PMath 450/650 - Assignment 1 - Winter 2015.

Deadline: Thursday, January 22nd 11:05 am.

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1. Let $a < b$ be elements in $\mathbb{R}$. Prove that if $f : [a, b] \to \mathbb{R}$ is piecewise continuous (*i.e.* f is bounded, and there is a partition $\{a = a_0 < a_1 < \ldots < a_k = b\}$ such that, f is continuous on each interval $(a_{i-1}, a_i)]$, then it is Riemann-integrable.

2. Let I be an interval in $\mathbb{R}$, *i.e.* I is of the form $[a, b]$, $[a, b)$, $(a, b]$ or (a, b). Prove that $\lambda^*(I)$ is equal to the length of the interval. (**Hint:** You can use $\lambda^*([a, b]) = \ell([a, b])$, which was proved in the class.)

3. Recall that we say a subset A of $\mathbb{R}$ is *Lebesgue measurable* if for every subset $E \subseteq \mathbb{R}$,
$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A).$$
Use the definition of Lebesgue measurability to prove that
   (i) if $\lambda^*(E) = 0$ then E is Lebesgue measurable.
   (ii) every open interval (a, b) is Lebesgue measurable.
   (iii) a countable union of Lebesgue measurable sets is Lebesgue measurable.

4. Let A be a subset of $\mathbb{R}$.
   (i) Prove that for every $\epsilon > 0$, there exists an open set $\mathcal{O}$ containing A such that $\lambda^*(\mathcal{O}) \leq \lambda^*(A) + \epsilon$.
   (ii) A set of real numbers is said to be a G_δ-set provided it is the intersection of a countable collection of open sets. Prove that if A is bounded then there exists a G_δ-set E containing A such that $\lambda^*(E) = \lambda^*(A)$.

5. (i) Determine the Lebesgue measure of the Cantor set.
   (ii) Find the cardinality of the set of Lebesgue measurable sets.
   (iii) **Bonus:** Find the cardinality of the set of Borel sets.

Comments: The submitted solutions must be tidy and legible. You are to provide full solutions to the problems. You are allowed, and encouraged to collaborate with your classmates, but the write-ups should be done individually, without access to the papers of fellow students. Copying assignments or tests from any source, completely or partially, allowing others to copy your work, will not be tolerated.