THE EULER-LAGRANGE PDE

TARAS KOLOMATSKI

1. A Few Words on Boundary Values

In this paper, we will endeavour to find solutions to certain partial differential equations which satisfy given boundary data. For functions that can be continuously extended to the boundary of our domain, this notion is clearly well defined. Indeed, with continuous boundary data we will hope for solutions with at least this level of regularity. However, as is to be expected, we will have to broaden the function spaces in which we look for solutions, hence a more sophisticated notion of boundary values is necessary.

When $U \subset \mathbb{R}^n$ is open and bounded with a C^1 boundary, ∂U is a compact C^1 $(n-1)$-manifold which can inherit a metric induced from its embedding in $\mathbb{R}^n$. With this metric, it makes sense to speak of the spaces $L^p(\partial U)$.

Our function space will be $W^{1,p}(U)$ and we will make use of the following result:

Theorem 1.1. ***(Evans 5.5.1)*** *Let $U \subset \mathbb{R}^n$ be open and bounded with a C^1 boundary. There exists a bounded linear operator*

$$T : W^{1,p}(U) \to L^p(\partial U),$$

such that

i. *$Tu = u|_{\partial U}$, for all $u \in W^{1,p}(U) \cap C(\overline{U})$, and*

i. *$\|Tu\|_{L^p(\partial U)} \leqslant C\|u\|_{W^{1,p}(U)}$, for all $u \in W^{1,p}(U)$, where $C = C(p, U)$ is an absolute constant.* □

The function Tu is called the **trace** of u. T is necessarily unique as, in the context of $U \subset \mathbb{R}^n$ open and bounded with a C^1 boundary, $C^\infty\left(\overline{U}\right)$ is dense in $W^{1,p}(U)$ (**Evans 5.3.2**).

The space $W_0^{1,p}(U)$ is the closure of $\mathcal{D}(U)$ in $W^{1,p}(U)$. There are multiple useful results that hold only of $W_0^{1,p}(U)$. For example Poincaré's inequality, which states, for bounded U, that:

$$\|u\|_{L^p(U)} \leqslant C\|\nabla u\|_{L^p(U)}.$$

This result cannot hold of $W^{1,p}(U)$ as we can break the inequality by adding a sufficiently large constant to u. Notice that, by the linearity of T, adding a constant to u adds this constant to the trace. Thus this difficulty is avoided when we impose boundary values. Further, members of $\mathcal{D}(U)$ vanish outside a strict subset of U and have zero trace by the first condition in the above theorem. These together suggest a relationship between $W_0^{1,p}(U)$ and the class of zero trace functions.

The following theorem is pertinent as it will allow us to apply results about $W_0^{1,p}(U)$ to spaces of functions that satisfy fixed boundary data.

Theorem 1.2. *Let $U \subset \mathbb{R}^n$ be open and bounded with a C^1 boundary. Then for $u \in W^{1,p}(U)$, $Tu = 0$ if and only if $u \in W_0^{1,p}(U)$.* □

2. Motivation for the Euler-Lagrange Equation

It is difficult to make any general statements of solutions to partial differential equations in the form:

$$A[u] = 0,$$

when A is a non-linear differential operator. However, if there is a functional (here taken to mean not necessarily linear) I such that for some notion of differentiation:

$$A[u] = I'[u] = 0,$$

then it is suggestive that we will be able to find solutions if we can solve the problem of minimising I. It turns out that the latter problem is frequently tractable. Further, minimisation problems of this form are often themselves an object of interest in numerous disciplines, including physics.

We will restrict our attention to the following construction:

Fix a dimension n, and a set $U \subset \mathbb{R}^n$ that is open and bounded with a C^∞ boundary. Additionally fix a smooth function

$$L : \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R},$$

such that L extends continuously to $\mathbb{R}^n \times \mathbb{R} \times \overline{U}$. We call L the **Lagrangian**.

