

## lecture 17

Continue deriving RG flow equations for the sine-Gordon

In lecture 16 we ~~had~~ have obtained SF functional model for the slow modes.

$$S[\varphi_c] = \frac{1}{2} \int d^2x (\vec{\nabla} \varphi_c)^2 -$$

$$- y e^{-\frac{1}{2} G_s(0)} \int d^2x \cos(\omega \varphi_c(\vec{x})) -$$

$$- y^2 e^{-G_s(0)} \int d^2x d^2x' \left\{ \cos[\omega \varphi_c(\vec{x}) + \omega \varphi_c(\vec{x}')] \right. \\ \cdot [e^{-G_s(\vec{x}-\vec{x}')} - 1] + \cos[\omega \varphi_c(\vec{x}) - \omega \varphi_c(\vec{x}')] \\ \cdot \left. [e^{G_s(\vec{x}-\vec{x}')} - 1] \right\}$$

Second term is the renormalized frequency term:

$$y^1 = y \beta^{2-\frac{n}{4}}$$

$y^2$  terms renormalize the temperature.

$$\cos[\omega \varphi_c(\vec{x}) - \omega \varphi_c(\vec{x}')] \approx \cos[\omega (\vec{x}-\vec{x}') \cdot \vec{\nabla} \varphi_c(\vec{x})] \approx$$

$$\approx 1 - \omega^2 [(\vec{x}-\vec{x}') \cdot \vec{\nabla} \varphi_c(\vec{x})]^2 + \dots$$

$$\begin{aligned}
 & \cos[\bar{m}\varphi_c(\vec{x}) + \bar{m}\varphi_c(\vec{x}')] = \\
 & \approx \cos[\bar{m}\varphi_c(\vec{x}) + \bar{m}(\vec{x}' - \vec{x}) \cdot \vec{\nabla}\varphi_c(\vec{x}')] = \\
 & = \cos[\bar{m}\varphi_c(\vec{x})] \cos[\bar{m}(\vec{x}' - \vec{x}) \cdot \vec{\nabla}\varphi_c(\vec{x})] - \\
 & - \sin[\bar{m}\varphi_c(\vec{x})] \cdot \sin[\bar{m}(\vec{x}' - \vec{x}) \cdot \vec{\nabla}\varphi_c(\vec{x})] \approx
 \end{aligned}$$

The expansion of  $\sin$  will contain ~~odd~~ odd powers of  $(\vec{x}' - \vec{x}) \Rightarrow$  will vanish upon integration over  $\vec{x}'$ .

$$\approx \cos[\bar{m}\varphi_c(\vec{x})] \cdot [1 - \bar{m}^2((\vec{x}' - \vec{x}) \cdot (\vec{\nabla}\varphi_c(\vec{x})))^2]$$

Will neglect this since this is proportional to  $\cos(\bar{m}\varphi_c)$  - we have neglected such terms to begin with. ~~and neglect~~

Then  $T$  is renormalized as:

$$T' = T + \bar{m}^2 y^2 e^{-G_0(0)} \int d^2x x^2 [e^{G_0(\vec{x})} - 1]$$

$$\begin{aligned}
 f_0(\vec{x}) &= \frac{(\bar{m})^2}{T} \int \frac{d^2q}{(2\pi)^2} \frac{e^{i\vec{q} \cdot \vec{x}}}{q^2} = \\
 &= \frac{1}{T} \int_0^\infty dq \int_{-\pi/2}^{\pi/2} dq \cdot q \frac{e^{iqx \cos\varphi}}{q^2} = \\
 &= \frac{\bar{m}}{T} \int_{-\pi/2}^{\pi/2} dq \frac{J_0(qx)}{q} \underset{q \rightarrow 0}{\approx} -\frac{\bar{m} \times 1 \ln b}{T} \left. \frac{d K_0(z)}{dz} \right|_{z=1x}
 \end{aligned}$$

K<sub>0</sub>-modified Bessel Function.

