STONE-WEIERSTRASS BY FUNCTIONAL ANALYSIS

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In this talk I will use many theorems from functional analysis to prove Stone-Weierstrass.

Theorem 1. (Stone-Weierstrass): *If X is a compact Hausdorff space, and $\mathcal{A} \subseteq C_{\mathbb{R}}(X)$ is a uniformly closed sub-algebra that separates points (if $x \neq y \in X$, then there is $f \in \mathcal{A}$ with $f(x) \neq f(y)$) and has $g \in \mathcal{A}$ with $g(x) > 0$, $\forall x \in \mathcal{A}$, then $\mathcal{A} = C_{\mathbb{R}}(X)$.*

$C_{\mathbb{R}}(X)$ is a Banach space, and we will need to know its dual space; this is given by the Riesz Representation theorem. First, let's discus the space of regular Borel real measures on X.

Theorem 2. (Jordan Decomposition): *If μ is a real measure on (Σ, X), then there exists a measurable set P such that $\mu(E \cap P) \geq 0$ and $\mu(E \cap P^C) \leq 0$ for all measurable E. Then $\mu^+(E) = \mu(E \cap P)$, $\mu^-(E) = -\mu(E \cap P)$ are unique positive mutually singular measures such that $\mu = \mu^+ - \mu^-$.*

Definition 1. Define $M_{\mathbb{R}}(X)$ to be the set of finite regular real Borel measures on X. The total variation of μ is defined here as $|\mu| = \mu^+ + \mu^-$. Then $M_{\mathbb{R}}(X)$ is made into a Banach space by the norm $\|\mu\| = |\mu|(X) = \mu^+(X) + \mu^-(X)$.

Theorem 3. (Riesz Representation): *If X is a compact Hausdorff space, then $C_{\mathbb{R}}(X)^* \cong M_{\mathbb{R}}(X)$ and the isomorphism is isometric and is given by $\mu \in M_{\mathbb{R}}(X) \mapsto \phi \in C_{\mathbb{R}}(X)^*$ with $\phi(f) = \int_X f d\mu$.*

In particular, we see from this theorem that $\|\mu\| = \sup_{f \in b(C(X))} \left|\int_X f d\mu\right|$.

We will use a nice corollary of the Hahn-Banach theorem:

Theorem 4. (Hahn-Banach): *If A is a proper closed subspace of $C_{\mathbb{R}}(X)$, then there exists $\mu \in M_{\mathbb{R}}(X)$, $\mu \neq 0$, so that $\int_X f d\mu = 0$ for all $f \in A$.*

Now let's do some analysis on this dual space!

Definition 2. The ω^* topology on $C_{\mathbb{R}}(X)^*$ is defined by $\mu_\lambda \to \mu$ if and only if $\int_X f d\mu_\lambda \to \int_X f d\mu$ for all $f \in C_{\mathbb{R}}$. This is the weakest topology on $C_{\mathbb{R}}(X)^*$ so that every $f \in C_{\mathbb{R}}(X)$ is continuous.

A note is on order:

Theorem 5. *If A is a proper closed subspace of $C_{\mathbb{R}}(X)$, then define $A^\perp = \{\phi \in C_{\mathbb{R}}(X)^* : \phi(f) = 0 \ \forall f \in A\}$. Then $A^\perp \neq \{0\}$, and $A^\perp$ is ω^*-closed.*

Proof. $A^\perp \neq \{0\}$ by Hahn-Banach. If ϕ_λ is a net in $A^\perp$ and $\phi_\lambda \to \phi$ in the ω^* topology, then $\phi_\lambda(f) \to \phi(f)$ for all $f \in C_{\mathbb{R}}(X)$, in particular if $h \in A$, $0 = \phi_\lambda(h) \to \phi(h)$ so $\phi(h) = 0$ and thus $\phi \in A^\perp$, so $A^\perp$ is ω^*-closed. □

This topology has useful properties:

Theorem 6. (Banach Alaoglu): *The norm closed unit ball of* $C_{\mathbb{R}}(X)^*$, $b(C_{\mathbb{R}}(X)^*)$, *is* ω^**-compact.*

And for our final ingredient:

Definition 3. If $U \subset V$, V a vector space, then U is convex if whenever $x, y \in U$ and $\lambda \in [0,1]$ then $\lambda x + (1-\lambda)y \in U$. If U is convex, then we say $z \in U$ is an extreme point of U if whenever $x, y \in U$ and $\lambda \in (0,1)$ with $z = \lambda x + (1-\lambda)y$ we have $x = y = z$. We denote the set of extreme points of U as $Ext(U)$. [Draw diagram of quarter circle in $\mathbb{R}^2$.]