We will denote the arguments of L as $L(p, z, x)$ with $p \in \mathbb{R}^n$, $z \in \mathbb{R}$, and $x \in U$. Further we let L_p and L_x denote the gradients with respect to the n-dimensional arguments p and x, and L_z the partial derivative with respect to z.

Define a functional I, corresponding to L, on the space $C^\infty\left(\overline{U}\right)$ by:

$$I[w] = \int_U L\left(Dw(x), w(x), x\right) dx.$$

In the interest of brevity, we henceforth suppress the argument in Dw and w.

Suppose that we fix boundary data $g \in C(\partial U)$, and $u \in C^\infty\left(\overline{U}\right)$ attains the minimal value of I among functions $w \in C^\infty\left(\overline{U}\right)$ with $w|_{\partial U} = g$. Then for all $v \in \mathcal{D}(U)$, $(u + tv)|_{\partial U} = g$ for any $t \in \mathbb{R}$. We can construct the following smooth function:

$$\tau(t) = I[u + tv] = \int_U L\left(Du + tDv, u + tv, x\right) dx.$$

Which will have a minimum at $t = 0$, and hence $\tau'(0) = 0$. Expanding this explicitly,

$$\tau'(0) = \int_U \left(\sum_{i=1}^{n} L_{p_i}\left(Du + tDv, u + tv, x\right) v_i + L_z\left(Du + tDv, u + tv, x\right) v \right) dx$$

As we assumed that v had support that is a strict subset of U, and as $L_{p_i}\left(Du + tDv, u + tv, x\right)$ and $v_i(x)$ are smooth functions in x_i, we may apply integration by parts to the first term to shift the derivative, without incurring a boundary term.

$$\tau'(0) = \int_U \left(-\sum_{i=1}^{n} L_{p_i}\left(Du, u, x\right)_i + L_z\left(Du, u, x\right) \right) v \; dx = 0$$

As this is true of all test functions v, it follows that

$$-\sum_{i=1}^{n} L_{p_i}\left(Du, u, x\right)_i + L_z\left(Du, u, x\right) = 0.$$

(*For L^p functions this would be true almost everywhere but the above expression is continuous.*)

We have derived a partial differential equation that is solved (not exclusively) by smooth minimises of I. This is the **Euler-Lagrange equation**, and will here be the object of study.

3. Existence of Weak Solutions

Suppose we are given a Lagrangian L and boundary data g such that over some non-empty function space satisfying the boundary constraint, the infimum of I is finite. We can then form a sequence of functions u_k such that $I[u_k]$ decreases to the infimum. Ideally, we hope that we can make a compactness argument to extract a converging subsequence, which by continuity will attain the minimal value of our functional. Unfortunately, several difficulties arise:

Consider the functional $I[u] = \int_U e^u dx$ on $C^\infty((0,1))$ with boundary values of 1 at the endpoints. It is easy to see that the infimum of I is 0, but that functions that come close to attaining it must have arbitrarily large negative values for subsets of $(0,1)$ arbitrarily close in measure to 1. Clearly a smooth minimiser does not exist for this problem, and a lack of compactness is apparent.

If the Lagrangian is thought of as a cost function, then it seems reasonable to expect rewards for functions with tame values and derivatives. Indeed if we control at least the derivative, then we should expect that functions that come close to minimising the functional I are uniformly bounded. This is an assumption we will have to make:

Definition 3.1. A Lagrangian L satisfies the **coercivity condition** if there is a number $1 < q < \infty$ and constants $\alpha > 0, \beta \geqslant 0$ such that:

$$L(p, z, x) \geqslant \alpha |p|^q - \beta$$

at all points $(p, z, x) \in \mathbb{R}^n \times \mathbb{R} \times U$.

We will make note of the value of q that satisfies this definition and look for solutions in the space $W^{1,p}(U)$. For functions in this space I is well defined (but not necessarily finite) as it is just an integral of some measurable function. For all functions $w \in W^{1,p}(U)$, this inequality directly implies that:

$$I[w] \geqslant \alpha \|Dw\|^q_{L^q(U)} - \beta \text{ vol}(U).$$

Hence I will always have a finite infimum when coercivity holds.

