

## Lecture 13

In lecture 12 we started looking at the low-temperature ( $\epsilon = d - 2$ ) expansion for the NLSM.

$$S[\vec{\pi}] = \frac{1}{2T} \int d^d k \left[ \vec{\nabla} \pi^a \cdot \vec{\nabla} \pi^a + (\pi^a \vec{\nabla} \pi^a) (\pi^b \vec{\nabla} \pi^b) - \right. \\ \left. - \frac{T}{\alpha^d} \pi^2 \right]$$

Rewrite in Fourier space:

$$S[\vec{\pi}] = \frac{1}{2T} \int_K k^2 \pi^a(\vec{k}) \pi^a(-\vec{k}) - \\ - \frac{1}{2T} \int_{k_1, \dots, k_4} (2\pi)^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \vec{k}_2 \cdot \vec{k}_4 \pi^a(\vec{k}_1) \pi^a(\vec{k}_2) \\ \cdot \pi^b(\vec{k}_3) \pi^b(\vec{k}_4) - \frac{T}{\alpha^d} \cdot \frac{1}{2T} \int_K \pi^a(\vec{k}) \pi^a(-\vec{k})$$

As before, split into fast and slow modes:

$$\pi^a(\vec{k}) = \pi^a_<(\vec{k}) + \pi^a_>(\vec{k})$$

$$\pi^a_<(\vec{k}) = \begin{cases} \pi^a(\vec{k}), & 0 < |k| < 1/\epsilon \\ 0, & 1/\epsilon < |k| < \infty \end{cases}$$

$$\pi^a(\vec{k}) = \begin{cases} 0, & 0 < k < 1/6 \\ \pi^a(\vec{k}), & 1/6 < k < 1 \end{cases}$$

$$S[\vec{n}] = S_0[\vec{n}_<] + S_0[\vec{n}_>] + S_{\text{int}}[\vec{n}_<, \vec{n}_>]$$

Integrating over the fast modes, we obtain:

$$S'[\vec{n}_<] = S_0[\vec{n}_<] + \langle S_{\text{int}}[\vec{n}_<, \vec{n}_>] \rangle_o + \dots$$

First contribution will be enough here.

$$\langle \pi^a(\vec{k}_1) \pi^b(\vec{k}_2) \rangle = \delta_{ab} \cancel{\int d^d k} (\omega)^d \delta(\vec{k}_1 + \vec{k}_2) G_0(\vec{k}_1)$$

$$G_0(\vec{k}_1) = \frac{T}{\vec{k}^2} \quad \text{- ignore the contribution of the} \\ \frac{T}{\omega} \int_k \pi^a(\vec{k}) \pi^a(-\vec{k}) \text{ to the fast mode propagator.}$$

$$\langle S_{\text{int}}[\vec{n}_<, \vec{n}_>] \rangle = -\frac{1}{2T} \int_{k_1, \dots, k_4} (\omega)^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4).$$

$$\cdot \vec{k}_2 \cdot \vec{k}_4 \langle \pi^a(\vec{k}_1) \pi^a(\vec{k}_2) \pi^b(\vec{k}_3) \pi^b(\vec{k}_4) \rangle_o =$$

$$= -\frac{1}{2T} \int_{k_1, \dots, k_4} (\omega)^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \vec{k}_2 \cdot \vec{k}_4 \cdot$$

$$\pi^a(\vec{k}_1) \pi^a(\vec{k}_2) \pi^b(\vec{k}_3) \pi^b(\vec{k}_4) -$$

$$-\frac{1}{2T} \int_{k_1, \dots, k_4} (\bar{m})^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \vec{k}_2 \cdot \vec{k}_4 .$$

$$\cdot \pi_L^q(\vec{k}_1) \bar{\pi}_L^q(\vec{k}_3) \langle \pi_S^q(\vec{k}_2) \bar{\pi}_S^q(\vec{k}_4) \rangle_{\text{os}} -$$

$$-\frac{1}{2T} \int_{k_1, \dots, k_4} (\bar{m})^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \vec{k}_2 \cdot \vec{k}_4 .$$

$$\cdot \langle \pi_S^q(\vec{k}_1) \bar{\pi}_S^q(\vec{k}_3) \rangle \pi_L^q(\vec{k}_2) \bar{\pi}_L^q(\vec{k}_4)$$

Here I have used the fact that momenta  $\vec{k}_2$  and  $\vec{k}_4$  have to be either both fast or both slow, otherwise the corresponding term vanishes upon integration over fast momenta.