$$\begin{aligned} T' &= T - \pi^2 y^2 \int dx x^2 \frac{\pi x \lambda}{T} \left. \frac{d K_0}{dx} \right|_{x=\lambda} \ln b \\ &= T + \frac{1}{2T} \left( \frac{(4\pi)^2 y}{\lambda^2} \right)^2 \ln b \end{aligned}$$

Define dimensionless frequency:

$$\tilde{y} = y \frac{(4\pi)^2}{\lambda^2}$$

Then we obtain:

~~$\tilde{y}' = \tilde{y} b^{2-\frac{\pi}{T}}$~~

$$T' = T + \frac{1}{2T} \tilde{y}^2 \ln b$$

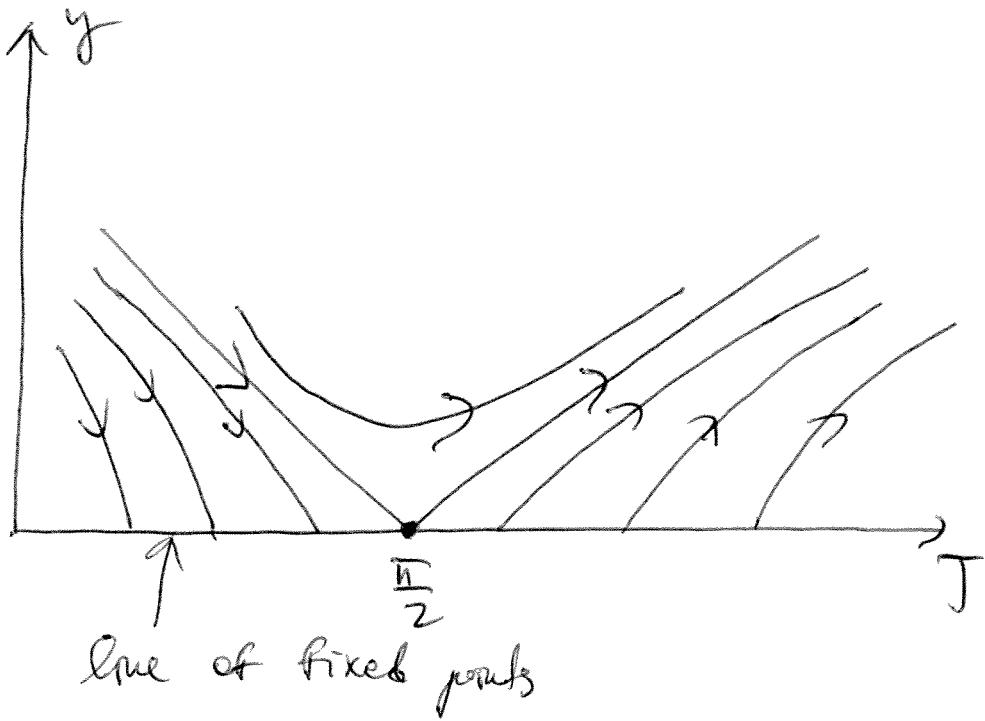
$$b = e^{\Delta l} \approx 1 + \Delta l$$

Then we obtain the following RG flow equations:

$$\begin{cases} \frac{dy}{dl} = \left(2 - \frac{\pi}{T}\right) y \\ \frac{dT}{dl} = \frac{\tilde{y}^2}{2T} \end{cases}$$

I dropped tilde for  $\tilde{y}$ .

Fixed point  $T^* = \frac{\pi}{2}$ ,  $y^* = 0$ .



$T < T_{\text{fc}}$ :  $y$  is irrelevant

$T > T_{\text{fc}}$ :  $y$  is relevant.

$$y = e^{-\frac{E_c}{T}}$$

$y \rightarrow 0$  means diverging vortex core energy  $\Rightarrow$  vanishing probability for vortices to appear.

When  $T < T_{\text{fc}}$ , vortices exist only in finite size vortex-antivortex pairs. At short length scales there is then a finite vortex density. As we perform coarse-graining RG transformation, vortex density decreases, and eventually disappears when we coarse grain over length scales much larger than the vortex-antivortex pair size. This is why  $y$  flows to zero when  $T \ll T_{\text{fc}}$ .