When we introduce the boundary data, our function space, called the **admissible set**, becomes

$$\mathcal{A} = \left\{ u \in W^{1,p}(U) : Tu = g \right\}.$$

With coercivity we will be able to show that a sequence u_k such that $I[u_k]$ converges to the infimum has uniformly bounded norms $\|u_k\|_{L^q(U)}$ and $\|Du_k\|_{L^q(U)}$. Boundedness does not imply compactness in $L^q(U)$, however as $L^q(U)$ is reflexive $(p > 1)$, and thus itself a dual space, bounded weakly closed sets are weakly compact by Banach-Alaoglu (the weak and weak* topologies coincide). Thus we may pass to a subsequence that converges weakly. Is it true, though, that I is continuous with respect to weak limits?

This is our second difficulty; functions with arbitrarily small L^q norm could take on arbitrarily large values, for example, so convergence in the Lebesgue or weak Lebesgue sense seems incompatible with I. Indeed, we will be unable to demonstrate that $I[u_k] \to I[u]$ for arbitrary weak limits $u_k \to u$. However, under certain assumptions we will be able to establish the following property.

Definition 3.2. A functional I on $W^{1,q}(U)$ is **weakly lower semicontinuous** if, whenever $u_k \to u$ weakly,

$$I[u] \leqslant \liminf_{k \to \infty} I[u_k].$$

We assume that $L(p, z, x)$ is convex in the variable p for fixed z, x. This assumption is reasonable as some work involving the second derivative of $\tau(t)$ yields that $[D_{p_i,p_j} L(Du, u, x)]_{i,j}$ is a positive semidefinite matrix for minimising functions u. We are assuming this holds on a larger set, though.

A characterization of convexity for smooth functions is that they lie above their supporting hyperplanes. Thus if $L(p,z,x)$ is convex in p, for fixed z,x we have:

$$L\left(p_2,z,x\right) \geqslant L\left(p_1,z,x\right)+L_p\left(p_1,z,x\right)\cdot\left(p_2-p_1\right).$$

With these ingredients, we will now begin producing results.

Theorem 3.3. *If L is bounded bellow and convex in p, then L is weakly lower semicontinuous on $W^{1,q}(U)$.*

Proof. Fix a sequence $u_k \in W^{1,q}(U)$ such that $u_k \to u$ weakly in $W^{1,q}(U)$. Let

$$l=\liminf_{k\in\mathbb{N}} I[u_k].$$

By passing to a subsequence, we can assume without loss of generality that

$$l=\lim_{k\in\mathbb{N}} I[u_k].$$

To establish the theorem, it suffices to show $I[u] \leqslant l$.

Setup: Weak convergence implies boundedness (see the appendix), so

$$\sup_{n\in\mathbb{N}}\|u_k\|_{W^{1,q}(U)}<\infty.$$

By the Kondrakov-Rellich compactness theorem (**Evans 5.7.1**), the inclusion $W^{1,q}(U) \to L^q(U)$ is compact. Thus we can pass to a subsequence such that $u_k \to u$ converges in $L^q(U)$. Norm convergence in $L^q(U)$ implies that we can pass to yet another subsequence that additionally converges pointwise almost everywhere. Finally, as $\overline{U}$ is compact and thus has finite Lebesgue measure, Ergoff's theorem tells us that our sequence converge almost uniformly. That is, for all $\varepsilon>0$, there is a measurable subset $E_\varepsilon \subset U$ with $\lambda(U\backslash E_\varepsilon)<\varepsilon$ such that $u_k \to u$ uniformly almost everywhere on E_ε.

Additionally, for all $\varepsilon>0$, let $F_\varepsilon=\left\{x \in U : |u(x)|+|Du(x)|<\frac{1}{\varepsilon}\right\}$, defined almost everywhere. Notice that $\lambda(U\backslash E_\varepsilon) \to 0$ as $\varepsilon \to 0$ by the monotone convergence theorem applied to the indicator functions. Combining the above two constructions, define $G_\varepsilon=E_\varepsilon \cap F_\varepsilon$. By construction, $\lambda(U\backslash G_\varepsilon) \to 0$ as $\varepsilon \to 0$.

