COMPACT GROUPS

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1. Preliminaries on Compact Groups

Definition 1.1. Let $(G, \cdot)$ be a group, and let τ be a topology on G. We say that τ is a **group topology**, and that $(G, \cdot, \tau)$ is a **topological group**, if the maps $x \mapsto x^{-1}$ and $(x, y) \mapsto x \cdot y$ are continuous under τ, with the latter taken as map from $(G \times G, \tau \times \tau)$. For brevity the terms **locally compact group** and **compact group** will refer to topological groups with the stated properties. We will henceforth suppress the group operation, and denote the identity element of the group by e. Further, all our topological groups are assumed Hausdorff.

Proposition 1.2. *Let (G, τ) be a topological group, U be open, and $x \in G$, then $xU = \{xs : s \in U\}$ is open. If K is compact, then K^{-1} is compact, and if A and B are compact, then so is $AB = \{ab : a \in A, b \in B\}$.*

Proof. The map $s \mapsto xs$ is continuous and has a continuous inverse $s \mapsto x^{-1}s$, using the continuity properties in the above definition. Thus it is a homeomorphism and takes open sets to open sets.

Next if K is compact, then we can easily see that K^{-1} is too as if U_i is an open cover for K^{-1}, then U_i^{-1} is an cover for K and its members are open by a similar argument as above. Applying compactness of K we see that K^{-1} is compact.

Finally, suppose A and B are compact. Then the set $A \times B \subseteq G \times G$ is compact. Then AB is compact as it is the continuous image of a compact set. □

Definition 1.3. Let (G, τ) be a locally compact group, a **left Haar measure** is a nonzero Radon measure (a fortiori real and positive) μ on G that satisfies $\mu(xE) = \mu(E)$ for every Borel set $E \subset G$ and $x \in G$.

Proposition 1.4. (Existence and uniqueness) *Every locally compact group G possesses a left Haar measure. If μ and λ are two Haar measures on G then there exists $c \in (0, \infty)$ so that $\mu = c\lambda$.*

Proposition 1.5. *Let G be a locally compact group, and $U \subseteq G$ be a nonempty open set, then if μ is a Haar measure on G, $\mu(U) > 0$.*

Proof. Suppose that $\mu(U) = 0$. Then let $K \subseteq G$ be any compact set. Fix $x \in U$. Then for any $y \in K$, $y \in (yx^{-1})U$. We have proven that this translate is open, so K is covered by open left translates of U. By compactness, we can find a finite collection $\{s_i : i \in [N]\}$ so that $K \subseteq \bigcup_{i=1}^N s_iU$. Then

$$\mu(K) \leqslant \sum_{i=1}^{N} \mu(s_iU) = \sum_{i=1}^{N} \mu(U) = 0.$$

But a Radon measure is inner regular, so

$$\mu(G) = \sup\{\mu(K) : K \subseteq G, K \text{ is compact}\} = 0.$$

This contradicts that a Haar measure is nonzero, so $\mu(U) > 0$. □

Proposition 1.6. *A locally compact group G is compact if and and only if, for any Haar measure μ on G, $\mu(G) < \infty$.*

Proof. A Radon measure is locally finite, and thus finite on compact sets. Thus any Haar measure on a compact group is finite. Conversely, suppose that G is not compact, fix μ. As μ is nonzero, we may find a compact subset $K \subseteq G$ so that $\mu(K) > 0$ by an argument in the previous theorem. Now KK^{-1} is compact, and G is not, so $G \backslash KK^{-1}$ is nonempty. Let $u_0 = e$, and if we pick $u_1 \in G \backslash KK^{-1}$, then I claim $u_1 K \cap K =$. Suppose $k \in u_1 K \cap K$, then $k = u_1 k'$, so $u_1 = kk'^{-1} \in KK^{-1}$, contradiction. Now recursively construct u_i so that $u_i \in G \backslash \bigcup_{j=1}^{n} u_j KK^{-1}$, which is nonempty as the union is compact. If $x \in u_i K \cap u_j K$, for $i > j$ then $x = u_i k_1 = u_j k_2$ and $u_i = u_j k_1 k_s^{-1} \in u_j KK^{-1}$, but $u_i \in G \backslash u_j KK^{-1}$. Thus all the $u_i K$ are disjoint, and we construct by axiom of choice a countable disjoint family of left translates of a compact set. By σ-additively, and as $\mu(K) > 0$,

$$\mu\left(\bigcup_{i=0}^{\infty} u_i K\right) = \sum_{i=0}^{\infty} \mu(u_i K) = \sum_{i=0}^{\infty} \mu(K) = \infty.$$ □

Remark 1.7. If G is a compact group, then we have a natural choice of Haar measure - the one that is a probability measure. Thus henceforth *the Haar measure* on G, a compact group, is a Haar measure μ such that $\mu(G) = 1$.

For the rest of this paper, we will restrict ourselves to the case that G is compact. This means that unless we explicitly state 'let G be a locally compact group', G is assumed compact.

We will need one final ingredient, which we state without proof:

Theorem 1.8. *The left Haar measure on G is also a right Haar measure. That is $\mu(Ex) = \mu(E)$ for all Borel sets $E \subset G$.*

2. The Unitarization Theorem

Definition 2.1. Let G be a locally compact group and $\mathcal{H}$ be a nonzero Hilbert space, then a **representation** of G on $\mathcal{H}$ is a map $\pi : G \to \mathcal{B}(\mathcal{H})$ satisfying:

- (**π is a group homomorphism**): for all $x, y \in G$ we have $\pi(xy) = \pi(x)\pi(y)$, and $\pi(e) = Id_{\mathcal{H}}$,
- (**Strong continuity**): for all $\xi \in \mathcal{H}$, the map $x \in G \mapsto \pi(x)\xi$ is continuous.

We say that π is unitary if $\pi(x)$ is a unitary operator (equivalent to $\pi(x^{-1}) = \pi(x)^*$, where * is the Hilbert space adjoint) for all $x \in G$.