Theorem 7. (Krein-Milman): *If* $K \subset C_{\mathbb{R}}(X)^*$ *is convex and* ω^**-compact, then* $K = \overline{Ext(K)}^{\omega^*}$.

Finally:

Theorem 8. (Stone-Weierstrass): *If* X *is a compact Hausdorff space, and* $\mathcal{A} \subseteq C_{\mathbb{R}}(X)$ *is a uniformly closed sub-algebra that separates points (if* $x \neq y \in X$*, then there is* $f \in \mathcal{A}$ *with* $f(x) \neq f(y)$*) and has* $g \in \mathcal{A}$ *with* $g(x) > 0$, $\forall x \in \mathcal{A}$*, then* $\mathcal{A} = C_{\mathbb{R}}(X)$.

Proof. Suppose $\mathcal{A} \neq C_{\mathbb{R}}(X)$. Then $\mathcal{A}$ is a proper closed subspace of $C_{\mathbb{R}}(X)$. Let $K = b_1(C_{\mathbb{R}}(X)^*) \cap \mathcal{A}^{\perp}$. Then K is the intersection of a ω^*-compact, convex, set and a ω^*-closed convex set, thus K is ω^*-compact, convex, and $K \neq \{0\}$.

By Krein-Milman, K has an extreme point μ. For any $\nu \in \mathcal{A}^{\perp}$ with $\nu \neq 0$, $\frac{\nu}{2|\nu|} + \frac{-\nu}{2|\nu|} = 0$, so 0 is not an extreme point of K and thus $\mu \neq 0$. In fact $\|\mu\| = 1$.

Suppose $\text{supp}(\mu) \neq \{x_0\}$, i.e. μ is not supported at one point. Then if $Y = \text{supp}(\mu)$, we can find $f \in \mathcal{A}$ which is not constant on Y as $\mathcal{A}$ separates points. Using our positive function g, we can find positive numbers so that $\widetilde{f} = \frac{f+cg}{d}$ satisfies $0 < \widetilde{f}(x) < 1$ for all $x \in X$, and $\widetilde{f}$ is not constant on Y. Hence assume $0 < f(x) < 1$ for all $x \in X$.

Thus if $\mu_1 = f\mu$, then $f\mu = f\mu^+ - f\mu^-$ and $f\mu^+$, $f\mu^-$ are positive mutually singular measures, thus by the uniqueness of the Jordan Decomposition, $|f\mu| = f\mu^+ + f\mu^- = f(\mu^+ + \mu^-) = f|\mu|$. Similarly, if $\mu_2 = (1-f)\mu$, then $|\mu_2| = (1-f)|\mu|$.

Further, if $h \in \mathcal{A}$, then $\mu_1(h) = \int_X hf d\mu$. But $\mathcal{A}$ is an algebra and $f, h \in \mathcal{A}$ so $fh \in \mathcal{A}$ and $\mu \in \mathcal{A}^{\perp}$, so $\mu_1(h) = 0$ and $\mu_1 \in \mathcal{A}^{\perp}$. Similarly $\mu_2 \in \mathcal{A}^{\perp}$.

Consider $\|\mu_1\| + \|\mu_2\| = \int_X d|\mu_1| + \int_X d|\mu_2| = \int_X fd|\mu| + \int_X (1-f)d|\mu| = \int_X d|\mu| = \|\mu\| = 1$. We have:

$$\|\mu_1\| \frac{\mu_1}{\|\mu_1\|} + \|\mu_2\| \frac{\mu_2}{\|\mu_2\|} = \mu_1 + \mu_2 = f\mu + (1-f)\mu = \mu$$

With $\frac{\mu_1}{\|\mu_1\|}, \frac{\mu_2}{\|\mu_2\|} \in K$. This is a convex combination. As f is not constant on Y, μ_1 is not a constant multiple of μ. Thus μ is not an extreme point - contradiction!

Thus $\text{supp}(\mu) = \{x_0\}$ and $\mu = \pm\delta_{x_0}$. Then for our positive g:

$$0 = \int_X g d\mu = \pm \int_X g d\delta_{x_0} = \pm g(x_0) \neq 0$$

Contradiction; thus $\mathcal{A} = C_{\mathbb{R}}(X)$.

□