

Lecture 8

In lecture 7 we started learning how to calculate averages ~~of~~ of products of complex fields with respect to gaussian statistical weight.

$$G(\gamma^*, \gamma) = e^{\sum_j \gamma_i^* M_{ij}^{-1} \gamma_j - \sum_j \gamma_i^* M_{ij} \gamma_j}$$

$$\langle \gamma_i \gamma_j^* \rangle = \frac{\int D\gamma \gamma_i \gamma_j^* e^{-\sum_j \gamma_i^* M_{ij} \gamma_j}}{\int D\gamma e^{-\sum_j \gamma_i^* M_{ij} \gamma_j}} =$$

$$= \left. \frac{\partial G(\gamma^*, \gamma)}{\partial \gamma_i^* \partial \gamma_j} \right|_{\gamma=0} = M_{ij}^{-1}$$

Generalize this to products of an arbitrary number of fields.

$$\langle \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_n} \gamma_{j_1}^* \dots \gamma_{j_n}^* \rangle =$$

$$= \frac{\int D\gamma \gamma_{i_1} \dots \gamma_{i_n} \gamma_{j_1}^* \dots \gamma_{j_n}^* e^{-\sum_j \gamma_i^* M_{ij} \gamma_j}}{\int D\gamma e^{-\sum_j \gamma_i^* M_{ij} \gamma_j}} =$$

$$= \left. \frac{\partial^m G(\gamma^*, \gamma)}{\partial \gamma_{i_1}^* \dots \partial \gamma_{i_n}^* \partial \gamma_{j_1} \dots \partial \gamma_{j_n}} \right|_{\gamma=0}$$

$$\frac{\partial^n}{\partial \gamma_{j_n} \dots \partial \gamma_{j_1}} e^{\sum_{ij} \gamma_i^* M_{ij}^{-1} \gamma_j} = \sum_{k_1, \dots, k_n} \gamma_{k_n}^* M_{k_n, j_n}^{-1} \dots \gamma_{k_1}^* M_{k_1, j_1}^{-1}$$

$$e^{\sum_{ij} \gamma_i^* M_{ij}^{-1} \gamma_j}$$

$$\frac{\partial^n}{\partial \gamma_{i_n}^* \dots \partial \gamma_{i_1}^* \partial \gamma_{j_n} \dots \partial \gamma_{j_1}} e^{\sum_{ij} \gamma_i^* M_{ij}^{-1} \gamma_j} \Big|_{\gamma=0} = \\ = \sum_P M_{i_{p_n}, j_n}^{-1} \dots M_{i_{p_1}, j_1}^{-1}$$

Here P is a permutation of integers from 1 to n .
Thus we have,

$$\langle \psi_{i_1} \dots \psi_{i_n} \psi_{j_n}^* \dots \psi_{j_1}^* \rangle = \sum_P M_{i_{p_n}, j_n}^{-1} \dots M_{i_{p_1}, j_1}^{-1}$$

This result is known as Wick's theorem.

Example :

$$\begin{aligned} \langle \psi_1 \psi_2 \psi_3^* \psi_4^* \rangle &= M_{13}^{-1} M_{24}^{-1} + M_{14}^{-1} M_{23}^{-1} = \\ &= \langle \psi_1 \psi_3^* \rangle \langle \psi_2 \psi_4^* \rangle + \langle \psi_1 \psi_4^* \rangle \langle \psi_2 \psi_3^* \rangle \end{aligned}$$

Apply these results to our case.

$$S'[\psi_k] = S_0[\psi_k] + \langle S_{\text{int}} \rangle_{0s} -$$

$$-\frac{1}{2} \left[\langle S_{\text{int}}^2 \rangle_{0s} - \langle S_{\text{int}} \rangle_{0s}^2 \right]$$

First calculate the average of a product of two fast fields:

$$\langle \psi_s(\vec{k}_1) \psi_s(\vec{k}_2) \rangle_0 = \langle \psi_s(\vec{k}_1) \psi_s^*(-\vec{k}_2) \rangle_0 =$$

$$= \frac{1}{Z_0} \int d\vec{k}_1 \psi_s(\vec{k}_1) \psi_s^*(-\vec{k}_2) \quad \text{(Integration over } \vec{k}_1 \text{)}.$$

$$e^{-\frac{1}{2V} \sum_{\vec{k}} (k^2 + \gamma) |\psi_s(\vec{k})|^2}$$

Here I used $\int \frac{d^d k}{(2\pi)^d} \rightarrow \frac{1}{V} \sum_{\vec{k}}$

~~Integration over \vec{k}~~

We obtain:

$$\langle \psi_s(\vec{k}_1) \psi_s(\vec{k}_2) \rangle_0 = \frac{V}{k_1^2 + \gamma} \delta_{\vec{k}_1, -\vec{k}_2} \equiv$$

$$\equiv V G_0(\vec{k}) \delta_{\vec{k}_1, -\vec{k}_2}$$

$$G_0(\vec{k}) = \frac{1}{k^2 + \gamma} \quad \begin{array}{l} \text{-fast mode propagator} \\ \text{(Green's function)} \end{array}$$