Theorem 2.2. *Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces, then there exists a unitary map $U : \mathcal{H} \to \mathcal{H}'$ (that is $\langle U\xi \mid U\eta\rangle_{\mathcal{H}'} = \langle \xi \mid \eta\rangle_{\mathcal{H}}, \forall \xi, \eta \in \mathcal{H}$) if and only if there exists a bounded invertible $S : \mathcal{H} \to \mathcal{H}'$.*

Proof. **Briefly:** Assume we have such an S, then well order an orthonormal basis for $\mathcal{H}$, $\{e_\alpha\}$, and apply Gram-Schmidt to the vectors $\{S(e_\alpha)\}$. Define U on the basis by transfinite recursion, sending e_α to the α-th vector constructed, and extend to a unique continuous linear map on $\mathcal{H}$ by the Bounded Linear Transformations theorem. □

Theorem 2.3. *Let $\pi : G \to \mathcal{B}(\mathcal{H})$ be a representation. Then there exists an invertible linear map $S \in \mathcal{B}(\mathcal{H})$, so that $\pi_S = S^{-1}\pi(\cdot)S$ is a unitary representation.*

Proof. Since G is compact and π is strongly continuous, the set $\{\pi(x)\xi : x \in G\} \subseteq \mathcal{H}$ is compact for each $\xi \in \mathcal{H}$, and is thus also bounded. By the Uniform Boundedness Principle, the operator norms are bounded too, that is $\sup_{x\in G} \|\pi(x)\| < \infty$. Now define for $\xi, \eta \in \mathcal{H}$,

$$[\xi, \eta] = \int_G \langle \pi(s)\xi \mid \pi(s)\eta\rangle \, ds.$$

By linearity of $\pi(s)$ and of integration, and by bilinearity of the inner product on $\mathcal{H}$, we see that this new form is bilinear. It is also easily seen to be conjugate symmetric $\left([\xi, \eta] = \overline{[\eta, \xi]}\right)$, and hence sesquilinear. We claim that it is positive.

$$[\xi, \xi] = \int_G \|\pi(s)\xi\|^2 ds,$$

where if we suppose $\xi \neq 0$, then $\|\pi(e)\xi\| = \|\xi\| \neq 0$. The map $s \mapsto \|\pi(s)\xi\|^2$ is continuous (by continuity of the norm, making note that joint continuity of the inner product does not necessarily hold). Thus we can find an open neighbourhood U of $e \in G$, such that $\|\pi(s)\xi\| > \frac{\|\xi\|}{2}$ for all $s \in U$. $m(U) > 0$, thus

$$[\xi, \xi] = \int_G \|\pi(s)\xi\|^2 ds > \frac{\|\xi\|}{2} m(U) > 0.$$

We have verified that $[\cdot \mid \cdot]$ gives an inner product. Let us call the resulting Hilbert space $\mathcal{H}'$, and let us denote its norm as $\|\cdot\|'$. Then

$$\|\xi\|' = \left(\int_G \|\pi(s)\xi\|^2 ds\right)^{\frac{1}{2}} \leqslant \left(\sup_{x\in G} \|\pi(x)\|\right) \|\xi\|.$$

We have proven that the supremum in parenthesise is finite. Thus the identity map $J : \mathcal{H} \to \mathcal{H}'$ is continuous. It is also bijective, and thus a homeomorphism by the Open Mapping Theorem. By the previous proposition there exists a unitary $U : \mathcal{H} \to \mathcal{H}'$. Fix such a U, then $[U\xi \mid U\eta] = \langle \xi \mid \eta\rangle$.

Define $S = J^{-1}U$, then for all $x \in G$, $\xi, \eta \in \mathcal{H}$ we have (note that J is suppressed as we think of the forms as functions on the underlying sets of the Hilbert spaces):

$$\begin{aligned}
\left\langle S^{-1}\pi(x)S\xi \mid \eta\right\rangle &= \left\langle U^{-1}\pi(x)U\xi \mid U^{-1}U\eta\right\rangle = [\pi(x)U\xi \mid U\eta] \\
&= \int_G \langle \pi(sx)U\xi \mid \pi(s)U\eta\rangle \, ds = \int_G \left\langle \pi(t)U\xi \mid \pi(tx^{-1})U\eta\right\rangle dt \\
&= \left[U\xi \mid \pi(x^{-1})U\eta\right] = \left\langle U^{-1}U\xi \mid U^{-1}\pi(x^{-1})U\eta\right\rangle \\
&= \left\langle \xi \mid S^{-1}\pi(x^{-1})S\eta\right\rangle = \left\langle \left(S^{-1}\pi(x^{-1})S\right)^* \xi \mid \eta\right\rangle.
\end{aligned}$$

Note in particular that all the above constructions were to enable the reparametrization of the integral, just as in the case for finite groups. As this holds for all ξ, η, we have that for $\pi_S = S^{-1}\pi(\cdot)S$, $\pi_S(x) = \pi_S(x^{-1})^*$, for all $x \in G$. It is easily checked that π_S is a representation, and thus it is a unitary representation, as desired. □

3. Complete Reducibility

Lemma 3.1. *Let $\pi : G \to \mathcal{H}(\mathcal{U})$ be a unitary representation. Then given $\xi \in \mathcal{H}\backslash\{0\}$, the operator given by:*

$$k_\xi \eta = \int_G \langle \eta \mid \pi(s)\xi \rangle \pi(s)\xi ds$$

is positive, nonzero, compact, self-adjoint, and $k_\xi \pi(x) = \pi(x) k_\xi$ for all $x \in G$.

Proof.

$$\langle k_\xi \eta \mid \eta \rangle = \int_G \langle \eta \mid \pi(s)\xi \rangle \langle \pi(s)\xi \mid \eta \rangle ds = \int_G |\langle \eta \mid \pi(s)\xi \rangle|^2 ds \geqslant 0$$

This calculation verifies positivity. It is easy to see that k_ξ is nonzero from this.

$$\langle k_\xi \xi \mid \xi \rangle = \int_G |\langle \xi \mid \pi(s)\xi \rangle|^2 ds$$

For nonzero ξ, $|\langle \xi \mid \pi(e)\xi \rangle|^2 = |\xi|^2 > 0$. Thus by continuity the integral is nonzero.