When  $T > T_{kr}$ , free vortices appear.

In this case  $R^*$  makes the distance between the free vortices shorter ~~constant~~ and their density larger  $\Rightarrow$  Singularity grows.

Find the behavior of the correlation length for  $T > T_{kr}$ .

Normally this is related to eigenvalues of the stability matrix at the fixed point.

$$\beta_T(T, y) = \frac{y^2}{2T}$$

$$\beta_y(T, y) = \left(2 - \frac{\pi}{T}\right) y$$

$$\frac{\partial \beta_T}{\partial T} \Big|_{y^*, T^*} = - \frac{y^{*2}}{2T^{*2}} = 0$$

$$\frac{\partial \beta_T}{\partial y} \Big|_{y^*, T^*} = \frac{y^*}{T^*} = 0$$

$$\frac{\partial \beta_y}{\partial T} \Big|_{y^*, T^*} = \frac{\pi}{T^{*2}} y^* = 0$$

$$\frac{\partial \beta_y}{\partial y} \Big|_{y^*, T^*} = 2 - \frac{\pi}{T^*} = 0$$

This stability matrix supply vanishing of the fixed point. This signals that the correlation length does not diverge as a simple power law of the transition.

Have to integrate flow equations near the fixed part.

Introduce new variables:

$$z = y^2, x = 2 - \frac{\pi}{T} ; z^* = x^* = 0$$

$$\frac{dz}{dt} = 2x z$$

$$\frac{dx}{dt} = \frac{\pi}{T^2} \quad \frac{dt}{dl} = \frac{\pi}{T^2} \quad \frac{y^2}{2T} \cong \frac{\pi}{2 \left(\frac{\pi}{2}\right)^3} z = \left(\frac{2}{\pi}\right)^2 z$$

$$\begin{cases} \frac{dz}{dt} = 2x z \\ \frac{dx}{dt} = \left(\frac{2}{\pi}\right)^2 z \end{cases}$$

Dividing first equation by the second, we obtain:

$$\frac{dz}{dx} = 2 \left(\frac{\pi}{2}\right)^2 x = \frac{\pi^2}{2} x$$

Thus  $z = \left(\frac{\pi}{2}\right)^2 x^2 + \text{scattered } \sigma$

~~scattered~~

~~scattered~~

$$\text{y}^2 = \left(\frac{\pi}{2}\right)^2 x^2 + \sigma$$

$\sigma$  parameterizes a family of hyperbolic curves.

$\sigma \geq 0$  gives the two asymptotes:

$$y = \pm \frac{\pi}{2} x$$

$$x < 0, y = -\frac{\pi}{2} x$$

$$x > 0, y = \frac{\pi}{2} x$$

• Trajectories corresponding to  $\sigma > 0$ , lie above the two asymptotes.

Trajectories with  $\sigma < 0$  lie below and intersect the line  $y=0$  at points  $x = \pm \frac{2}{\pi} \sqrt{-\sigma}$ .

Clearly we can then identify  $\sigma$  with the deviation from the critical point:

$\sigma \sim \frac{T - T_{KT}}{T_{KT}}$ , line  $y = -\frac{\pi}{2} x$  is the critical line.

Now solve for  $x$ :

$x = \frac{2}{\pi} \sqrt{z - \sigma}$ , and integrate along the flow trajectory:

$$\frac{dz}{dl} = 2xz = \frac{4}{\pi} z \sqrt{z - \sigma}$$

$$\int_0^l dl' = \frac{\pi}{4} \int_{z(0)}^{z(l)} \frac{dz'}{z' \sqrt{z'^2 - \sigma}}$$

Take  $x(0) = 0$ , then  $z(0) = \sigma > 0 \Rightarrow T > T_{\text{cr}}$ .

Take  $z(l) = 1$ .