$$\langle S_{\text{out}}[\vec{\pi}_L, \bar{\pi}_S] \rangle_{\text{os}} = -\frac{1}{2T} \int_{k_1, \dots, k_4} (\bar{m})^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) .$$

$$\cdot \vec{k}_2 \cdot \vec{k}_4 \pi_L^q(\vec{k}_1) \bar{\pi}_L^q(\vec{k}_3) \pi_S^q(\vec{k}_2) \bar{\pi}_S^q(\vec{k}_4) +$$

$$+ \frac{1}{2T} \int_k \pi_L^q(\vec{k}) \bar{\pi}_L^q(-\vec{k}) \int_{1/6}^{\infty} \frac{d^d q}{(\bar{m})^d} q^2 f_0(\vec{q}) +$$

$$+ \frac{1}{2T} \int_k k^2 \pi_L^q(\vec{k}) \bar{\pi}_L^q(-\vec{k}) \int_{1/6}^{\infty} \frac{d^d q}{(\bar{m})^d} \cancel{f_0}(\vec{q})$$

$$I_1 = \int_{\lambda/6}^{\lambda} \frac{d^d q}{(\bar{m})^d} G_0(\vec{q}) = \frac{S_d}{(\bar{m})^d} \int_{\lambda/6}^{\lambda} dq q^{d-1} \frac{T}{q^2} = \\ = \frac{S_d T}{(\bar{m})^d} \int_{\lambda/6}^{\lambda} dq q^{d-3} = \frac{S_d T}{(\bar{m})^d} \frac{1}{d-2} \left[ \lambda^{d-2} - \left( \frac{\lambda}{6} \right)^{d-2} \right]$$

$$b = e^{\Delta l}$$

$$1 - \left( \frac{1}{b} \right)^{d-2} \approx (d-2) \Delta l$$

$$I_1 = \frac{S_d \lambda^{d-2} T}{(\bar{m})^d} \Delta l$$

$$I_2 = \int_{\lambda/6}^{\lambda} \frac{d^d q}{(\bar{m})^d} q^2 f_0(\vec{q}) = \frac{S_d T}{(\bar{m})^d} \int_{\lambda/6}^{\lambda} dq q^{d+1} =$$

$$= \frac{S_d \lambda^{d+1} T}{(\bar{m})^d} \Delta l$$

~~$$I_1 = \frac{S_d \lambda^{d-2} T}{(\bar{m})^d} \Delta l$$~~
~~$$I_2 = \frac{S_d \lambda^{d+1} T}{(\bar{m})^d} \Delta l$$~~

Collecting all the terms, we obtain:

$$S'[\bar{n}_c] = \frac{1}{2\pi} \int_{\vec{k}} k^2 \bar{n}_c^a(\vec{k}) \bar{n}_c^a(-\vec{k}) +$$

$$+ \frac{1}{2\pi} I_1 \int_{\vec{k}} k^2 \bar{n}_c^a(\vec{k}) \bar{n}_c^a(-\vec{k}) +$$

$$+ \frac{1}{2\pi} I_2 \int_{\vec{k}} \bar{n}_c^a(\vec{k}) \bar{n}_c^a(-\vec{k}) -$$

$$- \frac{1}{2\pi} \int_{k_1, \dots, k_4} (\bar{m})^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \vec{k}_2 \cdot \vec{k}_4 \cdot$$

$$\cdot \bar{n}_c^a(\vec{k}_1) \bar{n}_c^a(\vec{k}_2) \bar{n}_c^a(\vec{k}_3) \bar{n}_c^a(\vec{k}_4) -$$

$$- \frac{1}{2} \left( \int_0^{N/6} \frac{dq^d}{(\bar{m})^d} + \int_{N/6}^{\infty} \frac{dq^d}{(\bar{m})^d} \right) \int_{\vec{k}} \bar{n}_c^a(\vec{k}) \bar{n}_c^a(-\vec{k})$$

In the last term I am using:

$$\left(\frac{\bar{m}}{a}\right)^d = \int_0^{\bar{m}} \cancel{dq^d} dq^d$$

$$\int_{N/6}^{\infty} \frac{dq^d}{(\bar{m})^d} = \cancel{\frac{S_d}{(\bar{m})^d}} \int_{N/6}^{\infty} dq^d q^{d-1} = \frac{S_d N^d}{(\bar{m})^d d} \Delta l =$$

$$= \frac{S_d N^d}{(\bar{m})^d} \Delta l = \frac{I_2}{T}$$

Thus the term  $-\frac{1}{2} \int_{\mathbb{R}^d} \frac{d^d q}{(2\pi)^d} \int_{\vec{k}} \pi_{\vec{c}}^q(\vec{k}) \pi_{\vec{c}}^q(-\vec{k})$   
 cancels the term, proportional to  $I_2$ .

