

Lecture 6

Lectures: calculated fluctuation correction to MFT result on the behavior of the specific heat near a continuous transition.

$d > 4$: $\delta C_V = \text{const}$ - MFT result $\alpha = 0 \Rightarrow$ unphysical.

$$d < 4: \delta C_V \propto \left(\frac{a}{R}\right)^4 |t|^{d-\frac{4}{2}}, \alpha = 2 - \frac{d}{2}$$

Any for $d < 4$ MFT always breaks down near the transition.

Ginzburg criterion: ~~$\delta C_V \ll 1$~~ - MFT is OK.

$$\Rightarrow \left(\frac{a}{R}\right)^4 |t|^{-d} \ll 1$$

$$|t|^{\frac{d}{2}-2} \ll \left(\frac{R}{a}\right)^4 \text{ - always breaks down close enough to } T_c \text{ in } d < 4.$$

$$\delta C_V = 2\delta^2 \left(\frac{a}{R}\right)^4 \int_{\pi} \frac{d^d Q}{(2\pi)^d} \frac{1}{[Q^2 + (\frac{a}{\zeta})^2]^2}$$

The breakdown of MFT is due to small-momentum = long-wavelength fluctuations.

This suggests that we should focus on ~~δC_V~~ long-distance properties of the functional $S[\varphi]$:

$$S[\varphi] = \frac{1}{2T} \sum_i \varphi_i T_{ij}^{-1} \varphi_j - \sum_i \ln \left[2 \cosh \left(\frac{\varphi_i}{T} \right) \right]$$

The exact microscopic form for the functional $S[\varphi]$ contains a lot of unnecessary details, which only obscure the essentials.

First, since the correlation length ξ diverges near the transition, we do not need the information about $\langle \varphi_j \rangle$ on the scale of the lattice constant, ~~and~~ we only need the information contained in the small-magnitude expansion of $\mathcal{I}(q)$ (equivalent to assuming that φ_i varies slowly on the scale of a):

$$\mathcal{I}(q) \approx Y - \frac{1}{2} K q^2$$

Second, ~~near the transition~~ we can approximate $\ln[2 \cosh(\frac{\varphi_i}{T})]$ by a couple of leading terms in its expansion with respect to $(\frac{\varphi_i}{T})$, since near the transition $\langle \varphi_i \rangle$ is small.

Let's obtain a simpler form for $S[\varphi]$ in this way.

$$\text{Use } \ln[2 \cosh(x)] \approx \ln 2 + \frac{x^2}{2} - \frac{x^4}{12}$$

We obtain:

$$S[\varphi] \approx \frac{1}{2T} \sum_{ij} \varphi_i \mathcal{Y}_{ij} \varphi_j - \frac{1}{2} \sum_i \left(\frac{\varphi_i}{T} \right)^2 + \frac{1}{12} \sum_i \left(\frac{\varphi_i}{T} \right)^4$$

I have neglected the constant term $\approx \ln 2$.

: Consider the quadratic term:

$$S_0[\varphi] = \frac{1}{2T^2} \sum_{ij} \varphi_i (T J_{ij}^{-1} - \delta_{ij}) \varphi_j.$$

Diagonalize by Fourier transform:

$$\varphi_i = \frac{1}{N} \sum_{\vec{q}} \varphi(\vec{q}) e^{i\vec{q} \cdot \vec{r}_i}$$

$$J_{ij}^{-1} = \frac{1}{N} \sum_{\vec{q}} J(\vec{q}) e^{i\vec{q} \cdot (\vec{r}_i - \vec{r}_j)}$$

We obtain:

$$S_0[\varphi] = \frac{1}{2T_N} \sum_{\vec{q}} \varphi(-\vec{q}) \left[\frac{T}{J(\vec{q})} - 1 \right] \varphi(\vec{q})$$