Now, as G is compact and $x \mapsto \pi(x)\xi$ is continuous, for all $\epsilon > 0$ we can cover the compact image of G with finitely many ϵ balls. Specifically, for ϵ given, we can find a partition of $G = \bigsqcup_{i \in [N]} E_i$ for $N \in \mathbb{N}$ and E_i Borel, and points $x_i \in E_i$ such that

$$\|\pi(x)\xi - \pi(x_i)\xi\| < \epsilon, \forall x \in E_i.$$

Then define

$$k_\xi^\epsilon \eta = \sum_{i=1}^{N} \mu(E_i) \langle \eta \mid \pi(x_i)\xi \rangle \pi(x_i)\xi.$$

Note that k_ξ^ϵ is finite rank. We will argue that k_ξ is a limit of these finite rank operators, and is hence compact. For $x \in E_i$,

$$\begin{aligned}
&\| \langle \eta \mid \pi(x_i)\xi \rangle \pi(x_i)\xi - \langle \eta \mid \pi(x)\xi \rangle \pi(x)\xi \| \\
&\qquad = \| \langle \eta \mid \pi(x_i)\xi - \pi(x)\xi \rangle \pi(x_i)\xi + \langle \eta \mid \pi(x)\xi \rangle (\pi(x_i)\xi - \pi(x)\xi) \| \\
&\qquad \leqslant | \langle \eta \mid \pi(x_i)\xi - \pi(x)\xi \rangle | \cdot \|\pi(x_i)\xi\| + | \langle \eta \mid \pi(x)\xi \rangle | \cdot \|\pi(x)\xi - \pi(x_i)\xi\| \\
&\qquad < \|\eta\| \cdot \epsilon \cdot \|\pi(x_i)\xi\| + \|\eta\| \cdot \|\pi(x)\xi\| \cdot \epsilon \\
&\qquad = 2\epsilon \|\eta\| \|\xi\|
\end{aligned}$$

With the last equality arising as π is a unitary representation. For $\|\eta\| \leqslant 1$, we perform the following bound. In the first step, we split the integral over the partition, then bring constants into the integral.

$$\begin{aligned}\|k_\xi\eta - k_\xi^\epsilon\eta\| &= \left\|\sum_{i=1}^N \left(\int_{E_i} (\langle \eta \mid \pi(s)\xi\rangle \pi(s)\xi)\, ds - \mu(E_i)\langle \eta \mid \pi(x_i)\xi\rangle \pi(x_i)\xi\right)\right\| \\ &= \left\|\sum_{i=1}^N \int_{E_i} (\langle \eta \mid \pi(s)\xi\rangle \pi(s)\xi - \langle \eta \mid \pi(x_i)\xi\rangle \pi(x_i)\xi)\, ds\right\| \\ &\leqslant \sum_{i=1}^N \int_{E_i} \|\langle \eta \mid \pi(x)\xi\rangle \pi(x)\xi - \langle \eta \mid \pi(x_i)\xi\rangle \pi(x_i)\xi\| ds \\ &< \sum_{i=1}^N \mu(E_i) 2\epsilon \|\eta\|\|\xi\| \leqslant 2\epsilon\|\xi\|\end{aligned}$$

Thus $\|k_\xi - k_\xi^\epsilon\| < 2\epsilon\|\xi\|$, and $k_\xi^\epsilon \to k_\xi$ as $\epsilon \to 0$. Thus k_ξ is compact.

Next, for self-adjointness,

$$\begin{aligned}\langle \zeta \mid k_\xi\eta\rangle &= \int_G \langle \zeta \mid \langle \eta \mid \pi(s)\xi\rangle \pi(s)\xi\rangle\, ds = \int_G \overline{\langle \eta \mid \pi(s)\xi\rangle}\langle \zeta \mid \pi(s)\xi\rangle\, ds \\ &= \overline{\int_G \langle \eta \mid \pi(s)\xi\rangle \overline{\langle \zeta \mid \pi(s)\xi\rangle} ds} = \overline{\int_G \langle \eta \mid \langle \zeta \mid \pi(s)\xi\rangle \pi(s)\xi\rangle\, ds} \\ &= \overline{\langle \eta \mid k_\xi\zeta\rangle} = \langle k_\xi\zeta \mid \eta\rangle.\end{aligned}$$

Finally,

$$\begin{aligned}\pi(x)k_\xi\eta &= \int_G \langle \eta \mid \pi(s)\xi\rangle \pi(xs)\xi ds = \int_G \left\langle \eta \mid \pi(x^{-1}t)\xi\right\rangle \pi(t)\xi dt \\ &= \int_G \langle \eta \mid \pi(x)^*\pi(t)\xi\rangle \pi(t)\xi dt = \int_G \langle \pi(x)\eta \mid \pi(t)\xi\rangle \pi(t)\xi dt = k_\xi\pi(x)\eta. \qquad \square\end{aligned}$$

Definition 3.2. Let $\pi : G \to \mathcal{B}(\mathcal{H})$ be a representation. Then a **π-invariant subspace** is a *closed* subspace $\mathcal{L} \subseteq \mathcal{H}$ such that $\pi(x)\mathcal{L} \subseteq \mathcal{L}$. We say that π is **irreducible** if the only π-invariant subspaces are $\{0\}$ and $\mathcal{H}$.

We say that π is **completely reducible** if there exists a family $\{\mathcal{L}_\alpha\}_{\alpha\in A}$ of nonzero closed subspaces of $\mathcal{H}$ satisfying:

i) $\mathcal{L}_\alpha \cap \mathcal{L}_{\alpha'} = \{0\}$ for $\alpha \neq \alpha'$,
ii) each $\mathcal{L}_\alpha$ is π-invariant with each $\pi|_{\mathcal{L}_\alpha}$ irreducible, and
iii) $\bigoplus_{\alpha\in A} \mathcal{L}_\alpha = \{\sum_{i=1}^n \xi_i : \xi_i \in \mathcal{L}_{\alpha_i}\}$ is dense in $\mathcal{H}$.

