

## lecture 19

Considering small fluctuations about the mean-field superconducting state.

$$S[\Phi, \vec{A}] = \frac{1}{T} \int d^d x \left[ |\nabla - ie\vec{A})\Phi|^2 + \right. \\ \left. + \alpha |\Phi|^2 + \frac{\beta}{2} |\Phi|^4 + \frac{1}{2} (\nabla \times \vec{A})^2 \right]$$

$\alpha$  and  $\beta$  coefficients are also redefined accordingly.

Let's first set  $e=0$  - neutral order parameter, like e.g. in superfluid He.

$$\Phi(x) = e^{i\theta(x)} (\Phi_0 + \psi(x))$$

Expand to second order in fluctuation.

$$|\nabla \Phi|^2 = \Phi_0 (\nabla \theta)^2 + (\nabla \psi)^2$$

$$|\Phi|^2 = \Phi_0^2 + 2\Phi_0 \psi + \psi^2$$

$$|\Phi|^4 \approx \Phi_0^4 + 4\Phi_0^2 \psi^2 + 6\Phi_0^2 \psi^2$$

Terms linear in  $\psi$ , ~~cancel~~ cancel.  
Then we obtain

$$S[\theta, \psi, \vec{A}] = \frac{1}{T} \int d^d x \left[ \Phi_0 (\nabla \theta)^2 + 2[\alpha \psi^2 + \right. \\ \left. + \frac{1}{2} (\nabla \times \vec{A})^2] \right]$$

- Phase fluctuations are the Goldstone modes - energy cost vanishes when  $k \rightarrow 0$ .  
 Amplitude fluctuations have finite energy cost in the  $k \rightarrow 0$  limit.

~~(iii)~~ Vector potential fluctuations are decoupled from phase fluctuations and also have vanishing energy cost in the  $k \rightarrow 0$  limit - these correspond to massless photons when quantized.

Now consider the case  $e \neq 0$ .

In this case we obtain :

$$S[\theta, \psi, \vec{A}] = \frac{1}{T} \int d^d x \left[ \Phi_0^2 (\vec{\nabla} \theta - e \vec{A})^2 + (\vec{\nabla} \psi)^2 + 2|a|\psi^2 + \frac{1}{2} (\vec{\nabla} \times \vec{A})^2 \right]$$

As before, we remove  $\vec{\nabla} \theta$  by a gauge transformation of  $\vec{A}$ :

$$\vec{A} \rightarrow \vec{A} + \frac{1}{e} \vec{\nabla} \theta$$

Then we obtain :

$$S[\theta, \psi, \vec{A}] = \frac{1}{T} \int d^d x \left[ \Phi_0^2 e^2 \vec{A}^2 + (\vec{\nabla} \psi)^2 + 2|a|\psi^2 + \frac{1}{2} (\vec{\nabla} \times \vec{A})^2 \right]$$

New Goldstone modes have disappeared.

This is Anderson-Higgs phenomenon - in systems with ~~gauge symmetry~~ "charged" order parameter there are no Goldstone modes.

Now let us consider superconductor-normal state transition as a function of temperature.

In MFT  $|\Phi| \sim \sqrt{T_c - T}$  as usual in mean-field theory. In most ~~superconductors~~ ~~superconducting~~ superconductors MFT actually works very well - deviations from it appear only so close to  $T_c$  that they are not observable. However, for the sake of being rigorous, we still need to consider fluctuations - they do have a significant influence on what happens very close to  $T_c$ .

Let us first consider "rough" type-I superconductors:

$$\mathcal{R} = \frac{\lambda}{\zeta} \ll 1$$

In this limit we can expect the fluctuations of  $\vec{A}$  to be much more important than fluctuations of  $\Phi$  so we can assume that  $\Phi$  is uniform, like in MFT.

Then we obtain:

$$S[\Psi, \vec{A}] = \frac{1}{T} \int d^3x \left[ e^2 \vec{A}^2 |\Phi|^2 + a |\Phi|^2 + \frac{1}{2} |\vec{A}|^4 + \frac{1}{2} (\vec{\nabla} \times \vec{A})^2 \right]$$

We want to integrate over  $\vec{A}$ . To do this we need to choose a particular gauge.

