf{x} + \frac{\mathbf{d}}{2^i} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\left(\mathbf{x} + \frac{\mathbf{d}}{2^{i-1}}\right) \implies \frac{1}{2}\left[g\left(\mathbf{x} + \frac{\mathbf{d}}{2^{i-1}}\right) - g(\mathbf{x})\right] = \frac{g\big(\mathbf{x} + \frac{\mathbf{d}}{2^{i-1}}\big) - g(\mathbf{x})}{2^{-(i-1)}\mathbf{d}}$$

Hence,

$$g\left(\mathbf{x} + \frac{\mathbf{d}}{2^i}\right) - g(\mathbf{x}) \geq g'(\mathbf{x}), \qquad \forall\, i \geq 0, \forall\, i \in \mathbb{Z}.$$

Hence,

$$\underbrace{g(\mathbf{x} + \mathbf{d})}_{g(\mathbf{y})} - g(\mathbf{x}) \geq \underbrace{\mathbf{d}}_{\mathbf{y}-\mathbf{x}} g'(\mathbf{x}).$$

□

Theorem 7.1.39: Let $f : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^n$ is convex. Let f be differentiable on S. Then, f is convex if and only if

$$f(\mathbf{y}) - f(\mathbf{x}) \geq \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}), \qquad \forall\, \mathbf{x}, \mathbf{y} \in S. \tag{7.1.2}$$

This inequality is called the ***sub-gradient inequality***.

Proof: $\impliedby$: This is true by above Theorem 7.1.38 2 $\implies$ 1.

$\implies$: Consider any $\mathbf{x}, \mathbf{y} \in S$. Let $\mathbf{d} = \mathbf{y} - \mathbf{x}$. Consider $g : [0, 1] \to \mathbb{R}$ given by $g(t) = f(\mathbf{x} + t\mathbf{d})$ for all $t \in [0, 1]$. This is well-defined since S is convex. Since g is convex, then by Theorem 7.1.38 we have

$$\underbrace{g(1)}_{f(\mathbf{y})} - \underbrace{g(0)}_{f(\mathbf{x})} \geq \underbrace{g'(0)}_{\nabla f(\mathbf{x})^\top \mathbf{d}}.$$

□

Theorem 7.1.40 (Optimality conditions for convex programs I)**:** Let $S \subseteq \mathbb{R}^n$ be convex. Consider the convex program

$$\begin{aligned} \text{(C-P)}: \min \quad & f(\mathbf{x}), \\ \text{subject to } & \mathbf{x} \in S, \\ \text{where} \quad & f : S \to \mathbb{R} \text{ is convex and differentiable on } S. \end{aligned}$$

Let $\overline{\mathbf{x}} \in S$. Then, $\overline{\mathbf{x}}$ is an optimal solution to (C-P) if and only if $\nabla f(\overline{\mathbf{x}})^\top (\mathbf{y} - \overline{\mathbf{x}}) \geq 0$, for all $\mathbf{y} \in S$.

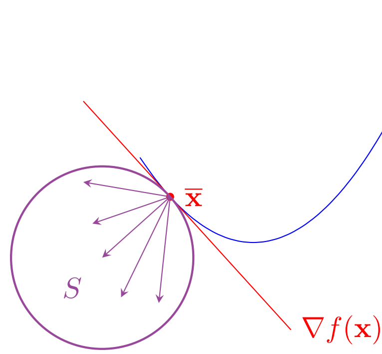

Figure 7.1.3

Proof: $\Longleftarrow$: By Theorem 7.1.39 (by sub-gradient inequality), we have

$$f(\mathbf{y}) - f(\overline{\mathbf{x}}) \geq \nabla f(\overline{\mathbf{x}})^\top (\mathbf{y} - \overline{\mathbf{x}}), \qquad \forall\, \mathbf{y} \in S.$$

So $f(\mathbf{y}) \geq f(\overline{\mathbf{x}})$ for all $\mathbf{y} \in S$.