Definition 3.3. If $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation, then we will require more when we say that π is completely reducible. Specifically, that there exist a family $\{\mathcal{L}_\alpha\}_{\alpha\in A}$ of nonzero closed subspaces of $\mathcal{H}$ satisfying the above conditions and also:

i) $\mathcal{L}_\alpha \perp \mathcal{L}_{\alpha'}$ for $\alpha \neq \alpha'$,
ii) $\mathcal{H} = l^2 - \bigoplus_{\alpha\in A} \mathcal{L}_\alpha = \left\{\sum_\alpha \xi_i : \xi_i \in \mathcal{L}_{\alpha_i}, \sum_\alpha \|\xi_\alpha\|^2 \leqslant \infty\right\}$.

The second condition follows from the other assumptions by basic Hilbert space arguments. This is a stronger set of conditions, but we will show that every unitary representation of a (compact) group is completely reducible in the second sense, hence also in the first, and thus these definition coincide on the class of unitary representations of compact groups.

Proposition 3.4. *Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation, and $\mathcal{L} \subseteq \mathcal{H}$ be a closed subspace. Then*

i) $\mathcal{L}$ is π-invariant if and only if $P_{\mathcal{L}}\pi(x) = \pi(x)P_{\mathcal{L}}$,

ii) $\mathcal{L}$ is π-invariant if and only if $\mathcal{L}^{\perp}$ is π-invariant.

Where $P_{\mathcal{L}}$ denotes the operator of orthogonal projection onto $\mathcal{L}$.

Proof. i) Suppose that $\mathcal{L}$ is π-invariant, then for all x in G, $P_{\mathcal{L}}\pi(x)P_{\mathcal{L}} = \pi(x)P_{\mathcal{L}}$, and thus

$$P_{\mathcal{L}}\pi(x) = (\pi(x)^*P_{\mathcal{L}})^* = \left(\pi(x^{-1})P_{\mathcal{L}}\right)^* = \left(P_{\mathcal{L}}\pi(x^{-1})P_{\mathcal{L}}\right)^* = (P_{\mathcal{L}}\pi(x)^*P_{\mathcal{L}})^* = P_{\mathcal{L}}\pi(x)P_{\mathcal{L}} = \pi(x)P_{\mathcal{L}}.$$

Conversely, if $P_{\mathcal{L}}\pi(x) = \pi(x)P_{\mathcal{L}}$, then $P_{\mathcal{L}}\pi(x)P_{\mathcal{L}} = \pi(x)P_{\mathcal{L}}$ which implies that $\mathcal{L}$ is π invariant.

ii) $P_{\mathcal{L}^{\perp}} = I - P_{\mathcal{L}}$ then if $P_{\mathcal{L}}\pi(x) = \pi(x)P_{\mathcal{L}}$,

$$P_{\mathcal{L}^{\perp}}\pi(x) = \pi(x) - P_{\mathcal{L}}\pi(x) = \pi(x) - \pi(x)P_{\mathcal{L}} = \pi(x)P_{\mathcal{L}^{\perp}}.$$

The converse follows as $(\mathcal{L}^{\perp})^{\perp} = \mathcal{L}$ when $\mathcal{L}$ is a closed subspace. □

Lemma 3.5. *If $\pi : G \to \mathcal{U}(\mathcal{H})$ is a representation and $\mathcal{H}$ is finite dimensional, then π is completely reducible.*

Proof. Because $\mathcal{H}$ is finite dimensional, we may pick a nonzero subspace $\mathcal{L}$ of minimal dimension. Then $\pi\mid_{\mathcal{L}}$ is irreducible. Either $\mathcal{L} = \mathcal{H}$, or $\mathcal{L}^{\perp}$ is a nonzero invariant subspace, and we may repeat this argument for the representation $\pi\mid_{\mathcal{L}^{\perp}}$. As the dimension of our representation is decreasing with each restriction, the process will terminate. □

Theorem 3.6. *Let G be a compact group, then:*

i) If π is irreducible, then π is finite dimensional.

ii) Every representation π is completely reducible.

Proof. By the unitarization theorem, it suffices to to consider unitary representations. Note the remarks above on the stronger conditions on complete reducibility, and also clearly that π is irreducible if and only if π_S is for isomorphisms S.

Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation. For $\xi \in \mathcal{H}\backslash\{0\}$, let k_ξ be the positive nonzero compact self-adjoint operator from Lemma 3.1. By the spectral theorem for self-adjoint compact operators on a Hilbert space, there is an orthonormal basis for $\mathcal{H}$ of eigenvectors, which have at most countable many eigenvalues. The set of eivenvalues is bounded and may only cluster about 0, and the eigenspace of any any non-zero eigenvalue is finite dimensional. Thus we may find an index set $|A| \leqslant \omega$, distinct non-zero $\lambda_\alpha \in \mathbb{C}$, and nonzero pairwise orthogonal finite-rank projection operators P_α for $\alpha \in A$, such that $k_\xi = \sum_{\alpha\in A}\lambda_\alpha P_\alpha$. Further because k_ξ is positive, $\lambda_\alpha \in \mathbb{R}^{>0}$.

Additionally, suppose $k_\xi T = Tk_\xi$, for some $T \in \mathcal{B}(\mathcal{H})$. Then for $\alpha \in A$, let $\mathcal{L}_\alpha = P_\alpha(\mathcal{H})$. Fix $\beta \in A$ and let $\eta \in \mathcal{L}_\beta$, then uniquely express $T\eta = \sum_\alpha \eta_\alpha + \eta^{\perp}$ with $\eta_\alpha \in \mathcal{L}_\alpha$ and $\eta_{\perp} \perp \mathcal{L}_\alpha$ for all α. Then $Tk_\xi\eta = T(\lambda_\beta\eta) = \sum_\alpha \lambda_\beta\eta_\alpha + \beta\eta^{\perp}$, whereas $k_\xi T\eta = k_\xi\left(\sum_\alpha \eta_\alpha + \eta^{\perp}\right) = \sum_\alpha \lambda_\alpha\eta_\alpha$. Applying P_α to both sides, we see $\lambda_\beta\eta_\alpha = \lambda_\alpha\eta_\alpha$. For $\alpha \neq \beta$, $\lambda_\alpha \neq \lambda_\beta$ so $\eta_\alpha = 0$. Applying $P^{\perp}$, projection onto the orthogonal complement of the closed span of the $\mathcal{L}_\alpha$s, we get $\lambda_\beta \eta^\perp = 0$, but as $\lambda_\beta \neq 0$, $\eta^\perp = 0$. Thus T maps $\mathcal{L}_\beta$ to itself, so $TP_\beta = P_\beta T$.