Use transverse gauge:  $\vec{\nabla} \cdot \vec{A} = 0$

$$(\vec{\nabla} \times \vec{A})^2 = \vec{A} \cdot \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) - \vec{\nabla} \cdot (\vec{A} \cdot \vec{\nabla} \times \vec{A}) \xrightarrow{?} =$$

$$= \vec{A} \cdot \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \vec{A} \cdot \nabla^2 \vec{A} = -\vec{A} \cdot \nabla^2 \vec{A}$$

Then, going to Fourier space, we obtain:

$$S[\Phi, \vec{A}] = \frac{1}{2\pi V} \sum_{\vec{k}} \left( k^2 + 2e^2 |\Phi|^2 \right) \vec{A}(\vec{k}) \cdot \vec{A}(-\vec{k}) +$$

$$+ \frac{V}{2} \left[ \alpha |\Phi|^2 + \frac{\beta}{2} |\Phi|^4 \right]$$

Integrating over  $\vec{A}$ , we obtain:

$$Z = e^{-\frac{V}{2} \left[ \alpha |\Phi|^2 + \frac{\beta}{2} |\Phi|^4 \right]} \det \mathcal{F}$$

$$\text{where } G_{ij}(\vec{k}) = \langle A_i(\vec{k}) A_j(-\vec{k}) \rangle =$$

$$\frac{\delta_{ij} - \hat{k}_i \hat{k}_j}{k^2 + 2e^2 |\Phi|^2}$$

The factor  $\delta_{ij} - \hat{k}_i \hat{k}_j$  takes care of  $\vec{k} \cdot \vec{A}(\vec{k}) = 0$   
condition.

Reexpanding  $\det \mathcal{F}$  and using  $\ln \det \mathcal{F} =$

$$S[\phi] = \frac{V}{T} \left[ \alpha |\phi|^2 + \frac{\beta}{2} |\phi|^4 \right] + \\ + \sum_{\vec{k}} \ln(k^2 + 2e^2 |\phi|^2)$$

This corresponds to the free energy density:

$$F(\phi) = \alpha |\phi|^2 + \frac{\beta}{2} |\phi|^4 + \\ + T \cdot \frac{V}{T} \sum_{\vec{k}} \ln(k^2 + 2e^2 |\phi|^2) \approx \\ \approx \alpha |\phi|^2 + \frac{\beta}{2} |\phi|^4 + T_c \int \frac{d^3k}{(2\pi)^3} \ln(k^2 + 2e^2 |\phi|^2)$$

$$\frac{df}{d|\phi|} = 2\alpha |\phi| + 2\beta |\phi|^3 + \\ + 4e^2 |\phi| T_c \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + 2e^2 |\phi|^2}$$

Write the integrand as:

$$\frac{1}{k^2 + 2e^2 |\phi|^2} = \frac{1}{k^2} - \frac{2e^2 |\phi|^2}{k^2 (k^2 + 2e^2 |\phi|^2)}$$

Then we obtain:

$$\frac{df}{d|\Phi|} = 2a|\Phi| + 2b|\Phi|^3 + 4e^2 T_c |\Phi| \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} -$$

$$- 8e^4 T_c |\Phi|^3 \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 (k^2 + 2e^2 |\Phi|^2)}$$

The third term is unimportant - just renormalizes  $T_c$ .

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 (k^2 + 2e^2 |\Phi|^2)} =$$

$$= \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k^2 + 2e^2 |\Phi|^2} \approx *$$