$\Longrightarrow$: Suppose, for contradiction, $\nabla f(\overline{\mathbf{x}})^\top (\mathbf{y} - \overline{\mathbf{x}}) < 0$ for some $\mathbf{y} \in S$. Then, by prop 2 (and since S is convex), there exists $\delta > 0$ such that for all $t \in (0, \min(\delta, 1)]$ we have

$$S \ni f(\overline{\mathbf{x}} + t(\mathbf{y} - \overline{\mathbf{x}})) < f(\overline{\mathbf{x}}).$$

Hence $\overline{\mathbf{x}}$ is not an optimal solution. □

Proposition 7.1.41: Suppose $S = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$ (polyhedron) with $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ and $f : S \to \mathbb{R}$ is convex and differentiable. Then, $\overline{\mathbf{x}}$ is an optimal solution to $\min f(\mathbf{x})$ such that $\mathbf{x} \in S$ if and only if there exsits $\boldsymbol{\lambda} \in \mathbb{R}^m_+$ such that $-\nabla f(\overline{\mathbf{x}})^\top = \boldsymbol{\lambda}^\top A$ and $\overline{\mathbf{x}}, \boldsymbol{\lambda}$ satisfy the complementary slackness (CS) conditions (i.e. $\lambda_i = 0$ or $(A\overline{\mathbf{x}})_i = b_i$ for all $i \in [m]$).

Note that this generalizes LP duality.

Proof: Exercise. ◁

7.1.3 Application: Separating Hyperplanes

Theorem 7.1.42: Let $S \subseteq \mathbb{R}^n$ be a such that $S \neq \varnothing$ is closed and convex. Let $\mathbf{z} \notin S$. Then, there exists a hyperplane separating $\mathbf{z}$ from S. i.e. $\exists\, \boldsymbol{\alpha} \in \mathbb{R}^n$ such that $\boldsymbol{\alpha}^\top \mathbf{x} < \boldsymbol{\alpha}\mathbf{z}$ for all $\mathbf{x} \in S$.

Proof: Let $\overline{\mathbf{x}}$ be the unique point in S nearest to $\mathbf{z}$. Then, $\overline{\mathbf{x}}$ is an optimal solution to the convex program

$$\begin{aligned} \text{(C-P)} : \min \qquad & f(\mathbf{x}) \underset{\text{def}}{=} \|\mathbf{x} - \mathbf{z}\|^2, \\ & \text{subject to } \mathbf{x} \in S. \end{aligned}$$

Claim 7.1.43*:* f is convex where $\nabla f(\mathbf{x}) = 2(\mathbf{x} - \mathbf{z})$.

Proof: Exercise. ◂

Hence, by Theorem 7.1.40, we have

$$\nabla f(\overline{\mathbf{x}})^\top(\mathbf{y}-\overline{\mathbf{x}}) \geq 0, \qquad \mathbf{y} \in S.$$

Hence we have $2(\overline{\mathbf{x}}-\mathbf{z})^\top(\mathbf{y}-\overline{\mathbf{x}}) \geq 0$ for all $\mathbf{y} \in S$. Let $\boldsymbol{\alpha} = \mathbf{z}-\overline{\mathbf{x}}$. Hence we have

$$\boldsymbol{\alpha}^\top\mathbf{y} \leq \boldsymbol{\alpha}^\top\overline{\mathbf{x}} \qquad \forall\, \mathbf{y} \in S \implies \boldsymbol{\alpha}^\top\mathbf{z} = \|\mathbf{z}-\overline{\mathbf{x}}\|^2 + \boldsymbol{\alpha}^\top\overline{\mathbf{x}} > \boldsymbol{\alpha}^\top\overline{\mathbf{x}} \qquad \text{since } \mathbf{z} \neq \overline{\mathbf{x}}.$$

□

Definition 7.1.44: We say that $f : \mathbb{R}^n \to \mathbb{R}$ is ***affine*** if f is of the form $f(\mathbf{x}) = \boldsymbol{\alpha}^\top\mathbf{x} + \beta$ where $\boldsymbol{\alpha} \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$. ◁

Remark 7.1.45: A non-linear program with affine functions are equivalent to linear programs. If $f(\mathbf{x}) = \boldsymbol{\alpha}^\top\mathbf{x} + \beta$, then we have $\nabla f(\mathbf{x}) = \boldsymbol{\alpha}$ for all $\mathbf{x} \in \mathbb{R}^n$. ◁