i) Suppose π is irreducible, fix $\xi \in \mathcal{H}\backslash\{0\}$, and express $k_\xi = \sum_{\alpha \in A} \lambda_\alpha P_\alpha$ satisfying the above conditions. As k_ξ is nonzero, $A \neq \emptyset$. We have that $k_\xi \pi(x) = \pi(x) k_\xi$, so also $P_\alpha \pi(x) = \pi(x) P_\alpha$, for all $\alpha \in A$ and all $x \in G$. Thus $\mathcal{L}_\alpha$ is π-invariant by 3.4. By irreducibility, $|A| = 1$, and by a similar argument, k_ξ has no kernel. Thus $\dim(\mathcal{H}) = \dim(\mathcal{L}_\alpha) < \infty$, as P_α is finite rank. Thus π is finite dimensional.

ii) We will make this argument with Zorn's Lemma. Let

$$\Lambda = \left\{(\mathcal{H}', \{\mathcal{L}_\alpha\}_{\alpha \in A}) : \mathcal{H}' \text{ is a } \pi\text{-invariant subspace and } \pi\,|_{\mathcal{H}'} \text{ is completely reduced by the } \mathcal{L}_\alpha\right\}.$$

In the above construction, each P yields a finite dimensional invariant subspace $\mathcal{L}$, and $\pi\,|_{\mathcal{L}}$ is completely reducible by 3.5. Thus $\Lambda \neq \emptyset$ (pedantry: note by our convention, the restriction of π to $\{0\}$ is not a representation and is thus not completely reducible).

Let $(\mathcal{H}_\gamma, \{\mathcal{L}_\alpha\}_{\alpha \in A_\gamma}) \in \Lambda$, indexed by a well ordered set $\gamma \in \Gamma$ be a nonempty ascending chain under inclusion in both components. Then let $\mathcal{H}_\Gamma = \overline{\bigcup_\gamma \mathcal{H}_\gamma}$. We claim

$$\left(\mathcal{H}_\Gamma, \bigcup_{\gamma \in \Gamma} \{\mathcal{L}_\alpha\}_{\alpha \in A_\gamma}\right) \in \Lambda.$$

For any $\xi \in \mathcal{H}_\Gamma$, ξ is the limit of a sequence of elements, each lying in some partial sum $\mathcal{H}_\lambda = \overline{\bigcup_{\gamma \leqslant \lambda} \mathcal{H}_\gamma}$. So we can first approximate arbitrarily well ξ by some member of a partial sum, and then approximate further to any degree of precision by a member of the span of $\{\mathcal{L}_\alpha\}_{\alpha \in A_\lambda}$. Thus the span of the union is dense in $\mathcal{H}_\Gamma$. Further members of the union remain pairwise orthogonal and a fortiori satisfy the other required properties to witness complete reducibility. This completes the claim.

Thus by Zorn's lemma, there is a maximal $\mathcal{H}' \in \Lambda$. If $\mathcal{H}' \neq \mathcal{H}$, then we can find a completely reducible subspace $\mathcal{H}''$ of $\mathcal{H}'^\perp$. Then $\overline{\mathcal{H}' \cup \mathcal{H}''} \in \Gamma$, violating maximality. Thus $\mathcal{H}' = \mathcal{H} \in \Lambda$, so π is completely reducible. □

4. Schur's Lemma and Orthogonality Relations

Theorem 4.1. ***(Schur's Lemma)*** *Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a finite dimensional representation. Then:*

i) π is irreducible if and only if $\pi(G)' = \{T \in \mathcal{B}(\mathcal{H}) : T\pi(\cdot) = \pi(\cdot)T\}$ is $\mathbb{C}I$.

ii) If $\pi' : G \to \mathcal{U}(\mathcal{H}')$ is another finite dimensional representation and both π and π' are irreducible, then each linear $A : \mathcal{H} \to \mathcal{H}'$ so that A intertwines, i.e. that $\pi'(\cdot)A = A\pi(\cdot)$ is of the form cU, $c \in \mathbb{R}^{\geqslant 0}$ and U unitary.

Proof. i) Suppose π is irreducible. If $T \in \pi(G)'$ then so is T^*, as $T^*\pi(x) = (\pi(x^{-1})T)^* = (T\pi(x^{-1}))^* = \pi(x)T^*$. Then the self adjoint operators $T + T^*$ and $i(T - T^*)$ are in $\pi(G)'$. Recall that these are finite-dimensional operators, so the spectral theory of rudimentary linear algebra applies. If $T + T^* \neq cI$ for $c \in \mathbb{C}$, then $T + T^*$ induces a non-trivial decomposition of $\mathcal{H}$ into eigenspaces and $T + T^*$ is a linear combination of pairwise orthogonal projections. We argued in the previous theorem that each such projection must be in $\pi(G)'$. But no proper projection can be in $\pi(G)'$ if π is irreducible, thus both $T + T^* = c_1 I$ and

$i(T - T^*) = c_2 I$. Then $T = (c_1 - ic_2)I \in \mathbb{C}I$. Conversely, if π is irreducible, then let $\mathcal{L}$ be a proper irreducible subspace, then $P_{\mathcal{L}} \in \pi(G)'$ is not of the form cI.

ii) Suppose that $A : \mathcal{H} \to \mathcal{H}'$ intertwines irreducible π and π'. That is $\pi'(\cdot)A = A\pi(\cdot)$. Then $\ker A$ is π-invariant, and $\operatorname{ran} A$ is π'-invariant. We thus see that either $A = 0$ or A is an isomorphism.

