

Lecture 9

In lecture 8 we started discussing RG flow equations:

$$\begin{cases} \frac{dr}{dl} = 2r + \frac{\gamma}{2} \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{r + \Lambda^2} \\ \frac{du}{dl} = (\gamma - d)u - \frac{3u^2}{2} \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{(r + \Lambda^2)^2} \end{cases}$$

In addition to the Gaussian fixed point $r^* = u^* = 0$, these ~~equations~~ have a nontrivial fixed point with $r^*, u^* \sim \varepsilon \equiv \gamma - d$.

My our calculations, which are perturbative in ε , are justified when ε is small.

Let us now obtain the contribution proportional to ε^2 .
This turns out to be somewhat nontrivial.

$$S'[\varphi_L] = S_0[\varphi_L] + \langle S_{\text{int}}[\varphi_L, \varphi_S] \rangle_{0>} - \frac{1}{2} \left[\langle S^2_{\text{int}}[\varphi_L, \varphi_S] \rangle_{0>} - \langle S_{\text{int}}[\varphi_L, \varphi_S] \rangle_{0>}^2 \right] + \dots$$

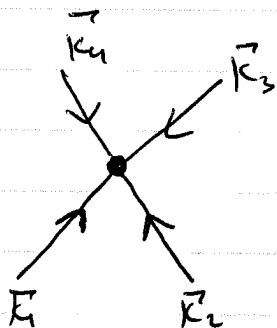
Brute-force evaluation of the second cumulant is very hard - huge number of terms.

To organize the calculation use graphical representation ~~---~~ - Feynman diagrams.

$$\text{Sout} [\varphi_L, \varphi_S] = \frac{u}{4!} \int_{\vec{k}_1, \dots, \vec{k}_4} [\varphi_L(\vec{k}_1) + \varphi_S(\vec{k}_1)] \cdot [\varphi_L(\vec{k}_2) + \varphi_S(\vec{k}_2)] \cdot [\varphi_L(\vec{k}_3) + \varphi_S(\vec{k}_3)] \cdot [\varphi_L(\vec{k}_4) + \varphi_S(\vec{k}_4)] \cdot (2\pi)^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)$$

Represent different terms that appear in this expression graphically.

With each term we will associate a vertex with 4 legs:



- represents $\frac{u}{4!} (2\pi)^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)$

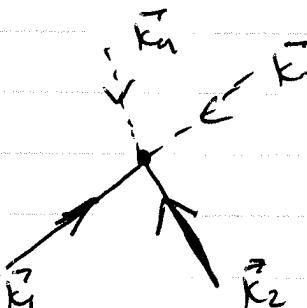
Each leg represents fast or slow field at a given momentum.

We will distinguish slow and fast fields by denoting slow fields by solid lines and fast field by dashed lines.

For example :

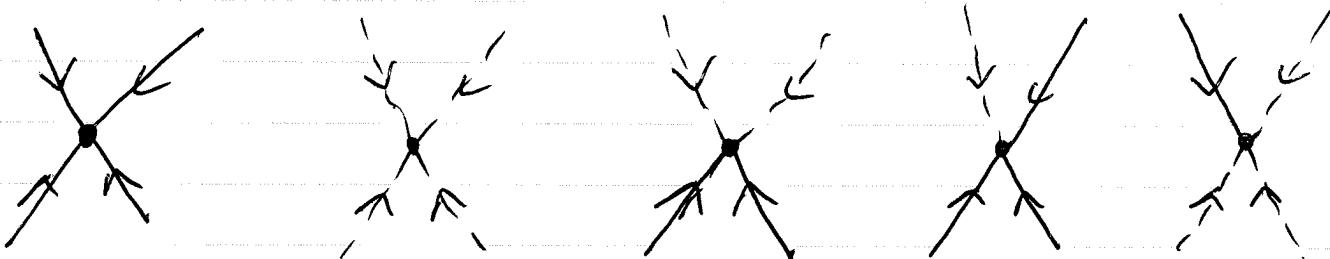
$$\frac{u}{4!} \int_{\vec{k}_1, \dots, \vec{k}_4} [\varphi_L(\vec{k}_1) \varphi_L(\vec{k}_2) \varphi_S(\vec{k}_3) \varphi_S(\vec{k}_4)]$$

$\cdot (2\pi)^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)$ will be represented as :



Arrows represent momenta "flowing in" each vertex, so that total momentum at each vertex sums up to zero.