This means that we can simply omit the last term in  
~~and~~  $S'[\pi_{\vec{c}}]$  and ~~cancel~~ omit the  $I_2$  term at every  
 RG step, since all these terms do cancel each other.

Thus we have:

$$S'[\pi_{\vec{c}}] = \frac{1}{2T} \int_{\vec{k}} (1 + I_1) k^2 \pi_{\vec{c}}^q(\vec{k}) \pi_{\vec{c}}^q(-\vec{k}) -$$

$$- \frac{1}{2T} \int_{\vec{k}_1, \dots, \vec{k}_4} (2\pi)^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \vec{k}_1 \cdot \vec{k}_4 \cdot$$

$$\cdot \pi_{\vec{c}}^q(\vec{k}_1) \pi_{\vec{c}}^q(\vec{k}_2) \pi_{\vec{c}}^q(\vec{k}_3) \pi_{\vec{c}}^q(\vec{k}_4)$$

Rescale momenta and the slow fields,

$$\vec{k}' = \vec{k}/b$$

$$\tilde{\pi}_{\vec{c}}^q(\vec{k}') = z^{-1} \pi_{\vec{c}}^q(\vec{k})$$

$$S'[\tilde{\pi}_{\vec{c}}^q] = \frac{1}{2T} \int_{\vec{k}'} b^{-d+2} z^2 k'^2 \pi_{\vec{c}}^{qa}(\vec{k}') \pi_{\vec{c}}^{qa}(-\vec{k}').$$

$$\cdot (1 + I_1) - \frac{1}{2T} \int_{\vec{k}_1', \dots, \vec{k}_4'} b^{-4d} b^d b^{-2} z^4 (2\pi)^d \cdot$$

$$\cdot \delta(\vec{k}_1' + \vec{k}_2' + \vec{k}_3' + \vec{k}_4') \cdot \vec{k}_1' \cdot \vec{k}_4' \pi_{\vec{c}}^{qa}(\vec{k}_1') \pi_{\vec{c}}^{qa}(\vec{k}_2') \pi_{\vec{c}}^{qb}(\vec{k}_3') \pi_{\vec{c}}^{qb}(\vec{k}_4')$$

$$S'[\vec{u}'] = \frac{Z^2}{2Tb^{d+2}} \int_{K'} \text{[redacted]} |k'|^2 \pi'^a(\vec{k}_1) \pi'^a(-\vec{k}_1) (1 + I_1) -$$

$$- \frac{Z^4}{2Tb^{3d+2}} \int_{K'_1, \dots, K'_4} (\vec{u}')^d \delta(\vec{k}'_1 + \vec{k}'_2 + \vec{k}'_3 + \vec{k}'_4) \vec{k}'_1 \cdot \vec{k}'_4 .$$

$$\cdot \pi'^a(\vec{k}'_1) \pi'^a(\vec{k}'_2) \pi'^b(\vec{k}'_3) \pi'^b(\vec{k}'_4)$$

Wavefunction renormalization  $Z$  is determined by the constraint  $\sigma^2 + \bar{\pi}^2 = 1$ .

We implemented this constraint by solving:

$$\sigma(\vec{x}) = \sqrt{1 - \bar{\pi}^2(\vec{x})}$$

See how this constraint is affected by the integration over fast modes.

Take the expectation value of  $\sigma(\vec{x})$  over fast modes at a particular point in space. Since all points are equivalent, can take  $\vec{x} = 0$ .

$$\langle \sigma(0) \rangle_0 = \left\langle \sqrt{1 - \bar{\pi}^2(0)} \right\rangle_0 \approx 1 - \frac{1}{2} \langle \bar{\pi}^2(0) \rangle_0$$

On the other hand,

$$\langle \sigma(0) \rangle_0 = \int_k \sigma_<(\vec{k}) + \int_k \cancel{\langle \sigma_>(\vec{k}) \rangle_0^0}$$

$\langle \sigma_>(\vec{k}) \rangle_0^0 \approx 0$  since ~~we expect it to be zero~~ we expect the expectation value of  $\sigma(\vec{x})$  to be uniform, i.e. occur at  $\vec{k} = 0$ .