Then we have:

$$l = \frac{\pi}{4} \int_{\sigma}^1 \frac{dz}{z \sqrt{z^2 - \sigma}} \approx \frac{\pi^2}{4\sqrt{\sigma}} \quad \text{for } \sigma \ll 1.$$

When  $z = y^2$  is small ~~reached beyond large scales~~  
~~and shorter than~~ the cutoff length scale is much  
 shorter than the separation between the ~~free~~ free  
 vortices = correlation length  $\xi$ .

When  $z = 1$  ~~reached beyond correlated~~ The cutoff  
 length scale have reached the correlation length.

$$\text{Thus } \xi \sim l = e^l = e^{\frac{\pi^2}{4\sigma}}$$

Thus we obtain:

$$\xi \sim e^{C \sqrt{\frac{T_{\text{cr}}}{T - T_{\text{cr}}}}} , T > T_{\text{cr}}$$

$C$  is a universal constant.

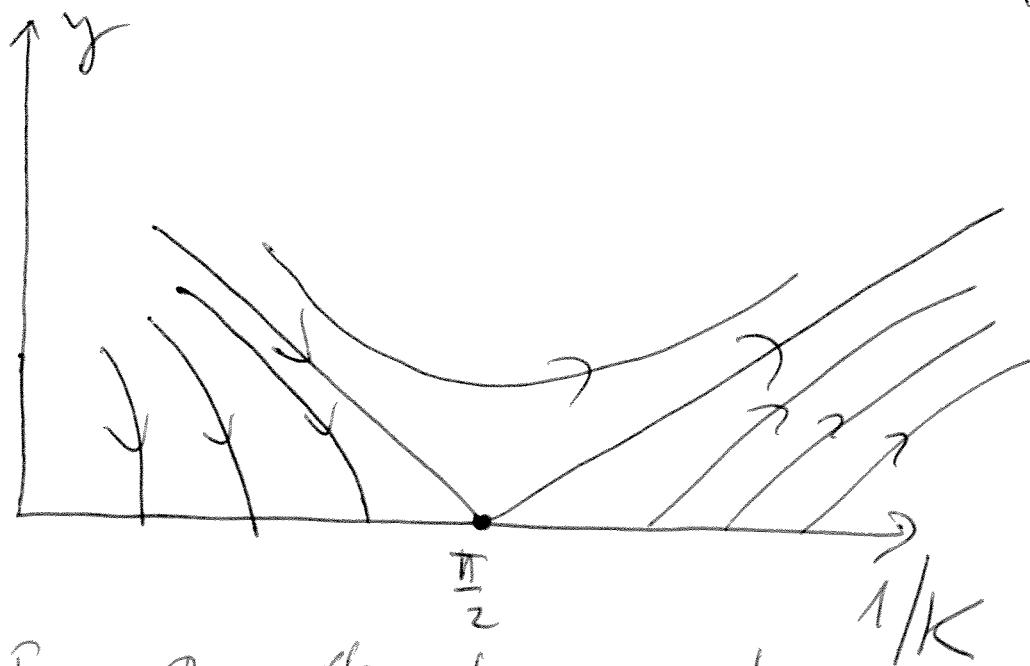
Thus  $\xi$  diverges faster than any power law at the KT transition.

~~•~~ Specific heat and susceptibility also exhibit non-power-law singularities!

Finally, let us see what happens with the generalized rigidity  $\zeta$  at the transition.



Let  $K(l=0) = \frac{\gamma}{T}$  - the microscopic rigidity, divided by temperature.