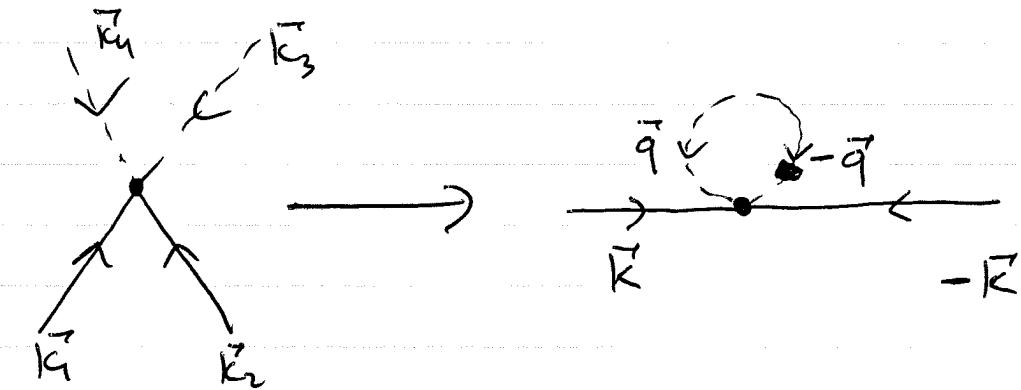
As we have seen before, Sout $[\varphi_c, \varphi_s]$ contains five distinct terms. They are represented graphically as:



We will depict the procedure of averaging ~~over~~ over fast field by connecting two dashed legs.

For example:

$$\frac{u}{4!} \int_{\vec{k}_1, \dots, \vec{k}_4} \varphi_c(\vec{k}_1) \varphi_c(\vec{k}_2) \langle \varphi_s(\vec{k}_3) \varphi_s(\vec{k}_4) \rangle_0 \cdot (m)^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)$$



Each dashed line corresponds to $\int \frac{d^d q}{(m)^d} g_0(\vec{q})$

Thus we get :

$$\frac{u}{q!} \int_K |\Psi_L(\vec{k})|^2 \int_{\Lambda/6}^{\Lambda} \frac{d^d q}{(2\pi)^d} \delta_0(\vec{q})$$

Second cumulant terms involve all possible terms with two vertices :

$$\frac{1}{2} \left[\langle S_{\text{out}}^2[\ell_L, \ell_S] \rangle_{0S} - \langle S_{\text{out}} [\ell_L, \ell_S] \rangle_{0S}^2 \right] = \\ = \frac{1}{2} \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right. + \left. \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right. + \dots \right\rangle_{0S} -$$

25 terms

$$- \frac{1}{2} \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right. + \left. \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right. + \dots \right\rangle_{0S} \cdot \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right. + \left. \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right. + \dots \right\rangle_{0S}$$