Assume A is an isomorphism $\mathcal{H} \to \mathcal{H}'$. Then $A^*A\pi(x) = A^*\pi'(x)A = (\pi'(x^{-1})A)^*A = (A\pi(x^{-1}))^*A = \pi(x)A^*A$. Thus $A^*A \in \pi(G)'$, and so $A^*A = cI$, by the previous part. But A^*A is positive, so $c \in \mathbb{R}^{\geqslant 0}$. If $c \neq 0$, then $U = \frac{1}{\sqrt{c}}A$ is unitary. □

Corollary 4.2. *If G is an abelian compact group, then every irreducible representation is 1 dimensional.*

Proof. Suppose $\pi : G \to \mathcal{U}(\mathcal{H})$ is irreducible. Then for all $x, y \in G$, $\pi(x)\pi(y) = \pi(xy) = \pi(yx) = \pi(y)\pi(x)$. Thus $\pi(x) = c_x I$ for $c_x \in \mathbb{C}$ by Schur's Lemma. If $\dim \mathcal{H} > 1$, then any proper subspace violates irreducibility. □

Definition 4.3. If $\pi : G \to \mathcal{U}(\mathcal{H})$, $\pi' : G \to \mathcal{U}(\mathcal{H}')$ are two irreducible representations, then we say that π and π' are equivalent if there is a unitary U such that $U\pi(\cdot) = \pi'(\cdot)U$, or identically, $U\pi(\cdot)U^* = \pi'(\cdot)$. It can be easily seen that this is an equivalence relation. Thus define

$$\hat{G} = \{\pi : G \to \mathcal{U}(\mathbb{C}^n) : \pi \text{ is irreducible}\} /_{\sim},$$

where $\sim$ is the relation defined above. We will henceforth abuse notations by implicitly selecting representatives, and thus treating members of $\hat{G}$ as representations.

Definition 4.4. Given $\pi \in \hat{G}$, we let $\mathcal{T}_\pi = \{\langle \pi(\cdot)\xi \mid \eta\rangle : \xi, \eta \in \mathcal{H}_\pi\}$. Members of $\mathcal{T}_\pi$ are continuous functions on G so $\mathcal{T}_\pi \subset C(G)$. Invoking compactness, we will think of $\mathcal{T}_\pi$ as a subset of the space $L^2(G)$, the L^2 space that arises from the Haar measure.

Remark 4.5. This is well defined, as if $\pi \sim \pi'$ are two representatives, and $U\pi(\cdot) = \pi(\cdot)'U$, for Uunitary,

$$\mathcal{T}_\pi = \{\langle \pi(\cdot)\xi \mid \eta\rangle : \xi, \eta \in \mathcal{H}_\pi\} = \{\langle U^*\pi'(\cdot)U\xi \mid \eta\rangle : \xi, \eta \in \mathcal{H}_\pi\} = \{\langle \pi(\cdot)U\xi \mid U\eta\rangle : \xi, \eta \in \mathcal{H}_\pi\} = \mathcal{T}_{\pi'}.$$

Theorem 4.6. ***(Schur's Orthogonality Relations)***
Let $\pi, \pi' \in \hat{G}$,

i) If $\pi \neq \pi'$, then $\mathcal{T}_\pi \perp \mathcal{T}_{\pi'}$ in $L^2(G)$.
ii) If $\xi, \eta, \zeta, \gamma \in \mathcal{H}_\pi$, then

$$\int_G \langle \pi(s)\xi \mid \eta\rangle \overline{\langle \pi(s)\zeta \mid \gamma\rangle} ds = \frac{1}{d_\pi}\langle \xi \mid \zeta\rangle\langle \gamma \mid \eta\rangle.$$

In particular, if $\{e_1, \dots e_{d_\pi}\}$ is an orthonormal basis for $\mathcal{H}_\pi$, then $\{\sqrt{d_\pi}\{\langle \pi(\cdot)e_i \mid e_j\rangle\}$ is an orthonormal basis for $\mathcal{T}_\pi$.

Proof. Given linear $A : \mathcal{H}_\pi \mapsto \mathcal{H}_{\pi'}$, let

$$\tilde{A} = \int_G \pi'(s^{-1})A\pi(s) ds.$$

Then

$$\pi'(x)\tilde{A} = \int_G \pi'\left((sx^{-1})^{-1}\right) A\pi(s)ds = \int_G \pi'(t^{-1})A\pi(tx)ds = \tilde{A}\pi(x).$$

Hence if $\pi \nsim \pi'$, $\tilde{A} = 0$, and otherwise, using the same representative, $\pi = \pi'$ and $\tilde{A} = cI$ for $c \in \mathbb{C}$. Given $\xi, \eta \in \mathcal{H}_\pi$ and $\zeta, \gamma \in \mathcal{H}_{\pi'}$, let $A(\nu) = \langle \nu \mid \eta \rangle \gamma$ for $\nu \in \mathcal{H}$. Then,

$$\begin{aligned}\langle \tilde{A}\xi \mid \zeta \rangle &= \int_G \left\langle \pi'(s^{-1})A\pi(s)\xi \mid \zeta \right\rangle ds = \int_G \left\langle \pi'(s^{-1})\langle \pi(s)\xi \mid \eta \rangle \gamma \mid \zeta \right\rangle ds \\ &= \int_G \langle \pi(s)\xi \mid \eta \rangle \left\langle \pi'(s^{-1})\gamma \mid \zeta \right\rangle ds = \int_G \langle \pi(s)\xi \mid \eta \rangle \overline{\langle \pi'(s)\zeta \mid \gamma \rangle} ds.\end{aligned}$$

i) If $\pi \nsim \pi'$, then $\langle \tilde{A}\xi \mid \zeta \rangle = 0$, thus $\mathcal{T}_\pi \perp \mathcal{T}_{\pi'}$.

ii) If $\pi = \pi'$ then let for the given choices of vectors $\tilde{A} = c_{(\eta,\gamma)}I$. Then $\langle \tilde{A}\xi \mid \zeta \rangle = c_{(\eta,\gamma)}\langle \xi \mid \zeta \rangle$. Now if π is a unitary representation, and $\{e_1, \ldots e_{d_\pi}\}$ is an orthonormal basis, then:

$$\begin{aligned}\mathrm{Tr}\tilde{A} &= \sum_{i=1}^{d_\pi} \int_G \langle \pi(s^{-1})A\pi(s)e_i \mid e_i \rangle ds = \int_G \sum_{i=1}^{d_\pi} \langle A\pi(s)e_i \mid \pi(s)e_i \rangle ds \\ &= \int_G \mathrm{Tr}A \; ds = \mathrm{Tr}A.\end{aligned}$$

So $d_\pi c_{(\eta,\gamma)} = \langle \langle \gamma \mid \eta \rangle \gamma \mid \gamma \rangle = \langle \gamma \mid \eta \rangle$, and $\langle \tilde{A}\xi \mid \zeta \rangle = \frac{1}{d_\pi} \langle \xi \mid \zeta \rangle \langle \gamma \mid \eta \rangle$, as desired. □

5. The Peter-Weyl Theorem

Definition 5.1. The left regular representation of G is $\lambda : G \to \mathcal{U}(L^2(G))$ given by $(\lambda(x)f)(y) = f(x^{-1}y)$. For $f \in L^2(G)$, and $x, y \in G$. It is simple to check that this is unitary.