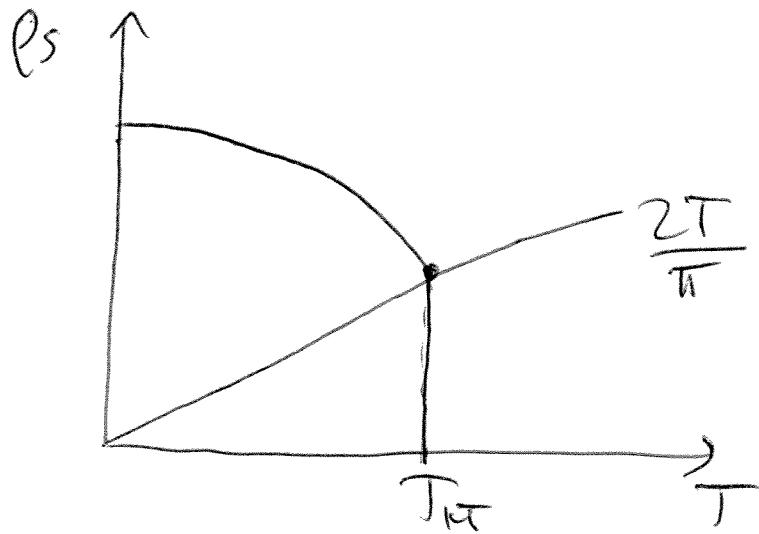
~~•~~ From the flow diagram we have:

$$\underset{\Delta T \rightarrow 0+}{\lim} \underset{l \rightarrow \infty}{\lim} [K(l, T_{KT} - \Delta T) - K(l, T_{KT} + \Delta T)] = \frac{2}{\pi}$$

$K(l \rightarrow \infty, T)$  is the microscopic rigidity at a

given temperature.

Thus  $K(T)$  exhibits a universal jump at  $T=T_{KT}$  of magnitude  $\frac{2}{T}$ .



$$K = \frac{C_s}{T}$$

This has been observed in experiments on filaments of superfluid He.

Landau functional for a superfluid:

$$S[\Phi] = \frac{1}{T} \int d^2x \left[ \frac{1}{2m} |\vec{\nabla} \Phi|^2 + \alpha |\Phi|^2 + \frac{\beta}{2} |\Phi|^4 \right]$$

$\Phi(\vec{x})$  is ~~the~~ a macroscopic wavefunction of the superfluid.

Sufficiently far below the mean-field transition temperature we can set:

$$\Phi(\vec{x}) = \Phi_0 e^{i\theta(\vec{x})}$$

$$S[\theta] = \frac{\Phi_0^2}{T} \int d^2x (\vec{\nabla} \theta)^2 - XY\text{-model.}$$

Start superconducting transition.

Ginzburg - Landau ~~free energy~~ free energy of a superconductor:

$$F = \int d^d x \left[ \frac{1}{2m^*} \left| (-i\hbar \vec{\nabla} + \frac{e^*}{c} \vec{A}) \Phi \right|^2 + \alpha (\Phi^2 + \frac{b}{2} |\Phi|^4 + \frac{\vec{B}^2}{8\pi} - \frac{\vec{B} \cdot \vec{H}}{4\pi} ) \right]$$

Here  $m^* = m$ ,  $e^* = 2e$ ,  $\vec{H}$  is the external magnetic field,  $\vec{B}$  is the ~~total~~ total field in the superconductor,  $\vec{B} = \vec{\nabla} \times \vec{A}$ .

$\Phi(\vec{x})$  is a macroscopic condensate wavefunction of the electron pairs.

Superconductivity is ~~not~~ superfluidity of electron pairs. However, unlike He atoms, electron pairs are charged. This is why there is coupling to electromagnetic field.

~~the~~ Thermodynamic equilibrium state of a superconductor is determined by minimizing  $F$ .

Vary  $F$  with respect to  $\Phi^*$ .

$$\delta_{\Phi^*} F = \int d^d x \left[ \alpha \Phi \delta \Phi^* + b \Phi |\Phi|^2 \delta \Phi^* + \frac{1}{2m^*} \left( i\hbar \vec{\nabla} \delta \Phi^* + \frac{e^*}{c} \vec{A} \delta \Phi^* \right) \cdot \left( -i\hbar \vec{\nabla} \Phi + \frac{e^*}{c} \vec{A} \Phi \right) \right]$$

$$\begin{aligned}
 & \int d^d x \vec{\nabla} \delta \Phi^* \cdot \left( -i\hbar \vec{\nabla} + \frac{e^*}{c} \vec{A} \right) \Phi = \\
 &= \int d^d x \vec{\nabla} \cdot \left[ \delta \Phi^* \left( -i\hbar \vec{\nabla} + \frac{e^*}{c} \vec{A} \right)^* \Phi \right] - \\
 & - \int d^d x \delta \Phi^* \vec{\nabla} \cdot \left( -i\hbar \vec{\nabla} + \frac{e^*}{c} \vec{A} \right) \Phi = \\
 &= - \int d^d x \delta \Phi^* \vec{\nabla} \cdot \left( -i\hbar \vec{\nabla} + \frac{e^*}{c} \vec{A} \right) \Phi
 \end{aligned}$$