The argument: As L is bounded bellow, by possibly adding a constant, we can assume $L \geqslant 0$.

$$I[u_k]=\int_U L\left(Du_k,u_k,x\right)dx \geqslant \int_{G_\varepsilon} L\left(Du_k,u_k,x\right)dx$$

$$\geqslant \int_{G_\varepsilon} L\left(Du,u_k,x\right)dx+\int_{G_\varepsilon} L_p\left(Du,u_k,x\right)\left(Du_k-Du\right)dx$$

For the first summand:

Du is bounded on G_ε so $\overline{Du(G_\varepsilon)}$ is compact.

$u_k \to u$ uniformly and u is bounded thus there is some N such that for all $k \geqslant N$,

$$\|u_k(x)\|_{L^\infty(G_\varepsilon)}<\|u\|_{L^\infty(G_\varepsilon)}+1<\infty.$$

Thus all u_k take values in a compact set, so $L(Du,u_k,x)$ is uniformly bounded ($\forall x \in U, n \in \mathbb{N}$) by some constant by the extreme value theorem. As $L(Du,u_k,x) \to L(Du,u,x)$ pointwise for each $x \in U$, the dominated convergence theorem implies:

$$\lim_{n\to\infty}\int_{G_\varepsilon} L\left(Du,u_k,x\right)dx=\int_{G_\varepsilon} L\left(Du,u,x\right)dx$$

For the second summand: $D_pL(Du,u_k,x) \to D_pL(Du,u_k,x)$ uniformly over $x \in G_\varepsilon$, and hence in $L^\infty(G_\varepsilon)$. Further $Du_k \to Du$ weakly in $L^q(G_\varepsilon)$. As G_ε has compact closure, $L^\infty(G_\varepsilon) \subset L^{q^*}(G_\varepsilon)$, so by joint continuity:

$$\lim_{n\to\infty}\int_{G_\varepsilon} L_p\left(Du,u_k,x\right)\left(Du_k-Du\right)dx=\int_{G_\varepsilon} L_p\left(Du,u,x\right)\left(Du-Du\right)dx=0.$$

Thus,

$$l = \lim_{n\to\infty} I[u_k] \geqslant \int_{G_\varepsilon} L\left(Du, u, x\right) dx$$

Letting $\varepsilon \to 0$ and applying the monotone convergence theorem (insert indicators), we see:

$$l \geqslant \int_U L\left(Du, u, x\right) dx = I[u]$$

□

Now that we have weak lower semicontinuity, we will be able to prove the existence of a minimiser of I.

Theorem 3.4. *Suppose L satisfies the coercivity inequality and convexity in p. If the admissible set $\mathcal{A}$ is non-empty for a given trace, then there exists $u \in \mathcal{A}$ such that*

$$I[u] = \inf_{w\in\mathcal{A}} I[w].$$

Proof. By adding a constant to L, we can set $\beta = 0$ in the coercivity inequality. So assume without loss of generality that $L \geqslant \alpha|Dw|^q$. Then for $w \in \mathcal{A}$,

$$I[w] \geqslant \alpha \int_U |Dw|^q dx \geqslant 0$$

Let $m = \inf_{w\in\mathcal{A}} I[w] \geqslant$. Suppose $m < \infty$, as otherwise the result trivially holds. Thus:

$$\sup_{w\in\mathcal{A}} \|Du_k\|_{L^q(U)} < \infty.$$

Now pick a sequence $u_k \in \mathcal{A}$ with $I[u_k] \to m$. We would like to work with $W_0^{1,q}(U)$ functions, and we can do so at the cost of a constant. Pick some $w \in \mathcal{A}$. Then $u_k - w \in W_0^{1,q}(U)$. Applying the Poincaré inequality:

$$\|u_k\|_{L^q(U)} \leqslant \|u_k - w\|_{L^q(U)} + \|w\|_{L^q(U)} \leqslant C\|D(u_k - w)\|_{L^q(U)} + \|w\|_{L^q(U)}$$

We have shown above that the norms $\|D(u_k - w)\|_{L^q(U)}$ are uniformly bounded, and the second term is a constant. Thus the norms $\|u_k\|_{L^q(U)}$ are uniformly bounded.