Then we obtain:

$$\sigma_L(0) = 1 - \frac{1}{2} \int_{\vec{k}} \langle \bar{n}_S^q(\vec{k}) n_S^q(-\vec{k}) \rangle_0 - \\ - \frac{1}{2} \int_{k_1 k_2} \bar{n}_S^q(\vec{k}_1) n_S^q(\vec{k}_2)$$

Crucial property:

Since  $\sigma = \sqrt{1 - \bar{n}^2}$ , we must have  $\sigma = 1$  when  $\bar{n} = 0$ .  
We want to preserve this property at each RG step.

Thus we must have  $\sigma_L(0) = 1$  when  $\bar{n}_L(0) = 0$ .

But this is clearly not the case:

$$\sigma_L(0) = 1 - \frac{1}{2} \int_{\vec{k}} \langle \bar{n}_S^q(\vec{k}) n_S^q(-\vec{k}) \rangle_0 \text{ when } \bar{n}_L(0) = 0.$$

We correct this by the wave function renormalization.

$$\sigma^*(\vec{k}') = z^{-1} \sigma_L(\vec{k}')$$

$$\bar{n}^*(\vec{k}') = z^{-1} \bar{n}_L(\vec{k}')$$

In real space we have:

$$\sigma_L(0) = \int_{\vec{k}} \sigma_L(\vec{k}) = \frac{z}{V^d} \int_{k'} \sigma^*(\vec{k}') = \frac{z}{V^d} \sigma^*(0)$$

$$\bar{n}_L(0) = \frac{z}{V^d} \bar{n}^*(0)$$

Substitute this into the constraint equation:

$$\frac{z}{B^d} \sigma'(0) = 1 - \frac{1}{2} \int_{\vec{k}} \langle \bar{\pi}_S^a(\vec{k}) \bar{\pi}_S^a(-\vec{k}) \rangle_{0S} - \\ - \frac{1}{2} \left( \frac{z}{B^d} \right)^2 \bar{\pi}_S^{1a}(0) \bar{\pi}_S^{1a}(0) + \dots$$

We require that  $\sigma'(0) = 1$  when  $\bar{\pi}'(0) = 0$ .

Then we obtain:

$$\frac{z}{B^d} = 1 - \frac{1}{2} \int_{\vec{k}} \langle \bar{\pi}_S^a(\vec{k}) \bar{\pi}_S^a(-\vec{k}) \rangle_{0S} = \\ = 1 - \frac{n-1}{2} I_1$$

The Landau functional of the slow modes is given by:

$$S'[\bar{\pi}^i] = \frac{z^2}{2TB^{d+2}} \int_{k^1} k^{12} \bar{\pi}^{ia}(\vec{k}^i) \bar{\pi}^{ia}(-\vec{k}^i) (1 + I_1)_+$$

Thus the temperature is renormalized as:

$$\frac{1}{T'} = \frac{1}{T} \left( \frac{z}{B^d} \right)^2 B^{d-2} (1 + I_1)$$

Substituting the expression for  $\left( \frac{z}{B^d} \right)$  obtained above, we get:

$$\frac{1}{T^1} = \frac{1}{T} B^{d-2} \left(1 + I_1\right) \left(1 - \frac{n-1}{2} I_1\right)^2$$

$I_1 \sim \Delta l \Rightarrow$  need only up to first order terms on  $I_1$ .

$$\frac{1}{T^1} \approx \frac{1}{T} B^{d-2} \left(1 + I_1\right) \left(1 - (n-1)I_1\right) \approx$$

$$\approx \frac{1}{T} B^{d-2} \left(1 + I_1 - (n-1)I_1\right) =$$

$$= \frac{1}{T} B^{d-2} \left(1 - (n-2)I_1\right)$$

$$B^{d-2} = e^{(\frac{d-2}{2})\Delta l} \approx 1 + \varepsilon \Delta l, \quad \varepsilon = \frac{d-2}{2}.$$

$$I_1 = \frac{S_d \Delta^{\varepsilon} T}{(\pi)^d} \Delta l$$

$$\frac{1}{T^1} = \frac{1}{T} \left(1 + \varepsilon \Delta l\right) \left[1 - (n-2) \frac{S_d}{(\pi)^d} \Delta^{\varepsilon} T \Delta l\right] =$$

$$= \frac{1}{T} + \frac{1}{T} \left[\varepsilon - (n-2) + \Delta^{\varepsilon} \frac{S_d}{(\pi)^d}\right] \Delta l$$