Definition 7.1.46: Consider a non-linear program

$$\begin{aligned} \text{(NLP)} : \min \quad & f(\mathbf{x}), \\ \text{subject to } & g_i(\mathbf{x}) \leq 0, \quad \forall\, i \in [m]. \end{aligned}$$

We say that $\hat{\mathbf{x}}$ is a ***slater point*** of

1. $\hat{\mathbf{x}}$ is feasible for NLP,
2. $g_i(\hat{\mathbf{x}}) < 0$ for all $i \in [m]$ such that g_i is not affine. ◁

Remark 7.1.47: The standard definition of slater points require $g_i(\hat{\mathbf{x}}) < 0$ for all $i \in [m]$ under ANY constraints (for g_i affine and not affine). ◁

Theorem 7.1.48 (Karush-Kuhn-Tucker (KKT) optimality conditions)**:** Consider a convex program

$$\begin{aligned} \text{(C-P)} : \min \quad & f(\mathbf{x}), \\ \text{subject to } & g_i(\mathbf{x}) \leq 0, \quad \forall\, i \in [m], \\ \text{where } \quad & f, g_i : \mathbb{R}^n \to \mathbb{R} \text{ are convex and differentiable.} \end{aligned}$$

Let $\overline{\mathbf{x}}$ be a feasible solution to (C-P). Define $J(\overline{\mathbf{x}}) = \{i \in [m] \mid g_i(\overline{\mathbf{x}}) = 0\}$ (these are the tight constraints).

1. (Sufficiency) If $-\nabla f(\overline{\mathbf{x}}) \in \operatorname{cone}\{\nabla g_i(\overline{\mathbf{x}}) \mid i \in J(\overline{x})\}$, then $\overline{\mathbf{x}}$ is an optimal solution to (C-P). i.e. there exists $\boldsymbol{\lambda} \in \mathbb{R}^m_+$ such that

$$-\nabla f(\overline{\mathbf{x}}) = \sum_{i\in[m]} \lambda_i \nabla g_i(\overline{\mathbf{x}}) \text{ and } \underbrace{\lambda_i g_i(\overline{\mathbf{x}}) = 0}_{\text{C-S conditions}}, \qquad \forall\, i \in [m].$$

   These conditions are called the ***KKT conditions***.

2. (Necessity) If $\overline{\mathbf{x}}$ is an optimal solution and if there exists a feasible solution $\hat{\mathbf{x}}$ such that $g_i(\hat{\mathbf{x}}) < 0$ for all $i \in J(\overline{\mathbf{x}})$ such that g_i is not affine then the KKT conditions hold.

Note that if (C-P) has a slater point then there exists a feasible solution $\hat{\mathbf{x}} < 0$ for all $i \in J(\overline{\mathbf{x}})$ such that g_i is not affine. We will refer the condition in blue in 2 as the *extra condition* in the proof.

Proof: 1: Suppose we have

$$-\nabla f(\overline{\mathbf{x}}) = \sum_{i\in[m]} \lambda_i \nabla g_i(\overline{\mathbf{x}}), \text{ where } \lambda_i \geq 0, \qquad \forall\, i \in J(\overline{\mathbf{x}}).$$

Then, for all feasible $\mathbf{y}$ and for all $i \in J(\overline{\mathbf{x}})$ by sub-gradient inequality we have

$$0 \geq \underbrace{g_i(\mathbf{y})}_{\leq 0} - \underbrace{g_i(\overline{\mathbf{x}})}_{=0} \geq \nabla g_i(\overline{\mathbf{x}})^\top(\mathbf{y}-\overline{\mathbf{x}}).$$

Hence

$$\nabla f(\overline{\mathbf{x}})^\top(\mathbf{y}-\overline{\mathbf{x}}) = -\sum_{i\in J(\overline{\mathbf{x}})} \underbrace{\lambda_i}_{\geq 0} \underbrace{\nabla g(\overline{\mathbf{x}})^\top(\mathbf{y}-\overline{\mathbf{x}})}_{\leq 0} \geq 0, \qquad \forall \text{ feasible } \mathbf{y}.$$

Hence by Theorem 7.1.40 we have $\overline{\mathbf{x}}$ is an optimal solution.