$$e^2 |\Phi|^2 = \frac{1}{2^2}$$

Assume  $\lambda \gg 1$

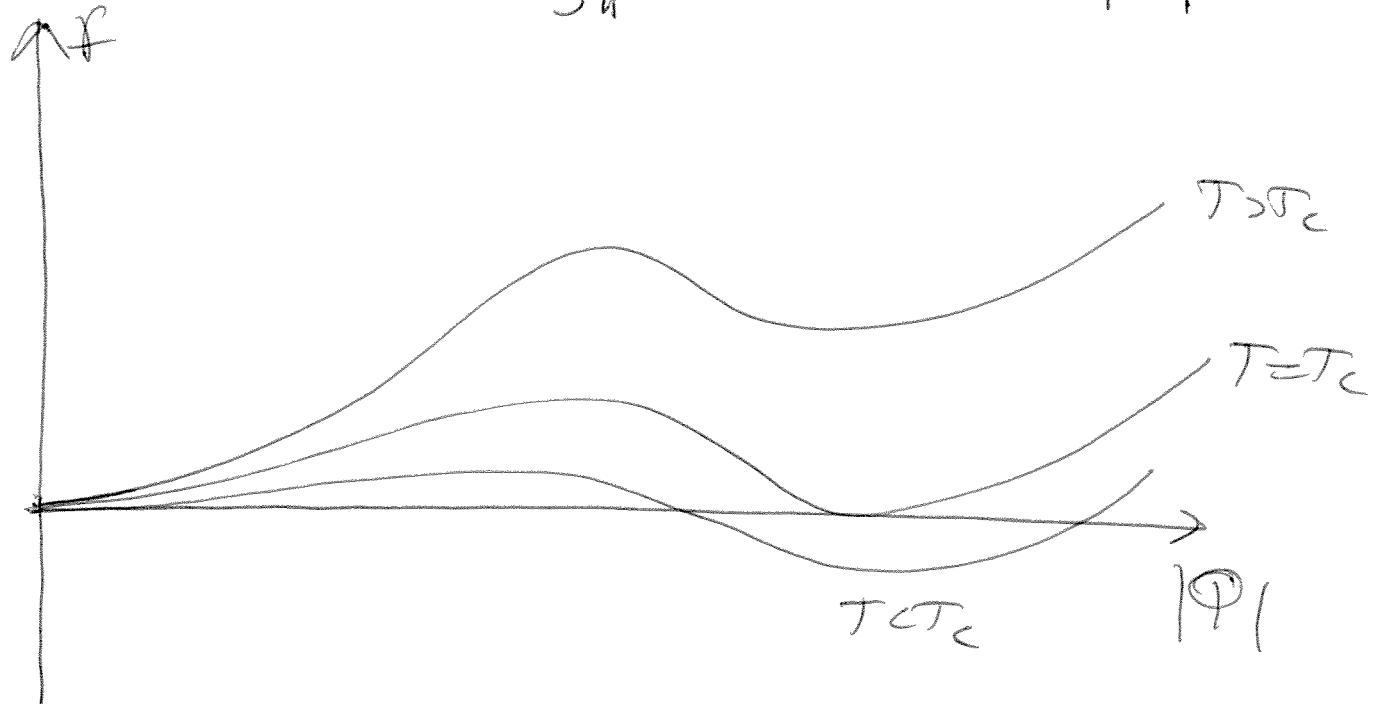
$$\approx \frac{1}{2\pi^2} \int_0^\infty \frac{dk}{k^2 + 2e^2 |\Phi|^2} = \frac{1}{4\pi\sqrt{2} e |\Phi|}$$

Then we obtain:

$$\frac{df}{d|\Phi|} = 2a|\Phi| + 2b|\Phi|^3 - \frac{\sqrt{2} e^3 T_c}{\pi} |\Phi|^2$$

This means that  $f$  is given by:

$$f(\Phi) = \alpha |\Phi|^2 - \frac{\sqrt{2} e^3 T_c}{3\pi} |\Phi|^3 + \frac{k}{2} |\Phi|^4$$



Coupling of  $\Phi$  to fluctuating vector potential  $\vec{A}$   
leads to fluctuation-induced first order transition.

The size of the jump is extremely small and is not observable experimentally.

The above was true for extreme type-I superconductors.

Now let us consider a general value of  $\chi = \frac{1}{3}$ .

In general, both fluctuations of the order parameter and the fluctuations of  $\vec{A}$  are important. Can try RG.

$$S[\Phi, \vec{A}] = \int d^4x \left[ \left| (\vec{\nabla} - ie\vec{A})\Phi \right|^2 + \alpha |\Phi|^2 + \frac{6}{2} |\Phi|^4 + \frac{1}{2} (\vec{\nabla} \times \vec{A})^2 \right]$$