As mentioned above we are interested only in the long-distance properties of $J_{ij}^{-1} \Rightarrow$ ~~we take~~ we use the small-momentum expansion:

$$J(\vec{q}) \approx J - \frac{1}{2} K q^2 = J \left[1 - \frac{1}{2} \frac{K}{J} q^2 \right]$$

$$\frac{dK}{J} = R^2$$

$$J(\vec{q}) \approx J \left[1 - \frac{1}{2d} R^2 q^2 \right]$$

$$\frac{T}{J(\vec{q})} - 1 = \frac{T}{J} \left[1 - \frac{1}{2d} q^2 R^2 \right]^{-1} - 1 =$$

$$\approx \frac{T}{T_c} - 1 + \frac{T}{T_c} \frac{1}{2d} q^2 R^2 =$$

$$= \frac{T}{T_c} - 1 + \frac{T}{T_c} \frac{1}{2d} q^2 R^2 \approx \frac{T}{T_c} - 1 + \frac{1}{2d} q^2 R^2.$$

~~RECALL~~

$$= t + \frac{1}{2d} q^2 R^2$$

$$\frac{T}{T(\vec{q})} - 1 \approx t + \frac{1}{2d} q^2 R^2$$

$$S_0[\varphi] = \frac{1}{2T_c^2 N} \sum_{\vec{q}} \varphi(-\vec{q}) \left(t + \frac{1}{2d} q^2 R^2 \right) \varphi(\vec{q})$$

Redefine variables:

$$\tilde{\varphi}(\vec{q}) = \frac{R a^{d/2}}{\cancel{\sqrt{2d} T_c}} \varphi(\vec{q})$$

$$S_0[\tilde{\varphi}] = \frac{1}{2Na^d} \sum_{\vec{q}} \tilde{\varphi}(-\vec{q}) \left(\frac{2dt}{R^2} + q^2 \right) \tilde{\varphi}(\vec{q})$$

$$\text{Define } r = \frac{2dt}{R^2}$$

$r > 0$ when $T > T_c$, $r < 0$ when $T < T_c$.

~~RECALL~~ Dropping tildes, we obtain:

$$S_0[\varphi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \varphi(-\vec{q}) (r + q^2) \varphi(\vec{q})$$

~~Fourier transform back to real space:~~

$$\varphi(\vec{x}) = \int \frac{d^d q}{(2\pi)^d} \varphi(\vec{q}) e^{i\vec{q} \cdot \vec{x}}$$

$$S_0[\varphi] = \frac{1}{2} \int d\vec{x} [(\nabla \varphi)^2 + r \varphi^2]$$

Similarly we can derive the continuum limit of the quartic term.

$$S_1[\varphi] = \frac{1}{12T_c^4} \sum_i \varphi_i^4$$

$$\sum_i \rightarrow \frac{1}{a^d} \int d\vec{x}$$

$$\varphi_i = \tilde{\varphi}_i \frac{\sqrt{2d} T_c}{R a^{d/2}}$$

$$S_1[\tilde{\varphi}] = \frac{1}{12T_c^4} \frac{(2d)^2 T_c^4}{R^4 a^{2d}} \sum_i \tilde{\varphi}_i^4 \rightarrow$$

$$\rightarrow \frac{d^2}{3R^4 a^d} \int d\vec{x} \tilde{\varphi}^4(\vec{x})$$

Drop tildes and define $\frac{u}{q} = \frac{d^2}{3R^4 Q^d}$

All that is important is that $u > 0$.

Then we finally obtain:

$$S[\varphi] = \int dx \left[\frac{1}{2} (\vec{\nabla} \varphi)^2 + \frac{1}{2} \varphi^2 + \frac{u}{4} \varphi^4 \right]$$

This is continuum Landau-Ginzburg-Wilson theory.
Often also called φ^4 field theory.

$$MFT: \varphi(x) = \bar{\varphi}.$$

$$\text{Minimise } S[\bar{\varphi}] : r\bar{\varphi} + u\bar{\varphi}^3 = 0.$$

$$\bar{\varphi} = 0 \text{ for } r > 0, \quad \bar{\varphi} = \pm \sqrt{\frac{|r|}{u}} \text{ for } r < 0.$$

All results we have obtained so far could just as well have been obtained from the above functional:

~~$r \sim t \Rightarrow \bar{\varphi} \sim |t|^{\frac{1}{4}}$~~ and so on.