2: First, we note that the statement is false without the extra condition (fix). Consider the convex program

$$\begin{aligned} \text{(C-P)}: \min \quad & f(\mathbf{x}) = x, \\ \text{subject to } & g_1(\mathbf{x}) = x_1^2 - x_2 \leq 0, \\ & g_2(\mathbf{x}) = x_2 \leq 0. \end{aligned}$$

(C-P) has the graph

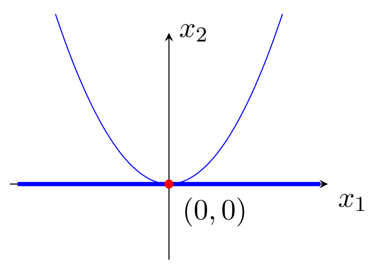

Figure 7.1.4: Illustration of (C-P).

Here the only feasible solution is $\overline{\mathbf{x}} = (0,0)$. Hence $\overline{\mathbf{x}}$ is the optimal solution to (C-P). We have $J(\overline{\mathbf{x}}) = \{1,2\}$. This gives us

$$\nabla f(\overline{\mathbf{x}}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \notin \operatorname{cone}\{\nabla g_1(\overline{\mathbf{x}}), \nabla g_2(\overline{\mathbf{x}})\}, \quad \text{where} \quad \begin{aligned} \nabla g_1(\overline{\mathbf{x}}) &= \begin{bmatrix} 2x_1 \\ -1 \end{bmatrix}\Bigg|_{\overline{\mathbf{x}}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \\ \nabla g_2(\overline{\mathbf{x}}) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Suppose $\overline{\mathbf{x}}$ is an optimal solution and

$$-\nabla f(\overline{\mathbf{x}}) \notin \operatorname{cone}\{\nabla g_i(\overline{\mathbf{x}}) \ \mid \ i \in J(\overline{\mathbf{x}})\} \equiv \nabla f(\overline{\mathbf{x}}) \notin \operatorname{cone}\{-\nabla g_i(\overline{\mathbf{x}}) \ \mid \ i \in J(\overline{\mathbf{x}})\}.$$

By Farkas' lemma (or duality), there exists $\mathbf{d} \in \mathbb{R}^n$ such that $-\nabla g_i(\mathbf{x})^\top \mathbf{d} \geq 0$ for all $i \in J(\overline{\mathbf{x}})$ and $\nabla f(\overline{\mathbf{x}})^\top \mathbf{d} < 0$. Since $\nabla f(\overline{\mathbf{x}})^\top \mathbf{d} < 0$, we want to show that there exists $\varepsilon > 0$ such that

$f(\overline{\mathbf{x}}+\varepsilon\mathbf{d}) < f(\overline{\mathbf{x}})$ (this follows from 1) and $\overline{\mathbf{x}}+\varepsilon\mathbf{d}$ is feasible. Since for $i \notin J(\overline{\mathbf{x}})$ we have $g_i(\overline{\mathbf{x}}) < 0$, then we can choose $\varepsilon > 0$ such that $g_i(\overline{\mathbf{x}}+\varepsilon\mathbf{d}) \leq 0$. Also, for $i \in J(\overline{\mathbf{x}})$ where g_i is affine we have

$$g_i(\overline{\mathbf{x}}+\varepsilon\mathbf{d}) = \underbrace{g_i(\overline{\mathbf{x}})}_{=0} + \varepsilon \underbrace{\mathbf{d}^\top \nabla g_i(\overline{\mathbf{x}})}_{\leq 0} \leq 0.$$

For $i \in J(\overline{\mathbf{x}})$, if g_i is not affine then the dot product of the gradient with $\mathbf{d}$ can be zero (i.e. $\nabla g_i(\overline{\mathbf{x}})^\top \mathbf{d} = 0$). So for all $\varepsilon > 0$, we could have $g_i(\overline{\mathbf{x}}+\varepsilon\mathbf{d}) > g_i(\overline{\mathbf{x}}) = 0$. Note that by the extra condition there exists a feasible $\hat{\mathbf{x}}$ such that $g_i(\hat{\mathbf{x}}) < 0$ for all $i \in J(\overline{\mathbf{x}})$ and g_i is not affine. We will use $\hat{\mathbf{x}}$ to come up with $\overline{\mathbf{d}} \in \mathbb{R}^n$ such that

a $\nabla f_i(\overline{\mathbf{x}})^\top \overline{\mathbf{d}} < 0$,

b $\nabla g_i(\overline{\mathbf{x}})^\top \overline{\mathbf{d}} < 0$, $\forall\, i \in J(\overline{\mathbf{x}})$ is not affine,

c $\nabla g_i(\overline{\mathbf{x}})^\top \overline{\mathbf{d}} \leq 0$, $\forall\, i \in J(\overline{\mathbf{x}})$ such that g_i is affine.