5 terms

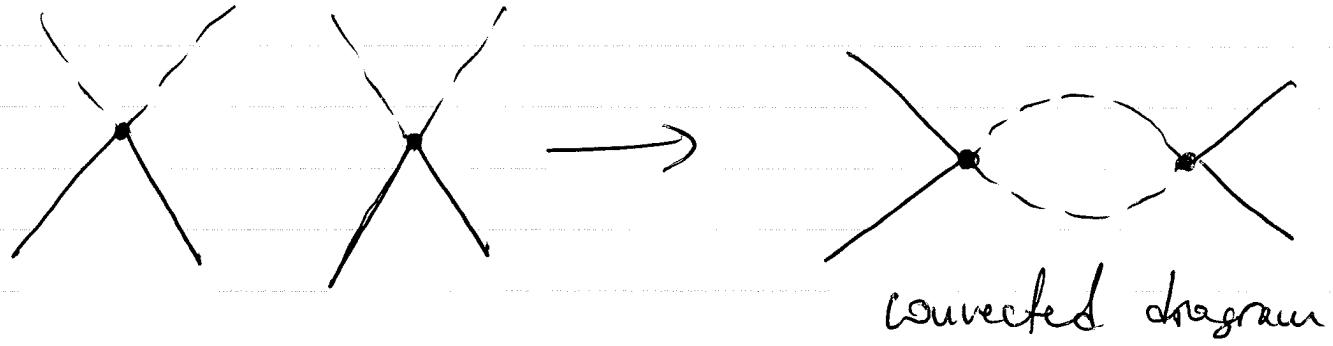
Averaging over fast fields means connecting dashed legs in all possible ways.

~~Important~~ An important thing to note is that terms of the type :



i.e. terms which consist of two disconnected pieces cancel between $\langle S^2_{\text{int}} \rangle_0$ and $\langle S_{\text{int}} \rangle_0$, since $\langle S_{\text{int}} \rangle_0^2$ consists of all possible disconnected terms.

Thus only connected diagrams contribute to the final result.



Cumulant expansion = summation of connected diagrams.

Thus to calculate the second cumulant contribution to $S'[\varphi_c]$ we need to draw all connected diagrams

with two vertices, with either two or four solid lines.

Diagrams with two solid lines correspond to terms with two slow fields — these contribute to the renormalization of σ and of the coefficient of the K^2 term.

Diagrams with 4 solid lines ~~contribute~~ correspond to renormalization of n .

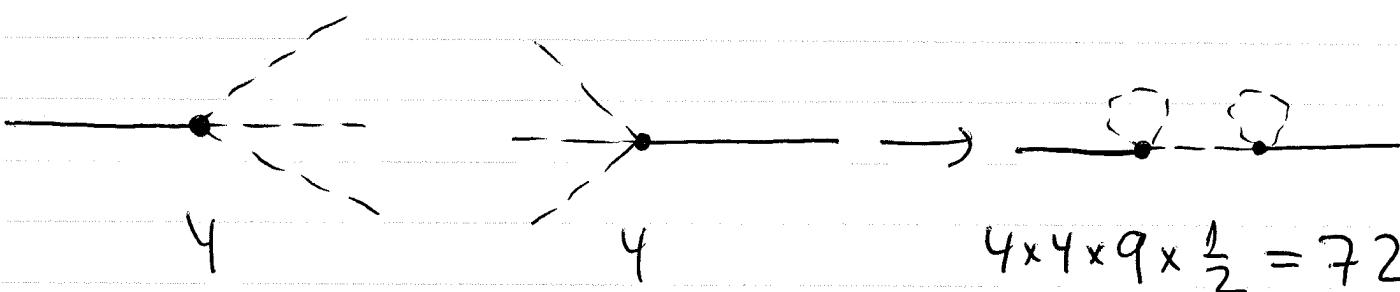


Possible fermis with 2 solid lines:

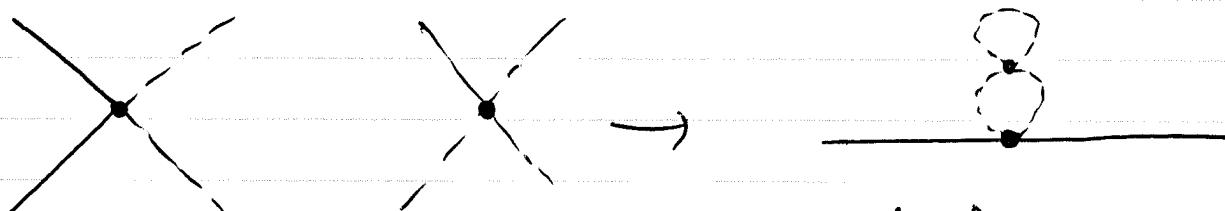


$$4 \times 4 \times 6 \times \frac{1}{2} = 48$$

6-number of ways dashed lines can be connected.
(Once two lines are connected, there are 2 possible ways
to connect the remaining 4 lines).