Banach-Alaoglu tells us that bounded weakly closed sets in $L^q(U)$ are weakly compact (we use here that $p > 1$ to ensure reflexivity). Recall, additionally, that norm closed convex subsets of a Banach space are weakly closed. Thus the norm closed balls in $W^{1,q}(U)$ are weakly compact. It follows from the above boundedness observations that there is a function $u \in W^{1,q}(U)$ such that passing to a subsequence we have $u_k \to u$ weakly in $L^q(U)$. We can pass to yet another subsequence such that the Du_k have a weak limit, which must evidently be Du. As $\mathcal{A}$ is a closed (hence convex) subspace, it is weakly closed so $u \in \mathcal{A}$.

By definition, $m \leqslant I[u]$. But as $u_k \to u$ weakly in $W^{1,q}(U)$ and $I[u_k] \to m$, our hypothesise allow us to invoke weak lower semicontinuity, so $I[u] \leqslant m$. Hence

$$I[u] = \inf_{w\in\mathcal{A}} I[w].$$

□

The Euler-Lagrange equation was derived such smooth minimizes of I are among it's solutions. We have shown that under certain assumptions, there exist weak minimizers of I, but it is not immediately apparent that these are solutions to the differential equation.

Unfortunately, this will require making additional assumptions that bound the growth of L from above. We require that for our q form coercivity, that:

$$|L(p, z, x)| \leqslant C\left(|p|^q + |z|^q + 1\right)$$
$$|L_p(p, z, x)| \leqslant C\left(|p|^{q-1} + |z|^{q-1} + 1\right)$$
$$|L_z(p, z, x)| \leqslant C\left(|p|^{q-1} + |z|^{q-1} + 1\right)$$

for some constant $C > 0$ and all points $p \in \mathbb{R}^n, z \in \mathbb{R}, x \in U$. We call this the **growth condition**.

Suppose that for a smooth function u, for all $v \in \mathcal{D}(U)$,

$$\int_U \left(-\sum_{i=1}^{n} L_{p_i} (Du, u, x)_i + L_z (Du, u, x) \right) v \; dx = 0.$$

Then, by density, this relation holds for all $v \in W_0^{1,p}(U)$.

More generally, if for $u \in W^{1,p}(U)$, this condition holds for all $v \in \mathcal{D}(U)$, then the growth condition implies that $L_{p_i}(Du, u, x), L_z(Du, u, x) \in L^{\frac{q}{q-1}}(U)$ - the conjugate space to $L^q(U)$. Thus the density argument also goes through. With these observations, it is reasonable to make the following definition.

Definition 3.5. We say $u \in \mathcal{A}$ is a weak solution of the Euler-Lagrange equation (derived from L, g), provided that for all $v \in W_0^{1,p}(U)$:

$$\int_U \left(-\sum_{i=1}^{n} L_{p_i} (Du, u, x)_i + L_z (Du, u, x) \right) v \; dx = 0.$$

Theorem 3.6. *Suppose that L satisfies the growth condition and $u \in \mathcal{A}$ is such that*

$$I[u] = \inf_{w \in \mathcal{A}} I[w].$$

Then u is a weak solution of the Euler-Lagrange equation.