The factor of $\frac{1}{2}$ in the exponential goes away because

$$\psi_s(-\vec{k}) = \psi_s^*(\vec{k}) \Rightarrow |\psi_s(\vec{k})|^2 = |\psi_s(-\vec{k})|^2$$

Going back to continuum notation for momenta:

$$\langle \psi_s(\vec{k}_1) \psi_s(\vec{k}_2) \rangle_0 = g_0(\vec{k}_1) (2\pi)^d \delta(\vec{k}_1 + \vec{k}_2)$$

Now let's calculate $\langle S_{\text{int}}[\psi_c, \psi_s] \rangle_0$.

Henceforth I'll use shorthand notation for multiple momentum integrals:

$$\int_{\vec{k}_1 \vec{k}_2 \dots \vec{k}_n} = \int_0^\infty \frac{d^d k_1}{(2\pi)^d} \dots \frac{d^d k_n}{(2\pi)^d}$$

Write out $S_{\text{int}}[\psi_c, \psi_s]$ explicitly:

$$S_{\text{int}}[\psi_c, \psi_s] = \frac{u}{4!} \int_{\vec{k}_1 \vec{k}_2 \vec{k}_3 \vec{k}_4} (2\pi)^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4).$$

$$[\psi_c(\vec{k}_1) + \psi_s(\vec{k}_1)] \cdot [\psi_c(\vec{k}_2) + \psi_s(\vec{k}_2)] \cdot$$

$$[\psi_c(\vec{k}_3) + \psi_s(\vec{k}_3)] \cdot [\psi_c(\vec{k}_4) + \psi_s(\vec{k}_4)] =$$

$$= \frac{u}{4!} \int_{k_1 \dots k_4} (2\pi)^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4).$$

$$[\psi_c(\vec{k}_1) \psi_c(\vec{k}_2) \psi_c(\vec{k}_3) \psi_c(\vec{k}_4)]^A +$$

$$+ \psi_s(\vec{k}_1) \psi_s(\vec{k}_2) \psi_s(\vec{k}_3) \psi_s(\vec{k}_4)]^B +$$

$$\begin{aligned}
 & + 4 \varphi_L(\vec{k}_1) \varphi_L(\vec{k}_2) \varphi_L(\vec{k}_3) \varphi_S(\vec{k}_4)^C + \\
 & + 6 \varphi_L(\vec{k}_1) \varphi_L(\vec{k}_2) \varphi_S(\vec{k}_3) \varphi_S(\vec{k}_4)^D + \\
 & + 4 \varphi_L(\vec{k}_1) \varphi_S(\vec{k}_2) \varphi_S(\vec{k}_3) \varphi_S(\vec{k}_4)^E]
 \end{aligned}$$

Here the factors 4 and 6 are the binomial coefficients from combinations:

$$4 = \binom{4}{1} = \frac{4!}{1! 3!} \quad \begin{array}{l} \text{— number of ways} \\ \text{to choose one object} \\ \text{out of 4.} \end{array}$$

$$6 = \binom{4}{2} = \frac{4!}{2! 2!} \quad \begin{array}{l} \text{— number of ways} \\ \text{to choose 2 out of 4.} \end{array}$$

It is clear that terms C and E, which have an odd number of φ_S fields, will vanish upon averaging (since $S_0[\varphi_S]$ is ~~invariant~~ invariant under $\varphi_S \rightarrow -\varphi_S$).

Thus we only need to consider A, B and D term.

A consists entirely of slow fields — nothing to average \Rightarrow This term remains as it is.

B depends only on fast fields — we are not interested in this term — It will only generate ~~a~~ an additive constant to $S'[\varphi_L]$.

Thus of the five terms only D is non-trivial.