We see that given $\overline{\mathbf{d}}$, we can choose $\varepsilon > 0$ (suitably small) such that $f(\overline{\mathbf{x}}+\varepsilon\overline{\mathbf{d}}) < f(\overline{\mathbf{x}})$, $\overline{\mathbf{x}}+\varepsilon\overline{\mathbf{d}}$ is feasible, which gives us a contradiction.

Consider $\hat{\mathbf{d}} = \hat{\mathbf{x}} - \overline{\mathbf{x}}$. For $i \in J(\overline{\mathbf{x}})$ such that g_i is not affine we have

$$0 > \underbrace{g_i(\hat{\mathbf{x}})}_{<0} - \underbrace{g_i(\overline{\mathbf{x}})}_{=0} \geq \nabla g_i(\overline{\mathbf{x}})^\top \hat{\mathbf{d}},$$

i.e. $\nabla g_i(\overline{\mathbf{x}})^\top \hat{\mathbf{d}} < 0$. For the affine constraint we have for $i \in J(\overline{\mathbf{x}})$ such that g_i is affine. This gives us

$$0 \geq g_i(\hat{\mathbf{x}}) - g_i(\overline{\mathbf{x}}) \geq \nabla g_i(\overline{\mathbf{x}})^\top \hat{\mathbf{d}}.$$

So we can take $\delta > 0$ suitable small such that $\overline{\mathbf{d}} = \mathbf{d} + \delta\hat{\mathbf{d}}$ satisfies a, b and c and we are done. $\square$

End of Lecture 23

Start of Lecture 24

We complete proof for Theorem 7.1.48. The proof is included at the end of previous lecture.

Recall 7.1.49: Recall the following.

1. If $\nabla f(\overline{\mathbf{x}})^\top \mathbf{d} < 0$ then $\exists\, \delta > 0$ such that $\forall\, t \in (0, \delta]$, $f(\overline{\mathbf{x}} + t\mathbf{d}) < f(\overline{\mathbf{x}})$.
2. $\forall\, \gamma > 0$, $\exists\, \delta > 0$ such that $|f(\overline{\mathbf{x}} + t\mathbf{d}) - f(\overline{\mathbf{x}})| \leq \gamma$, $\forall\, t \in [-\delta, \delta]$.

$\triangleleft$

Remark 7.1.50: We have shown that the KKT conditions are sufficient and necessary (under some conditions) for optimality. Recall that KKT conditions are

$$-\nabla f(\overline{\mathbf{x}}) \in \operatorname{cone}\{\nabla g_i(\overline{\mathbf{x}}) \;\mid\; i \in J(\overline{\mathbf{x}})\}.$$

Equivalently we have $\exists\, \boldsymbol{\lambda} \in \mathbb{R}^m_+$ such that

$$\nabla f(\overline{\mathbf{x}}) + \sum_{i \in [m]} \lambda_i \nabla g_i(\overline{\mathbf{x}}) = \nabla \left(f(\overline{\mathbf{x}}) + \sum_{i \in [m]} \lambda_i g_i(\overline{\mathbf{x}}) \right) = 0 \text{ and } \lambda_i g_i(\overline{\mathbf{x}}) = 0, \qquad \forall\, i \in [m].$$

These conditions generalizes the geometric statement of CS conditions ($c \in \operatorname{cone}\{\text{rows of } A^=\}$). $\triangleleft$

Corollary 7.1.51: $\overline{\mathbf{x}}$ is an optimal solution to $\min f(\mathbf{x})$ subject to $\mathbf{x} \in \mathbb{R}^n \iff \nabla f(\overline{\mathbf{x}}) = 0$.

This concludes the final lecture for CO 255 in Winter 2019.

End of Lecture 24

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