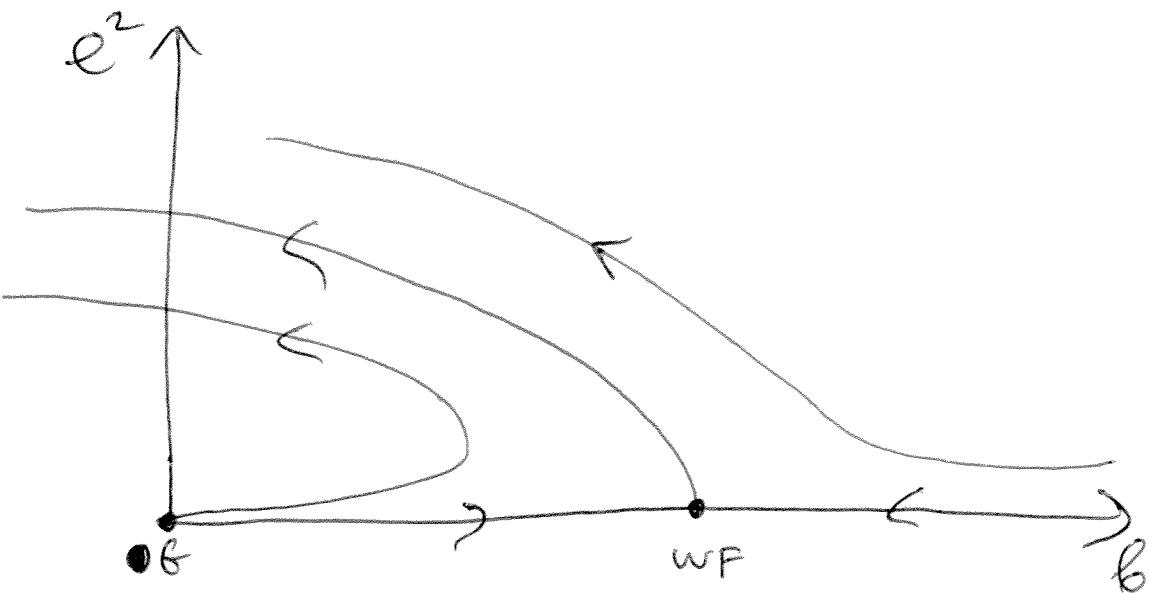
RF ~~oscillation~~ will generate flows of  $a$ ,  $b$  and  $e$ .  
The flow equations have the form:

$$\frac{db}{dl} = \varepsilon b + 6e^2 b - (N+4)b^2 - 6e^4$$

$$\frac{de^2}{dl} = \varepsilon e^2 - \frac{N}{3} e^4$$

Here  $N$  is the number of components of  $\Phi$  ( $N=1$  is the physical case). Flow diagram has different topology depending on  $N$ .

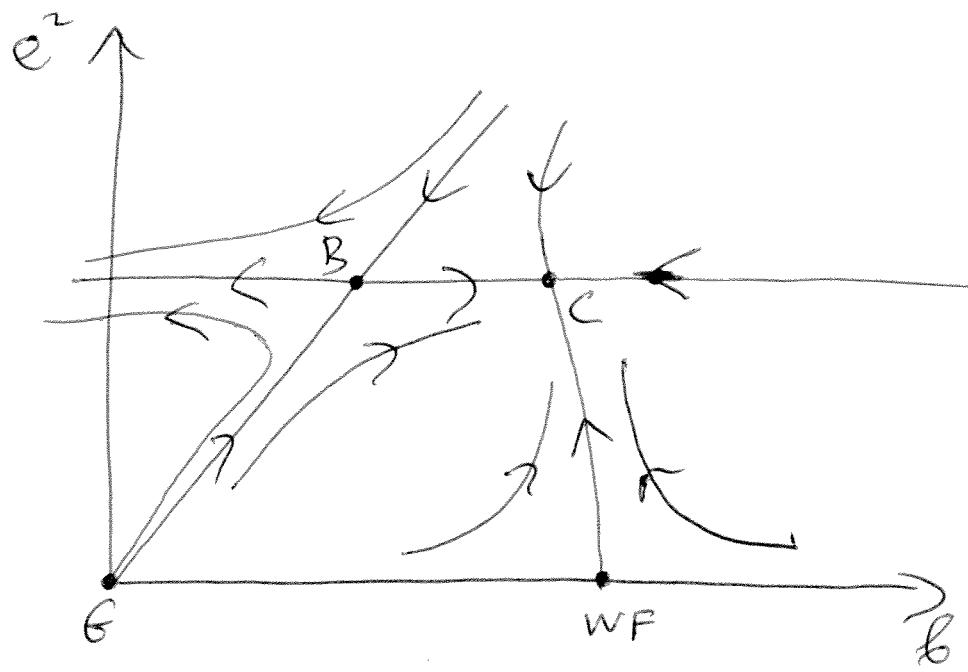
If  $N < N_c = 182.952$ , the flow diagram has the form,



There are two fixed points: Gaussian and Wilson-Fisher.  
Both are unstable and the flows ~~go~~ go in the  $\beta < 0$  direction.

$\beta < 0$  generally means a first order transition (recall homework).  
Thus such a flow diagram is interpreted as indicating a first-order transition.

For  $N > N_c$ , the flows have the form:



Hence in this case there is a stable fixed point (C) with  $e^2 > 0$  and  $\beta > 0$ .

Thus  $\epsilon$ -expansion predicts that the transition in the physical case  $N=1$  is always first order.

This is believed not to be true. We will show that the transition in extreme type-II case, i.e.  $N \gg 1$ , is actually second order.