Then we obtain:

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$$D = \frac{u}{u!} \cdot 6 \int_{k_1=k_4} (\omega)^{\perp} \delta(k_1 + k_2 + k_3 + k_4).$$

$$\langle \varphi_L(\vec{k}_1) \varphi_L(\vec{k}_2) \varphi_S(\vec{k}_3) \varphi_S(\vec{k}_4) \rangle_{03} =$$

$$= \frac{4}{\pi} \int_{k_1 - ik_0} \left(\omega \right)^d \delta \left(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4 \right) .$$

$$\Psi_L(E) \Psi_L(k_2) \langle \Psi_S(k_3) \rho_S(k_4) \rangle_{o_s} =$$

$$= \frac{u}{4} \int_{k_1 = -k_2} \left(2\pi\right)^d \delta\left(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4\right) .$$

$$\cdot \varphi_{\zeta}(\vec{k}_1) \varphi_{\zeta}(\vec{k}_2) \, g_0(\vec{k}_3) \, (2\pi)^d \delta(\vec{k}_3 + \vec{k}_4)$$

We have $\vec{k}_3 = -\vec{k}_n$ from the last δ -function \Rightarrow

$\Rightarrow \vec{K}_1 = -\vec{K}_2$ from the first one.

$$\text{let } \vec{k}_1 = -\vec{k}_2 = \vec{k}, \quad , \quad \vec{k}_3 = -\vec{k}_4 = \vec{q}$$

Then we obtain:

~~(K) $\int_{\mathbb{R}^d} \frac{1}{|q|^d} \delta_0(q) dq = 0$~~

$$D = \frac{u}{q} \int_K |\varphi_k(\vec{k})|^2 \int_{\substack{\wedge \\ \lambda/6}} \frac{d^d q}{(2\pi)^d} \delta_0(\vec{q})$$

$$\int_{\substack{\wedge \\ \lambda/6}} \frac{d^d q}{(2\pi)^d} \delta_0(\vec{q}) = \int_{\substack{\wedge \\ \lambda/6}} \frac{d^d q}{(2\pi)^d} \frac{1}{r+q^2} =$$

$$= \frac{S_d}{(2\pi)^d} \int_{\substack{\wedge \\ \lambda/6}} d^d q \frac{q^{d-1}}{r+q^2} \equiv I_1$$

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} - \text{surface area of a sphere}$$

of unit radius in d -dimensions.

The integral is easy to evaluate by expanding in Taylor series with respect to r :

$$\frac{1}{r+q^2} = \frac{1}{q^2} \left(1 - \frac{r}{q^2} + \dots \right)$$

$$I_1 = \frac{S_d}{(2\pi)^d} \int_{\substack{\wedge \\ \lambda/6}} d^d q q^{d-3} \left(1 - \frac{r}{q^2} + \dots \right) =$$

$$= \frac{S_d}{(\infty)^d} \left[\lambda^{d-2} - \frac{1 - \left(\frac{1}{6}\right)^{d-2}}{d-2} - r\lambda^{d-4} \frac{1 - \left(\frac{1}{6}\right)^{d-4}}{d-4} + \dots \right]$$

b is slightly greater than 1 \Rightarrow convenient to write as:

$$b = e^{\Delta l}, \Delta l \rightarrow 0.$$

~~(*)~~ Then we have:

$$1 - \left(\frac{1}{6}\right)^{d-2} = 1 - e^{-\Delta l \cdot (d-2)} \approx (d-2)\Delta l$$

Thus we obtain:

$$\begin{aligned} I_1 &= \frac{S_d}{(\infty)^d} \left[\lambda^{d-2} - r\lambda^{d-4} + \dots \right] \Delta l = \\ &= \frac{S_d}{(\infty)^d} \frac{\lambda^d \Delta l}{r + \lambda^2} \end{aligned}$$

Thus we have:

$$\cancel{\text{something}} = \frac{u}{4} I_1 \int_k |\Psi_L(k)|^2$$

Let us analyze what we have so far we will calculate the contribution of the second cumulant later.

$$\begin{aligned}
 S'[\varphi_c] &= S_0[\varphi_c] + \langle S_{\text{int}}[\varphi_c, \varphi_c] \rangle_{0>} + \dots \\
 &= \frac{1}{2} \int_K (k^2 + r) |\varphi_c(\vec{k})|^2 + \\
 &\quad + \frac{u}{4!} I_1 \int_K |\varphi_c(\vec{k})|^2 + \\
 &\quad + \frac{u}{4!} \int_{K_1 \dots K_n} (2\pi)^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \varphi_c(\vec{q}) \dots \varphi_c(\vec{k}_4) \\
 &= \frac{1}{2} \int_K \left(k^2 + r + \frac{u}{2} I_1 \right) |\varphi_c(\vec{k})|^2 + \\
 &\quad + \frac{u}{4!} \int_{K_1 \dots K_n} (2\pi)^d \delta(\vec{k}_1 + \dots + \vec{k}_n) \varphi_c(\vec{k}_1) \dots \varphi_c(\vec{k}_n)
 \end{aligned}$$

$\frac{u}{2} I_1$ term renormalizes r .