Our result from lecture 5 for the critical exponent ζ at $T > T_c$ corresponds to theory (GW functional with $u=0$, i.e. assuming gaussian (noninteracting) fluctuations). As mentioned in lecture 5, this also leads to incorrect results. This can be seen by simple dimensional analysis.

We will define the dimension of any physical quantity to be its dimensions in units of inverse length, or momentum.

$$\dim [K] = 1$$

$$\dim [x] = -1$$

Since $Z = \int d\varphi e^{-S[\varphi]}$, $S[\varphi]$ is obviously dimensionless.

~~$\int d^d x (\vec{\nabla} \varphi)^2$~~ is dimensionless \Rightarrow
 $\dim [x^d] + \dim [x^{-2}] + \dim [\varphi^2] = 0$.
 $-d + 2 + 2 \dim [\varphi] = 0$

$$\dim [\varphi] = \frac{d-2}{2}$$

$$\dim [r] + d \dim [x] + 2 \dim [\varphi] = 0.$$

$$\dim [r] = d - 2 \frac{d-2}{2} = 2$$

$$\dim [u] + 4 \dim [\varphi] + d \dim [x] = 0.$$

$$\dim [u] = d - 4 \frac{d-2}{2} = 4 - d$$

Using these results, rewrite $S[\varphi]$ in dimensionless variables.

Define dimensionless coordinates and dimensionless field $\tilde{\varphi}$ as:

$$x = \tilde{x} r^{-\frac{d}{2}} \quad \text{-assume } r > 0 \text{ or } T > T_c.$$

$$\varphi = \tilde{\varphi} r^{\frac{d-2}{4}}$$

\tilde{x} and $\tilde{\varphi}$ are dimensionless.

Then we obtain:

$$\begin{aligned} S[\tilde{\varphi}] &= \int d^d \tilde{x} r^{-\frac{d}{2}} \left[\frac{1}{2} r r^{\frac{d-2}{2}} (\tilde{\nabla} \tilde{\varphi})^2 + \right. \\ &\quad \left. + \frac{1}{2} r^{\frac{d-2}{2}} \tilde{\varphi}^2 + \frac{u}{4} r^{d-2} \tilde{\varphi}^4 \right] = \\ &= \int d^d \tilde{x} \left[\frac{1}{2} (\tilde{\nabla} \tilde{\varphi})^2 + \frac{1}{2} \tilde{\varphi}^2 + \frac{u}{4} r^{\frac{d-4}{2}} \tilde{\varphi}^4 \right] \end{aligned}$$

$$r \sim t$$

Thus the coefficient of the $\tilde{\varphi}^4$ term is given by:

$$g = \frac{u}{4} r^{\frac{d-4}{2}} \sim u t^{\frac{d-4}{2}}$$

Now for $d > 4$, $g \rightarrow 0$ as $T \rightarrow T_c$, gaussian ~~approx~~
Fluctuation approximation is OK, but ~~is~~ considering
Fluctuations is unnecessary since MFT works.

: For $d < 4$ ϕ diverges as $T \rightarrow T_c \Rightarrow$ ~~can't neglect~~
 can't neglect the ϕ^4 term \Rightarrow gaussian ~~theory~~
 Fluctuations give wrong results.

Thus for $d < 4$ neither MF nor perturbation theory
 in ϕ^4 term will work.

We are faced with the problem of evaluating
 $Z = \int D\phi e^{-S[\phi]}$

$$\text{where } S[\phi] = \int d^d x \left[\frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \phi^2 + \frac{1}{4!} \phi^4 \right]$$

and we can't make any more approximations in
 $S[\phi]$.