$$4 \times 4 \times 9 \times \frac{1}{2} = 72$$



$$6 \times \binom{4}{2} \times 2 \times 2 \times \frac{1}{2} =$$

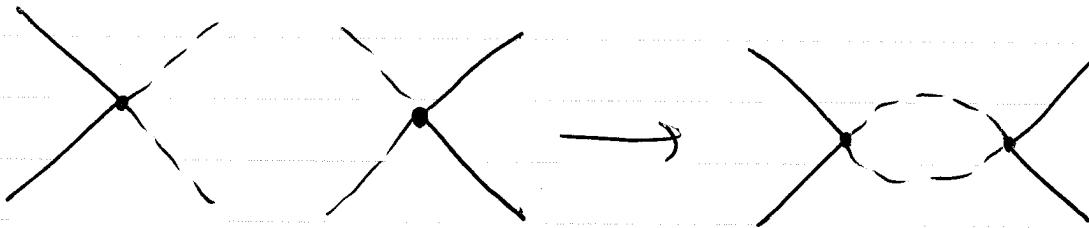
$$= 6 \times 6 \times 2 \times 2 \times \frac{1}{2} = 72$$

\uparrow
of ways to
choose 2 legs out
of 4

\uparrow
interchange left
and right vertex.

\uparrow
of ways
to connect two legs
on the right to two legs
on the left

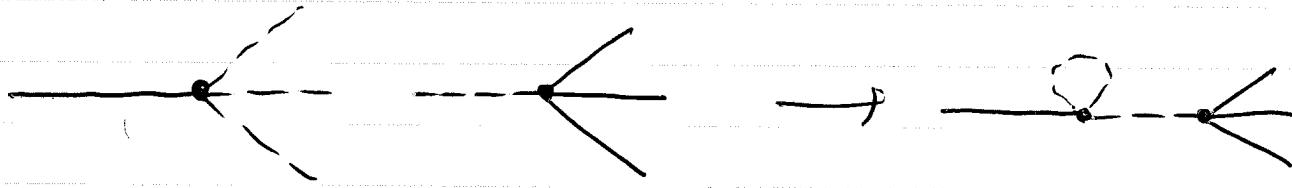
Possible terms with 4 solid lines:



6

6

$$6 \times 6 \times 2 \times \frac{1}{2} = \cancel{72} 36$$

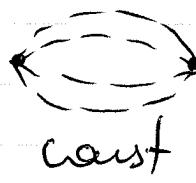
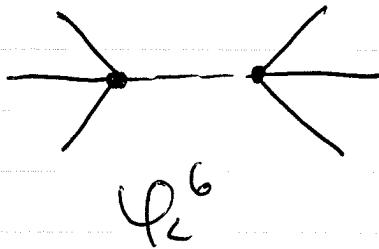


4

4

$$4 \times 4 \times 3 \times 2 \times \frac{1}{2} = 48$$

Which terms did we ignore?