Proof. We fix $v \in W_0^{1,p}(U)$ and define

$$\tau(t) = I[u + tv].$$

The growth condition implies that τ takes on finite values. Pick $t \neq 0$. We consider:

$$\frac{\tau(t) - \tau(0)}{t} = \int_U \frac{L(Du + tDv, u + tv, x) - L(Du, u, x)}{t} dx$$

and let

$$L^t(x) = \frac{L(Du + tDv, u + tv, x) - L(Du, u, x)}{t}$$

denote the integrand. For x fixed,

$$L^t(x) \to \sum_{i=1}^{n} L_{p_i} (Du, u, x) v_i + L_z (Du, u, x) v$$

almost everywhere, as this is just the derivative of a smooth function in t. Similarly, we can apply the fundamental theorem of calculus:

$$\begin{aligned} L^t(x) &= \frac{1}{t} \int_0^t \frac{d}{ds} L(Du + sDv, u + sv, x) ds \\ &= \frac{1}{t} \int_0^t \sum_{i=1}^{n} L_{p_i} (Du + sDv, u + sv, x) v_i + L_z (Du + sDv, u + sv, x) v \; ds. \end{aligned}$$

Consider a term in the integrand, by Young's Inequality

$$|L_{p_1} (Du + sDv, u + sv, x) v_1| \leqslant \frac{1}{q} |L_{p_1} (Du + sDv, u + sv, x)|^{\frac{q}{q-1}} + \frac{q-1}{q} |v_1|^q$$

Where $|L_{p_1} (Du + sDv, u + sv, x)|^{\frac{q}{q-1}} \leqslant C(|Du + sDv|^p + |u + sv|^p + 1)$ by the growth condition. Combining similar estimates for the remaining terms, we conclude

$$|L^t(x)| \leqslant C' \left(|Du|^q + |u|^q + |Dv|^q + |v|^q + 1 \right)$$

Uniformly over $t \in (0,1)$ as, for example, $|Du + sDv|^p \leqslant p^n(|Du|^p + s^p|Dv|^p) \leqslant p^n(|Du|^p + |Dv|^p)$. This bound is a constant that does not depend on s, and so can be moved out of the integral, which is over a probability measure.

Hence $|L^t(x)| \in L^1(U)$ and is dominated by $C'\left(|Du|^q + |u|^q + |Dv|^q + |v|^q + 1\right) \in L^1(U)$. Hence by the dominated convergence theorem,

$$\lim_{t\to 0^+} \frac{\tau(t) - \tau(0)}{t} = \lim_{t\to 0^+} \int_U L^t(x)dx = \int_U \lim_{t\to 0^+} L^t(x)dx = \int_U \sum_{i=1}^n L_{p_i}\left(Du, u, x\right) v_i + L_z\left(Du, u, x\right) v \ dx.$$

As this limit exists, it must equal 0 as u is a minimiser. This is true of all $v \in W_0^{1,p}(U)$. Thus u is a weak solution of the Euler-Lagrange equation. □

4. Regularity

Unfortunately, we will only be able to prove partial results on the regularity of weak solutions. We restrict our attention to Lagrangian functions that depend on p only, which have quadratic growth (i.e. $q = 2$). Particularly, we will assume that

$$I[w] = \int_U L(Dw) - fw \ dx$$

for some $f \in L^2(U)$. Mimimisers, $u \in W^{1,2}(U)$, must satisfy:

$$\int_U \sum_{i=1}^n L_{p_i}\left(Du\right) v_i dx = \int_U fv dx$$

for all $v \in W_0^{1,2}(U)$. One can show this implies u satisfies

$$-\sum_{i=1}^n \left(L_{p_i}\left(Du\right)\right)_i = f.$$

We will need some ingredients for the following proof. Define for $u \in W^{1,2}(U)$, the Newton quotient

$$D_k^h(u)(x) = \frac{u(x + he_k) - u(x)}{h}.$$

If $v \in W_c^{1,2}(U)$, then computation yields the following parts-like identity holds:

$$\int_U u D_k^h v \ dx = -\int_U v D_k^{-h} u \ dx.$$

We will require two additional assumptions. The first resembles elipticity:

$$\sum_{i,j=1}^n L_{p_i,p_j}(p)\xi_i\xi_j \geqslant \theta|\xi|^2$$

for some fixed $\theta > 0$ and all $\xi \in \mathbb{R}^n$. The second is that $|D^2L(p)|$ is globally bounded on $\mathbb{R}^n$.