Then we obtain :

$$\begin{aligned}
 \delta_{\Phi^*} F = & \int d^d x \left[ a \Phi + b |\Phi|^2 \Phi + \right. \\
 & \left. + \frac{1}{m^*} \left( -i\hbar \vec{\nabla} + \frac{e^*}{c} \vec{A} \right)^2 \Phi \right] \delta \Phi^* = 0
 \end{aligned}$$

Thus we get the first GL equation:

$$\frac{1}{m^*} \left( -i\hbar \vec{\nabla} + \frac{e^*}{c} \vec{A} \right)^2 \Phi + a \Phi + b |\Phi|^2 \Phi = 0$$

Look like a Schrödinger equation for a particle in a magnetic field, but nonlinear.

The magnetic field inside the sample is also a thermodynamic variable  $\Rightarrow$  also have to minimize  $F$  with respect to  $\vec{A}$ .

$$\begin{aligned}
 \delta_{\vec{A}} F = & \int d^d x \left[ \frac{1}{m^*} \frac{e^*}{c} \delta \vec{A} \Phi^* \cdot \left( -i \hbar \vec{\nabla} + \frac{e^*}{c} \vec{A} \right) \Phi + \right. \\
 & + \frac{1}{m^*} \left( i \hbar \vec{\nabla} + \frac{e^*}{c} \vec{A} \right) \Phi^* \cdot \frac{e^*}{c} \delta \vec{A} \Phi + \\
 & \left. + \frac{1}{q_0} \left( \vec{\nabla} \times \vec{A} \right) \cdot \left( \vec{\nabla} \times \delta \vec{A} \right) - \frac{1}{q_0} \vec{H} \cdot \left( \vec{\nabla} \times \delta \vec{A} \right) \right]
 \end{aligned}$$

Consider the last two terms:

$$\frac{1}{q_0} \int d^d x \left[ (\vec{\nabla} \times \vec{A}) - \vec{H} \right] \cdot (\vec{\nabla} \times \delta \vec{A})^* = *$$

Use Identity:

$$\vec{a} \cdot (\vec{\nabla} \times \vec{b}) = \vec{b} \cdot (\vec{\nabla} \times \vec{a}) - \vec{\nabla} \cdot (\vec{a} \times \vec{b})$$

$$\begin{aligned}
 * &= \frac{1}{q_0} \int d^d x \delta \vec{A} \cdot \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) - \xrightarrow{0} \\
 &- \frac{1}{q_0} \oint d \vec{s} \cdot \left[ \delta \vec{A} \times (\vec{\nabla} \times \vec{A} - \vec{H}) \right] = \\
 &= \frac{1}{q_0} \int d^d x \delta \vec{A} \cdot \vec{\nabla} \times (\vec{\nabla} \times \vec{A})
 \end{aligned}$$

Then we obtain :

$$\delta_{\vec{A}} F = \int d^4x = \left[ -\frac{i\hbar e^*}{m^* c} (\phi^* \vec{\nabla} \phi - \phi \vec{\nabla} \phi^*) + \right. \\ \left. + \frac{e^{*2}}{m^* c^2} |\phi|^2 \vec{A} + L \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) \right] \cdot \delta \vec{A} = 0$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}$$

Thus we obtain :

$$\vec{j} = \frac{i\hbar e^*}{m^*} (\phi^* \vec{\nabla} \phi - \phi \vec{\nabla} \phi^*) - \frac{e^{*2}}{m^* c} |\phi|^2 \vec{A} -$$

superconducting current.