Thus we obtain :

$$\frac{1}{2} \left[\langle S_{\text{int}}^2 \rangle_{0S} - \langle S_{\text{int}} \rangle_{0S}^2 \right] =$$

$$= 48$$



$$+ 72 - \cancel{?} +$$

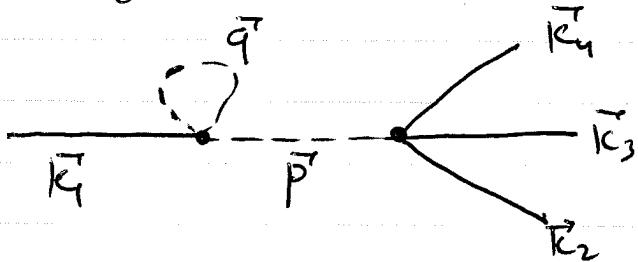
+

$$+72 \quad \text{Diagram} \quad +36 \quad \text{Diagram} \quad +$$

$$+48 \quad \text{Diagram}$$

First 3 terms contribute to the renormalization of γ ,
but only to $O(\epsilon^2)$ ~~are negligible~~ ignore.

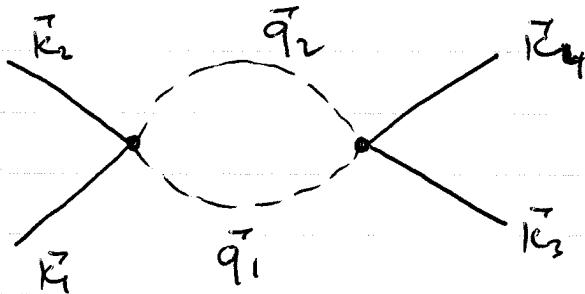
Only need to consider the last two terms.



$\vec{p} = -\vec{k}_1$ - impossible, since \vec{k}_1 is a "slow" momentum,
i.e. $0 < k_1 < 1/6$, while \vec{p} is a fast momentum,
 $1/6 < p < 1$.

Hence $\text{Diagram} = 0$.

After all we have only one diagram to evaluate.



$$\vec{k}_1 + \vec{k}_2 + \vec{q}_1 + \vec{q}_2 = 0$$

$$\vec{k}_3 + \vec{k}_4 - \vec{q}_1 - \vec{q}_2 = 0 \Rightarrow \vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4 = 0$$

$$\text{let } \vec{q}_1 = \vec{q}, \quad \vec{q}_2 = -\vec{q} - \vec{k}_1 - \vec{k}_2$$

Then we obtain:

$$\times \times = \left(\frac{n}{k!} \right)^2 \int_{k_1+k_2} (\omega)^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4).$$

$$= \Psi_L(\vec{k}_1) \Psi_L(\vec{k}_2) \Psi_L(\vec{k}_3) \Psi_L(\vec{k}_4) \cdot \int_{1/8}^{\infty} \frac{d^d q}{(\omega)^d} f_0(\vec{q}) \cdot$$

$$\cdot f_0(-\vec{k}_1 - \vec{k}_2 - \vec{q})$$

$$\text{let } I_2 = \int_{1/8}^{\infty} \frac{d^d q}{(\omega)^d} f_0(\vec{q}) f_0(-\vec{k}_1 - \vec{k}_2 - \vec{q})$$

\vec{k}_1 and \vec{k}_2 are slow momenta, \vec{q} is fast momentum – ignore \vec{k}_3 and \vec{k}_4 , will justify this ~~more~~ gradually later.

$$I_2 = \int_{1/8}^{\infty} \frac{d^d q}{(\omega)^d} f_0^2(\vec{q}) = \int_{1/8}^{\infty} \frac{d^d q}{(\omega)^d} \frac{1}{(r+q^2)^n} =$$

$$= \frac{S_d}{(\infty)^d} \int_{\lambda/\delta}^{\lambda} dq \frac{q^{d-1}}{(r+q^2)^2} =$$

$$= \frac{S_d}{(\infty)^d} \int_{\lambda/\delta}^{\lambda} dq \frac{q^{d-1}}{q^4} \left[1 - \frac{2r}{q^2} + \dots \right] =$$

$$= \frac{S_d}{(\infty)^d} \int_{\lambda/\delta}^{\lambda} dq \left[q^{d-5} - 2r q^{d-7} + \dots \right] =$$