Theorem 4.1. *If $u \in W^{1,2}(U)$ satisfies the differential equation:*

$$-\sum_{i=1}^n \left(L_{p_i}\left(Du\right)\right)_i = f$$

and the two conditions in the previous paragraph, then $u \in C^1(U)$.

Proof. Let open $V \subset\subset U$ be given. Find an open W with $V \subset\subset W \subset\subset U$, and let ζ be a smooth function such that $\text{supp}(\zeta) \subset W$ and $\zeta|_U = 1$. For sufficiently small $h \neq 0$, define $v = -D_k^{-h}(\zeta^2 D_k^h u)$.

Then $v \in W_0^{1,2}(U)$, so we substitute:

$$\int_U \sum_{i=1}^n L_{p_i}(Du)\left(-D_k^{-h}(\zeta^2 D_k^h u)\right)_i dx = \int_U \sum_{i=1}^n \left(D_k^h L_{p_i}(Du)\right)\left(\zeta^2 D_k^h u\right)_i dx$$
$$= -\int_U f\left(D_k^{-h}(\zeta^2 D_k^h u)\right) dx$$

Now we investigate:

$$\begin{aligned} D_k^h L_{p_i}(Du(x)) &= \frac{L_{p_i}(Du(x+he_k) - L_{p_i}(Du(x))}{h} \\ &= \frac{1}{h}\int_0^1 \frac{d}{ds} L_{p_i}(sDu(x+he_k) - (1-s)Du(x))\, ds \\ &= \frac{1}{h}\sum_{j=1}^n \left(\int_0^1 L_{p_i,p_j}(sDu(x+he_k) - (1-s)Du(x))\ \ ds\right)(u_j(x+he_k) - u_j(x)) \\ &= \sum_{j=1}^n a_{i,j}^h(x) D_k^h u_j(x) \end{aligned}$$

Where

$$a_{i,j}^h(x) = \int_0^1 L_{p_i,p_j}(sDu(x+he_k) - (1-s)Du(x)).$$

The left hand side becomes:

$$\sum_{i,j=1}^n \int_U a_{i,j}^h D_k^h u_j \left(\zeta^2 D_k^h u\right)_i dx = \sum_{i,j=1}^n \int_U a_{i,j}^h(x) D_k^h u_j \left(\zeta^2 D_k^h u_i + 2\zeta\zeta_i D_k^h u\right) dx.$$

The assumption that resembled elepticity yields, after transporting the sum through the integral:

$$\sum_{i,j=1}^n \int_U a_{i,j}^h \zeta^2 D_k^h u_j D_k^h u_i \geqslant \theta \int_U \zeta^2 |D_k^h(Du)|^2 dx.$$

Subsequent arguments establish:

$$\int_V |D_k^h Du|^2 dx \leqslant C \int_U f^2 + |Du|^2 dx$$

For all sufficiently small $h \neq 0$, which is argued to imply that Du is continuous, and hence that u is continuously differentiable. □

5. Appendix

Theorem 5.1. *Let X be a Banach space, and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X converging weakly to $x \in X$. Then $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$.*

Proof. Hahn-Banach tells us that for all $y \in X$, there is $\varphi \in X^*$ with $\|\varphi\| = 1$ that norms y, i.e. $\|\varphi(y)\| = \|y\|$; this is the statement that the inclusion $X \subset X^{**}$ is isometric. Now, suppose $x_n \to x$ weakly. For all $\varphi \in X^*$, $\varphi(x_n) \to \varphi(x)$, so $\sup_{n\in\mathbb{N}} \varphi(x_n) < \infty$. But if we think of x_n as a member of X^{**}, then $\sup_{n\in\mathbb{N}} x_n(\varphi) < \infty$ pointwise on X^*, so by the uniform boundedness principle, $\sup_{n\in\mathbb{N}} \|x_n\|_{X^{**}} = \sup_{n\in\mathbb{N}} \|x_n\| < \infty$. □

6. Source

This paper follows Chapter 8 of *Partial Differential Equations* by Lawrance C. Evans.