PM 453/653 **Assignment 2** **Due Monday October 19.**

1. (a) Show that $l_1^* = l_\infty$.

   (b) Describe all infinite matrices $T = \big[t_{ij}\big]_{i,j=1}^{\infty}$ which act as bounded operators from l_1 to itself.

2. Suppose that $T : \mathcal{H} \to \mathcal{H}$ is a linear map on a Hilbert space $\mathcal{H}$ such that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in \mathcal{H}$. Prove that T is bounded.

3. Let Y be a closed subspace of a Banach space X. Show that there are *isometric* isomorphisms $Y^* \simeq X^*/Y^\perp$ and $(X/Y)^* \simeq Y^\perp$.

4. Let $\varphi(f) = \int_0^1 f(t)\,dt$ for $f \in C_{\mathbb{R}}[0,1]$. Let Φ be any Hahn–Banach extension of φ to the Banach space $B_{\mathbb{R}}[0,1]$ of all bounded real-valued functions on $[0,1]$ with the sup norm.

   (a) Show that $\Phi(\chi_{[0,.5]}) = .5$.

   (b) Show that $\Phi(\chi_{\mathbb{Q}\cap[0,1]})$ may be any real number in $[0,1]$.

5. Let X be a *separable* Banach space.

   (a) Show that X is isometrically isomorphic to a subspace of l^∞. **Hint:** find a countable set of linear functionals $\{\varphi_n\}$ of norm one so that $\sup|\varphi_n(x)| = \|x\|$ for all $x \in X$.

   (b) Show that X is a quotient of l^1. **Hint:** define a norm one linear map of l^1 *onto* X so that the image of the unit ball of l^1 is dense in the ball of X. Prove that the quotient norm coincides with the norm on X.

6. (a) Prove that every finite dimensional subspace of a Banach space is closed.

   (b) Prove that no infinite dimensional Banach space has a countable basis *as a vector space* (i.e. a collection $\{e_n : n \ge 1\}$ so that every vector in X is a *finite* linear combination of $\{e_n : n \ge 1\}$).

Bonus Problem

7. A *basis* for a Banach space X is a sequence $\{e_n : n \ge 1\}$ such that for each $x \in X$, there are unique scalars $\{c_n\}$ such that $x = \lim_{n\to\infty} \sum_{i=1}^n c_i e_i$. For convenience, normalize so that $\|e_n\| = 1$ for $n \ge 1$.

   (a) Show that $\varphi_n(x) = c_n$ is a linear functional.

   (b) Define $S_n x = \sum_{i=1}^n c_i e_i$, and set $|||x||| = \sup_{n\ge1} \|S_n x\|$. Prove that $|||\cdot|||$ is a norm.

   (c) Show that $(X, |||\cdot|||)$ is complete.

   (d) Prove that the identity map T from $(X, |||\cdot|||)$ to $(X\|\cdot\|)$ is an isomorphism. Hence deduce that $\sup_{n\ge1}\|S_n\| = \|T^{-1}\|$ and that each φ_n is continuous.