~~$$= \frac{S_d}{(\infty)^d} \left[\frac{\lambda^{d-4}}{d-4} \left(1 - \left(\frac{1}{\delta} \right)^{d-4} \right) - \right.$$~~

$$\left. - 2r \frac{\lambda^{d-6}}{d-6} \left(1 - \left(\frac{1}{\delta} \right)^{d-6} \right) + \dots \right]$$

$$1 - \left(\frac{1}{\delta} \right)^n = 1 - e^{-n\delta l} \approx n\delta l$$

$$I_2 = \frac{S_d}{(\infty)^d} \left[\lambda^{d-4} - 2r \lambda^{d-6} + \dots \right] \delta l =$$

$$= \frac{S_d}{(\infty)^d} \frac{\lambda^d \delta l}{(r+\lambda^2)^2}$$

$$\cancel{X} = \left(\frac{u}{4!}\right)^2 I_2 \int_{k_1+k_2} (m)^d \delta(\vec{k}_1 + \dots + \vec{k}_4).$$

$$\cdot \varphi_c(\vec{k}_1) \varphi_c(\vec{k}_2) \varphi_c(\vec{k}_3) \varphi_c(\vec{k}_4)$$

Thus, collecting all terms up to second order in u , we obtain:

$$S'[\varphi_c] = \frac{1}{2} \int_K \left(k^2 + \frac{u}{2} I_1 \right) |\varphi_c(\vec{k})|^2 +$$

$$+ \left[\frac{u}{4!} - 36 \left(\frac{u}{4!} \right)^2 I_2 \right] \int_{k_1+k_2} (m)^d \delta(\vec{k}_1 + \dots + \vec{k}_4).$$

$$\cdot \varphi_c(\vec{k}_1) \varphi_c(\vec{k}_2) \varphi_c(\vec{k}_3) \varphi_c(\vec{k}_4)$$

Rescale momenta and slow fields:

$$\vec{k}' = \vec{k} b$$

$$\varphi'(\vec{k}') = b^{-\frac{d+2}{2}} \varphi_c(\vec{k})$$

We obtain:

$$\begin{cases} r' = \left(r + \frac{u}{2} I_1 \right) b^2 \\ u' = \left(u - \frac{3}{2} u^2 I_2 \right) b^{4-d} \end{cases}$$

Rewriting ~~the~~ The resource relations in differential form, we finally obtain:

$$\begin{cases} \frac{dr}{dl} = 2r + \frac{u}{2} \frac{S_d}{(\bar{m})^d} \frac{\lambda^d}{r+\lambda^2} \\ \frac{du}{dl} = (4-d)u - \frac{3u^2}{2} \frac{S_d}{(\bar{m})^d} \frac{\lambda^d}{(r+\lambda^2)^2} \end{cases}$$

We have derived RG flow equations for ϕ^4 LFW theory to order $\epsilon = 4-d$ (one-loop order).

Find the non-gaussian fixed point.

$$\frac{dr}{dl} = \frac{du}{dl} = 0$$

From the second equation:

$$u^* \approx \frac{2}{3}(4-d) \frac{(2\bar{m})^d}{S_d} \quad \lambda^{4-d} \approx \frac{2}{3}\epsilon \frac{(\bar{m})^4}{S_4}$$

$$S_4 = \frac{2\pi^{4/2}}{\Gamma(\frac{y}{2})} = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2$$

$$u^* = \frac{2}{3}\epsilon \cdot \frac{16\pi^4}{2\pi^2} = \frac{16\pi^2\epsilon}{3}$$

$$\rho^* = -\frac{u^*}{4} \frac{S_4}{(m)^4} \Lambda^2 = -\frac{\varepsilon}{6} \Lambda^2$$

$$\begin{cases} \rho^* = -\frac{\varepsilon}{6} \Lambda^2 \\ u^* = \frac{16\pi^2}{3} \varepsilon \end{cases} \quad - \text{Wilson-Fisher fixed point.}$$

Note that $\rho^* < 0$ at the critical point, unlike in the MF case.