• Introduce dimensionless temperature:

$$\tilde{T} = T \lambda^\varepsilon$$

$$\text{let } K_d = \frac{S_d}{(\hbar v)^d}$$

~~The~~ The differential form of the RG recursion relation then becomes:

$$\frac{d}{dl} \left( \frac{1}{\tilde{T}} \right) = \frac{1}{\tilde{T}} \left[ \varepsilon - (n-2) \tilde{T} K_d \right]$$

Using  $\frac{d}{dl} \left( \frac{1}{\tilde{T}} \right) = -\frac{1}{\tilde{T}^2} \frac{d\tilde{T}}{dl}$ , we finally obtain:

$$\frac{d\tilde{T}}{dl} = -\varepsilon \tilde{T} + (n-2) K_d \tilde{T}^2 \quad \begin{matrix} \text{-RG flow} \\ \text{equation for NLSM near } d=2. \end{matrix}$$

fixed points:

$$1. \tilde{T}^* = 0$$

$$2. \tilde{T}^* = \frac{\varepsilon}{K_d(n-2)}$$

Analyze the stability of the fixed points.

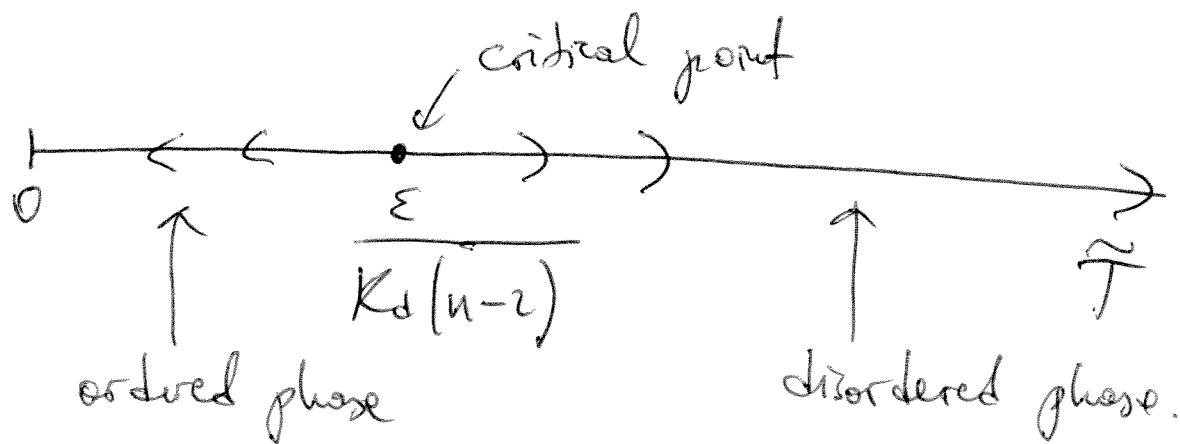
$$\beta(\tilde{T}) = -\varepsilon \tilde{T} + (n-2) K_d \tilde{T}^2$$

$$\frac{d\beta}{dT} = -\varepsilon + 2(n-2)K_d \tilde{T}$$

$$\left. \frac{d\beta}{dT^2} \right|_{T=0} = -\varepsilon$$

$$\left. \frac{d\beta}{dT} \right|_{T=\frac{\varepsilon}{K_d(n-2)}} = -\varepsilon + 2 K_d (n-2) \frac{\varepsilon}{K_d (n-2)} = \varepsilon$$

Thus  $\tilde{T}^{*} > 0$  fixed point is stable,  $\tilde{T}^{*} = \frac{\varepsilon}{K_d(n-2)}$   
 fixed point is unstable.



This is consistent with Mermin-Wagner theorem.

Particularly interesting is the behavior in  $d=2$ .

In this case we have:

$$\frac{d\tilde{T}}{d\ell} = (n-2) K_d \tilde{T}^2$$

$n > 2$  - always in disordered phase - Mean-Field Theory.

$n=1$  - Ising model → there is an ordered phase at low  $\tilde{T}$ .

$n=2$  -  $\beta$ -function vanishes (true to all orders in  $T$ ).

This signals that  $n=2$  in 2D is a special case - we will see that there is in fact a finite- $T$  transition, even though there is no long-range order at any finite  $T$ .