Definition 5.2. For $\pi \in \hat{G}$, π is a finite dimensional unitary representation, and we can define a conjugate representation $\overline{\pi} \in \hat{G}$. Given an orthonormal basis $\{e_1^\pi, \ldots, e_{d_\pi}^\pi\}$ for $\mathcal{H}_\pi$, $\pi(x)$ has matrix $[\pi_{i,j}(x)]_{i,j}$, and we define $\overline{\pi}(x)$ by its matrix $[\overline{\pi_{i,j}(x)}]_{i,j}$. Note that $\overline{\pi}(x)$ is unitary, and the conjugate representation is irreducible.

To see that this is well defined on $\hat{G}$, suppose $\pi \sim \pi'$. Then $\pi(\cdot) = U^*\pi'(\cdot)U$ for U unitary. Identify the Hilbert spaces with $\mathbb{C}^{d_\pi}$, then with the usual orthonormal basis,

$$\begin{aligned}\pi &= \left[\sum_{k,l=1}^{d_\pi} \overline{U_{k,i}} \pi'_{k,l} U_{l,j} \right]_{i,j}, \\ \overline{\pi} &= \left[\sum_{k,l=1}^{d_\pi} \overline{U_{l,j}} \overline{\pi'_{k,l}} U_{k,i} \right]_{i,j} = \left[\sum_{k,l=1}^{d_\pi} U_{k,i} (\overline{\pi'})_{k,l} \overline{U_{l,j}} \right]_{i,j}, \\ \overline{\pi} &= \overline{U}^* \; \overline{\pi'} \; \overline{U}.\end{aligned}$$

Thus $\overline{\pi} \sim \overline{\pi'}$, so this is a well defined operation on $\hat{G}$.

Theorem 5.3. ***(Peter-Weyl)*** *Let G be a compact group,*

i) For $\pi \in \hat{G}$, let $\{e_1^\pi, \ldots, e_{d_\pi}^\pi\}$ be an orthonormal basis for $\mathcal{H}_\pi$, and let $\pi_{i,j} = \langle \pi(\cdot)e_j^\pi \mid e_i^\pi \rangle \in L^2(G)$. Let $C_{\pi,j} = span\{\pi_{i,j} : i \in [d_\pi]\}$. Then $C_{\pi,j}$ is λ-invariant, and $\lambda|_{C_{\pi,j}} \sim \overline{\pi}$.

ii) Let $\mathcal{T}(G) = \bigoplus_{\pi \in \hat{G}} \mathcal{T}_\pi$. Then $\mathcal{T}(G)$ is uniformly dense in $C(G)$.

iii) $\mathcal{T}(G)$ is dense in $L^2(G)$, and $\mathcal{B} = \{\sqrt{d_\pi}\pi_{i,j}\}$ is an orthonormal basis for $L^2(G)$. Moreover, if π^k denotes the direct sum of k copies of π, then

$$\lambda \sim \bigoplus_{\pi \in \hat{G}} \pi^{d_\pi}.$$

Proof. The Schur Orthogonality Relations tell us that $\mathcal{B}$ is an orthonormal set in $L^2(G)$.

i) For $x, y \in G$, $\pi(x^{-1}y) = \pi(x)^*\pi(y)$. $\pi_{i,j}$ is a member of $C(G) \subset L^2(G)$, and

$$(\lambda(x)\pi_{i,j})(y) = \pi_{i,j}(x^{-1}y) = \sum_{k=1}^{d_\pi} \overline{\pi_{k,i}(x)}\pi_{k,j}(y)$$

$$\lambda(x)\pi_{i,j} = \sum_{k=1}^{d_\pi} \overline{\pi_{k,i}(x)}\pi_{k,j} \in C_{\pi,j},$$

as the sum is a linear combination of the basis of $C_{\pi,j}$. Thus $C_{\pi,j}$ is λ-invariant.

Define $U : \mathcal{H}_\pi \to C_{\pi,j}$ by $Ue_i^\pi = \sqrt{d_\pi}\pi_{i,j}$. Then,

$$\begin{aligned} U^*\lambda(x)Ue_i^\pi &= \sqrt{d_\pi}U^*\lambda(x)\pi_{i,j} = \sqrt{d_\pi}U^* \sum_{k=1}^{d_\pi} \overline{\pi_{k,i}(x)}\pi_{k,j} \\ &= \sum_{k=1}^{d_\pi} \overline{\pi_{k,i}(x)}U^*(\sqrt{d_\pi}\pi_{k,j}) = \sum_{k=1}^{d_\pi} \overline{\pi_{k,i}(x)}e_k^\pi = \overline{\pi}(x)e_i^\pi. \end{aligned}$$

Thus $\lambda|_{C_{\pi,j}} \sim \overline{\pi}$.

ii) We will argue by Stone-Weierstrass. G is compact and Hausdorff. We will show that the span of the matrix coefficients, $\mathcal{T}(G)$, is a conjugate closed and point separating subalgebra of $C(G)$, and is hence uniformly dense in $C(G)$.