Now let's carry out the last step of the RG transformation - rescale momentum variables and the ~~the~~ slow fields to bring $S'[\varphi_c]$ to the original form.

$$\vec{k}' = \vec{k} R$$

$$\varphi'(\vec{k}') = R^{-\frac{d+2-4}{2}} \varphi_c(\vec{k})$$

We obtain :

$$S'[\varphi'] = \frac{1}{2} \int_{K^1} b^{-d} \left(k^{12} b^{-2} + r + \frac{u}{2} I_1 \right).$$

$$\cdot b^{d+2-\gamma} |\varphi'(\vec{k}^1)|^2 +$$

$$+ \frac{u}{4!} \int_{k_1' \dots k_4'} b^{-4d} (2\pi)^d \delta(\vec{k}_1' + \dots + \vec{k}_4') b^d.$$

$$\cdot b^{2(d+2-\gamma)} \varphi'(\vec{k}_1') \dots \varphi'(\vec{k}_4')$$

Clearly $\gamma = 0$ at this level.

$$\begin{cases} r' = \left(r + \frac{u}{2} I_1 \right) b^2 \\ u' = u b^{4-d} \end{cases} \quad - \text{RG recursion relations.}$$

$$S'[\varphi'] = \frac{1}{2} \int_{K^1} (k^{12} + r') |\varphi'(\vec{k}')|^2 +$$

$$+ \frac{u'}{4!} \int_{k_1' \dots k_4'} \varphi'(\vec{k}_1') \dots \varphi'(\vec{k}_4') (2\pi)^d \delta(\vec{k}_1' + \dots + \vec{k}_4')$$

Analyze RG recursion relations.

$$\begin{aligned} r' &= \left(r + \frac{u}{2} I_1 \right) b^2 = \left(r + \frac{u}{2} I_1 \right) e^{2\alpha l} \approx \\ &\approx \left(r + \frac{u}{2} I_1 \right) (1 + 2\alpha l) = \end{aligned}$$

$$= \left(r + \frac{u}{2} \frac{s_d}{(\infty)^d} \frac{\lambda^d \Delta l}{r + \lambda^2} \right) (1 + 2\Delta l) \approx$$

$$\approx r + 2r\Delta l + \frac{u}{2} \frac{s_d}{(\infty)^d} \frac{\lambda^d \Delta l}{r + \lambda^2}$$

$$\frac{r' - r}{\Delta l} = 2r + \frac{u}{2} \frac{s_d}{(\infty)^d} \frac{\lambda^d}{r + \lambda^2}$$

Can consider r to be a function of ℓ , which parametrizes ~~the~~ steps of the RG procedure!

$$\text{Then } \frac{r' - r}{\Delta l} = \frac{dr}{d\ell} \quad \text{as } \Delta l \rightarrow 0.$$

Thus we obtain:

$$\frac{dr}{d\ell} = 2r + \frac{u}{2} \frac{s_d}{(\infty)^d} \frac{\lambda^d}{r + \lambda^2}$$

$$u' = u b^{4-d} = u \ell^{(4-d)\Delta l} \approx u (1 + (4-d)\Delta l)$$

Then we get:

$$\frac{du}{d\ell} = (4-d)u$$

We have derived RG flow equations to first order in u :

$$\begin{cases} \frac{dr}{dl} = 2r + \frac{\gamma}{2} \frac{S_d}{(2\pi)^d} \frac{\lambda^d}{r+\lambda^2} \\ \frac{du}{dl} = (4-d)u \end{cases}$$

These equations obviously have only one fixed point:
 $r^* = u^* = 0$. This is called gaussian fixed point.

To get a nontrivial fixed point, we need to calculate the contribution of the second cumulant.
 We will find:

$$\begin{cases} \frac{dr}{dl} = 2r + \frac{\gamma}{2} \frac{S_d}{(2\pi)^d} \frac{\lambda^d}{r+\lambda^2} \\ \frac{du}{dl} = (4-d)u - \frac{3u^2}{2} \frac{S_d}{(2\pi)^d} \frac{\lambda^d}{(r+\lambda^2)^2} \end{cases}$$

This ~~does not~~ has, in addition to gaussian fixed point,
 also a nontrivial fixed point:

~~$r^*, u^* \sim 4-d \equiv \varepsilon$~~

Cumulant expansion, which is an expansion in powers of ε ,
 is justified when ε is small \Rightarrow what we are
 doing is perturbation theory in deviation from ~~ε~~
 the upper critical dimension $d=4$.