The ~~standard~~ solution of this problem is the renormalization group (RG) procedure.

Basic idea - coarse graining. We have arrived at the continuum LFW form for $S[\phi]$ by assuming that $\phi(\vec{x})$ varies slowly on the scale of the lattice constant a . However, as we approach the transition, the only relevant length scale, the correlation length ξ - diverges. This means that in fact $\phi(\vec{x})$ varies slowly on much larger length scale than a and we can continue simplifying LFW functional by eliminating short-length-scale degrees of freedom. RG is a systematic procedure for performing this elimination.

$$S[\varphi] = \int d^d x \left[\frac{1}{2} (\vec{\nabla} \varphi)^2 + \frac{1}{2} \varphi^2 + \frac{u}{4!} \varphi^4 \right]$$

$$\varphi(\vec{x}) = \int_0^\Lambda \frac{d^d k}{(2\pi)^d} \varphi(\vec{k}) e^{i \vec{k} \cdot \vec{x}}$$

$\Lambda \sim \frac{\pi}{a}$ is the high-momentum cutoff.

Without this cutoff the continuum theory is not well-defined, momentum integrals will have ~~the~~ ultraviolet divergences.

\int_0^Λ means that the magnitude of \vec{k} is less than Λ .

Now we write $\varphi(\vec{x})$ as a sum of two terms:

$$\varphi(\vec{x}) = \varphi_L(\vec{x}) + \varphi_R(\vec{x})$$

$$\varphi_L(\vec{x}) = \int_0^{\Lambda/6} \frac{d^d k}{(2\pi)^d} \varphi(\vec{k}) e^{i \vec{k} \cdot \vec{x}}$$

$$\varphi_R(\vec{x}) = \int_{\Lambda/6}^\Lambda \frac{d^d k}{(2\pi)^d} \varphi(\vec{k}) e^{i \vec{k} \cdot \vec{x}}$$

Here B is a number, slightly larger than 1.

φ_L are called the "slow modes", φ_R the "fast modes".

Next we rewrite ~~the~~ (GW functional) $S[\varphi]$ in terms of the slow and fast modes.

: Generally $S[\varphi_L, \varphi_S]$ will consist of 3 terms:

$$S[\varphi_L, \varphi_S] = S_0[\varphi_L] + S_0[\varphi_S] + S_{\text{int}}[\varphi_L, \varphi_S]$$

Consider first the quadratic part of the LFW functional.

$$S_0[\varphi] = \int d^d x \left[\frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} \varphi^2 \right]$$

In ~~3D~~ momentum space this becomes:

$$S_0[\varphi] = \frac{1}{2} \int_0^\infty \frac{d^d k}{(2\pi)^d} \varphi(-\vec{k}) \varphi(\vec{k}) (k^2 + r)$$

$$\varphi(\vec{k}) \text{ is real} \Rightarrow \varphi(-\vec{k}) = \varphi^*(\vec{k})$$

$$\begin{aligned} S_0[\varphi] &= \frac{1}{2} \int_0^\infty \frac{d^d k}{(2\pi)^d} (k^2 + r) |\varphi(\vec{k})|^2 = \\ &= \frac{1}{2} \int_0^{1/6} \frac{d^d k}{(2\pi)^d} (k^2 + r) |\varphi_L(\vec{k})|^2 + \\ &\quad + \frac{1}{2} \int_{1/6}^\infty \frac{d^d k}{(2\pi)^d} (k^2 + r) |\varphi_S(\vec{k})|^2 \end{aligned}$$

$$S_0[\varphi_L] = \frac{1}{2} \int_0^{1/6} \frac{d^d k}{(2\pi)^d} (k^2 + r) |\varphi_L(\vec{k})|^2$$

$$S_0[\varphi_S] = \frac{1}{2} \int_{1/6}^\infty \frac{d^d k}{(2\pi)^d} (k^2 + r) |\varphi_S(\vec{k})|^2$$