Conjugate closed is immediate as $\overline{\pi_{i,j}}(x) = \overline{\pi}_{i,j}(x)$, so if we conjugate some linear combination of the basis, the resulting linear combination remains in the span. To see that this is a algebra, we will use tensor products. For $\pi_{i,j}, \pi'_{k,l}$, the product

$$\pi_{i,j}\pi'_{k,l}(x) = \langle \pi(x)e_j^\pi \mid e_i^\pi \rangle \langle \pi'(x)e_l^{\pi'} \mid e_k^{\pi'} \rangle = \langle (\pi \otimes \pi')(x)e_j^\pi \otimes e_l^{\pi'} \mid e_i^\pi \otimes e_k^{\pi'} \rangle.$$

Now, $\pi \otimes \pi'$ is not necessarily irreducible, but it is completely reducible. Say, $\pi \otimes \pi' = \bigoplus_{n=1}^N \pi_n$, with $\pi_n \in \hat{G}$, and let $P_n : \mathcal{H}^\pi \otimes \mathcal{H}^{\pi'} \to \mathcal{H}^{\pi_n} \subset \mathcal{H}^\pi \otimes \mathcal{H}^{\pi'}$ be the orthogonal projection onto the space on which the n-th representation acts. Then $\sum_{n=1}^N P_n = Id_{\mathcal{H}^\pi \otimes \mathcal{H}^{\pi'}}$ is a resolution of the identity. Thus,

$$\pi_{i,j}\pi'_{k,l}(x) = \langle (\pi \otimes \pi')(x) \sum_{n=1}^N P_n e_j^\pi \otimes e_l^{\pi'} \mid \sum_{m=1}^N P_m e_i^\pi \otimes e_k^{\pi'} \rangle = \sum_{n=1}^N \langle \pi_n(x) P_n e_j^\pi \otimes e_l^{\pi'} \mid P_n e_i^\pi \otimes e_k^{\pi'} \rangle$$

because the $\mathcal{H}^{\pi_n}$ are invariant and mutually orthogonal, so the cross terms vanish. But this is a sum of matrix coefficients of elements of $\hat{G}$, and is thus in $\mathcal{T}(G)$. This shows $\mathcal{T}(G)$ is an algebra.

Finally, we show point separation. First we prove λ is injective. Let $x \in G \backslash e$. We can find U open, $e \in U \subset G$, so that $x \notin UU^{-1}$ or identically, $xU \cap U = \varnothing$. Then consider the indicator $\chi_U \in L^2(G)$. $(\lambda(x)\chi_U)(y) = \chi_U(x^{-1}y) = \chi_{xU}(y)$, so $\lambda(x)\chi_U = \chi_{xU} \neq \chi_U$. Thus $\lambda(x) \neq Id_{L^2(G)}$. Now fix $x \neq y \in G$. By complete reducibility, $L^2(G)$ breaks down into spaces on which λ acts irreducibly. Thus there is some space $\mathcal{L} \subset L^2(G)$ on which λ acts irreducibly and $\lambda\mid_{\mathcal{L}}(x) \neq \lambda\mid_{\mathcal{L}}(y)$. Then $\pi = \lambda\mid_{\mathcal{L}} \in \hat{G}$ witnesses point separation.

iii) $C(G)$ is dense in $L^2(G)$ (in general, for X locally compact Hausdorff, $C_c(G)$ is dense in $L^p(G)$ for $1 \leqslant p < \infty$). Thus $\mathcal{T}(G)$ is dense in $L^2(G)$ as it is dense in $C(G)$. Thus $\mathcal{B}$ spans and is an orthonormal basis. Now $\mathcal{T}(G)$ has already been broken down into mutually orthogonal spaces on which λ acts irreducibly as $\overline{\pi}$, with multiplicity d_π, for each $\pi \in \hat{G}$. Thus, using that twice taking a conjugate representation of $\pi \in \hat{G}$ is the identity on $\hat{G}$ and hence that $\pi \mapsto \overline{\pi}$ is bijective,

$$\lambda \sim \bigoplus_{\pi \in \hat{G}} \overline{\pi}^{d_\pi} \sim \bigoplus_{\pi \in \hat{G}} \pi^{d_\pi}. \qquad \square$$

This brings us to the end of this journey. Complete reducibility gives us the structure of all representations of compact groups in terms of irreducible representations, which are tractable as they are finite dimensional. The totality of the information about the irreducible representations of a group is contained in the left regular representation. One could also develop character theory. Thus, compact groups are the natural setting in which the general theorems of finite group representation theory live.

A significant application of this theory occurs in harmonic analysis. The Fourier transform of $f \in L^1(G)$, indexed by $\hat{G}$, is

$$\hat{f} : \hat{G} \mapsto \bigsqcup_{\pi \in \hat{G}} \mathcal{B}(\mathcal{H}_\pi)$$
$$\hat{f}(\pi) = \int_G f(s)\pi(s^{-1})ds.$$

Both the coefficients and their values may seem complicated - for the Fourier transform of $\mathbb{R}$ we just calculate a complex valued integral with neither the need to understand the full representation theory of some nonabelian group nor the calculation of operator valued integrals!

Of course, representation theory sneakily underlies the Forurier theory of $\mathbb{R}$, for there is a abelian group of translations in $\mathbb{R}$ with one dimensional irreducible representations and characters e^{ixt}. Nor are the computations in the compact setting intractable, for the operator valued integrals are of finite complex matrices, so calculating a Fourier coefficient given π is not difficult. This theory is used regularly in physics for systems with continuous symmetry groups, and is just as versatile as vanilla Fourier theory in solving equations of motion.

I have to stop somewhere, so I will leave you with something to wonder about. The Lagrangian formulation of classical mechanics gives equations on tangent bundle of space, while the Hamiltonian formulation gives equations on the cotangent bundle. There is a connection between the operations of passing to a dual vector space, with a natural isomorphism to the double dual, and that of passing to the dual group of an abelian system, with a similar guarantee given by Pontryagin duality. This connection can make formal the process of solving eigenvalue problems for well behaved time evolution in frequency space. Modern gauge theories live on principal bundles as opposed to the vector bundle that is the tangent space, and there is a dual formulation of this theory using techniques in harmonic analysis.

6. References

This paper follows:

- *A Course in Abstract Harmonic Analysis* by G. Folland
- My lecture notes from a course following the above taught by Nico Spronk
- Section 3 of *254A - Hilbert's Fith Problem* by Terrance Tao